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Switching properties of multivariate Shewhart control charts†

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Abstract

We investigate the properties of multivariate Shewart control charts with VSI procedure for monitoring simultaneous monitoring mean vector and covariance matrix in term of ANSW (average number of switches), probability of switch and ASI (average sampling interval), ATS (average time to signal). From examining the ANSW values, we know that it does not switch frequently. The VSI control charts are superior to the corresponding FSI control charts in terms of ATS. And, it can be also seen that the VSI procedures have substantially fewer switches for small or moderate shifts of the mean vector and variances.

Keywords: Average number of switches, average run length, average sampling interval, average time to signal, switching property.

1. Introduction

The purpose of using a control chart is to detect assignable causes of variation in the process. We want to detect any departure as quickly as possible. In many process-monitoring applications the state of the process is characterized by p continuous random variables, and the joint distribution of these variables is assumed to be multivariate normal. We want to detect special causes that may change mean vector μ and/or covariancematrix Σ.

The first multivariate control charts were Shewhart-type charts developed by Hotelling (1947). General discussions of multivariate monitoring and multivariate Shewhart-type charts can be found in the review articles of Alt (1984), Wierda (1994), Lowry and Montgomery (1995), and Bersimis *et al.* (2007), or in quality control texts such as Montgomery (2009).

The traditional approach to sampling for process monitoring is to use fixed sampling rate (FSR) control charts based on taking samples of fixed size using a fixed-length sampling interval between samples. However, the ability to detect changes in the process can be enhanced by using a variable sampling rate (VSR) control chart. The basic idea of a VSR

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control chart is that the sampling rate should be increased when there is some indication of a change in the process and decreased when there is no indication of a change in the process.

A variable sampling interval (VSI) control chart is a particular type of VSR chart. In particular, a short sampling interval is used next if there is some indication of a problem with the process, and a long sampling interval is used next if there is no indication of a problem.

VSI procedures were first investigated by Arnold (1970). Smeach and Jernigan (1977) extend the work of the Arnold (1970), Reynolds and Arnold (1989) studied for the VSI Shewhart control charts. Chengalur-Smith *et al.* (1989) developed various types of multivariate VSI Shewhart charts for monitoring. Some recent work on the problem of simultaneously monitoring the process mean and variance on univariate VSI control charts by Reynolds and Stoumbos (2001), Stoumbos and Reynolds (2005), and Zhang and Wu (2006) has been studied. Chang and Cho (2005) studied CUSUM control charts for μ with VSI. Reynolds and Cho (2006, 2011), Jeong and Cho (2012a, 2012b) studied multivariate control charts for monotoring the mean vector μ and/or covariancematrix Σ. In this research, we evaluate various switching properties of the VSI multivariate Shewhart control charts for controlling mean vector and variance-covariance matrix. Chang and Heo (2012) studied switching properties of CUSUM control charts for controlling mean vector μ. Gwon and Cho (2015) studied the switching properties of bivariate Shewhart charts for the covariance matrix Σ.

2. Description of control procedures

2.1. Notation and assumptions

Suppose that the process of interest has quality characteristics represented by the X and the distribution is the multivariate normal with mean vector and covariance matrix. In addition to the mean vector μ and covariance Σ, it will be convenient to represent the standard deviations of the p variables using the vector $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_p)'$. Assume that μ_0, σ_0 and Σ_0 are the in-control values for μ, σ and Σ, respectively. We assume that (μ_0, Σ_0) are the known target values for (μ, Σ) and the in-control process covariance matrix is as follows;

$$\Sigma_0 = \begin{pmatrix} \sigma_{011} & \sigma_{012} & \cdots & \sigma_{01p} \\ \sigma_{021} & \sigma_{022} & \cdots & \sigma_{02p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{0p1} & \sigma_{0p2} & \cdots & \sigma_{0pp} \end{pmatrix}.$$

Suppose that we monitor the production process by using a sample of n independent observation vectors taken at each sampling point. Using only two possible sampling intervals simplifies the application of the VSI control chart, so here we consider VSI charts with two sampling intervals.

Let d_1 and d_2 represent the two possible sampling intervals, where $0 < d_1 < d < d_2$. And d is the average sampling interval when the process is in control.

At sampling point k, let X_{kij} represent observation j $(j = 1, 2, \cdots, n)$ for variable i

$(i = 1, 2, \cdots, p)$, and let the corresponding standardized observaiton be

$$Z_{kij} = \frac{(X_{kij} - \mu_{0i})}{\sigma_{0i}},$$

where μ_{0i} and σ_{0i} is the ith component of μ_0 and σ_{0i}, respectively. Also let $Z_{kj} = (Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})'$ be the vector of standardized observations. Let Σ_Z be the covariance matrix of Z_{kj}, and let Σ_{Z0} be the in-control covariance matrix of Σ_Z. Let $\hat{\Sigma}_{Zk}$ be the MLE of Σ_Z, where the (i, i') element of the estimated covariance matrix $\hat{\Sigma}_{Zk}$ is $\sum_{j=1}^{n} Z_{kij} Z_{ki'j}/n$.

Let $\bar{X}_{ki} = \sum_{j=1}^{n} X_{kij}/n$ be the sample mean, so the standardized sample mean is as follows

$$Z_{ki} = \sqrt{n}(\bar{X}_{ki} - \mu_{0i})/\sigma_{0i}, i = 1, 2, \cdots, p.$$

When the process is in control, Z_{ki} has a standard normal distribution, so Σ_{Z0} is the in-control correlation matrix of $(Z_{k1}, Z_{k2}, \cdots, Z_{kp})$.

2.2. Multivariate Shewhart control charts

The Shewhart control chart proposed by Hotelling (1947) for monitoring μ. If Σ_0 is known, then this Shewhart control chart is equivalent to a Shewhart control chart based on the control statistic

$$S_k = (Z_{k1}, Z_{k2}, \cdots, Z_{kp})\Sigma_{Z0}^{-1}(Z_{k1}, Z_{k2}, \cdots, Z_{kp})'.$$

Hotelling (1947) proposed a multivariate control chart for monitoring mean vector based on

$$\sum_{j=1}^{n}(Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})\Sigma_{Z0}^{-1}(Z_{k1j}, Z_{k2j}, \cdots, Z_{kpj})' = ntr(\hat{\Sigma}_{Zk}\Sigma_{Z0}^{-1}).$$

A signal is given if $S_k > h$, where h is the upper control limit. The VSI version of Shewhart chart uses the long sampling interval d_2 if $S_k \leq g$, and the short sampling interval d_1 if $g \leq S_k \leq h$, where $0 \leq g \leq h$. The in-control distribution of S_k is chi-squared with p degrees of freedom, so g and h can be determined using the quantiles of the chi-squared distribution. And let noncentrality parameter be $\tau^2 = n(\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0)$.

2.3. Properties of VSI procedure

By plotting some control statistic in time order, a control chart is maintained. We divide the in-control region into 2 regions I_1, I_2, where I_i is the region for which the sampling interval d_i is used $(i = 1, 2)$. In this paper, we assume that a control chart start at time 0 and d_0 is a fixed constant interval used before the first sample. Then the ARL and ATS are as follows

$$\begin{cases} ARL = 1 + \psi_1 + \psi_2, \\ ATS = d_0 + d_1\psi_1 + d_2\psi_2, \end{cases}$$

where ψ_i is the average number of samples of using sampling interval d_i before the chart signals. Then the ATS can be expressed as $ATS = d \cdot ARL$, d can be interpreted as the ASI of the chart to signal, and ρ_1 can be interpreted as the long-run propotion of sampling interval in which d_1 is used where

$$d = d_1\rho_1 + d_2(1-\rho_1) \text{ and } \rho_1 = \begin{cases} (\psi_1+1)/ARL & \text{if } d_0 = d_1, \\ \psi_1/ATS & \text{if } d_0 = d_2. \end{cases}$$

For theoretical and numerical comparisons, we know that VSI is substantially more efficient than FSI in terms of ARL and ATS. But we cannot get any switching information on VSI schemes from ARL and ATS. Hence, we define the number of switches (NSW) as the number of switches made from the start of the process to the signal, and let ANSW be the expected value of the NSW. The ANSW can be obtained by using Walds identity as follows

$$ANSW = ARL \cdot P(switch),$$

where the ARL can be approximated by a Markov chain approach for the central or noncentral $\chi^2(p)$ distribution for multivariate Shewhart control chart, and is equal to $1/P(signal)$ And, the probability of switch is given by

$$P(switch) = P(d_1) \cdot P(d_2|d_1) + P(d_2) \cdot P(d_1|d_2),$$

where $P(d_i)$ is the probability of using sampling interval d_i, and $P(d_i|d_j)$ is the conditional probability of using sampling interval in the current sample given that the sampling interval d_j $(d_i \neq d_j)$ was used in the previous sample. Following Amin and Lestinger (1991), the number of samples to switch (NSSW) as the number of samples taken from the time the process starts using the sampling interval d_i until a switch is made to sampling interval d_j $(i \neq j)$, and let the average number of samples until a switch (ANSSW) be the expected value of the number of samples until a switch. The number of samples until a switch has a geometric distribution with parameter $p = P(switch)$ when the process does not change. Thus the ANSSW is given by

$$ANSSW = \frac{1}{P(switch)}, \quad P(switch) > 0.$$

2.4. Measures of multivariate Shewhart control chart performance

When there is a sustained shift in a process parameter, the ability of different control charts to detect this shift can be evaluated using the expected time required for detection.

In evaluating the properties of the VSI control charts, it is usual to compare the performances of the VSI to the FSI control charts. The process parameters are choosed for both FSI and VSI control charts having the same in control ARL and ATS values because the ASI is equal to one unit.

In this computation, each control chart was calibrated so that the on-target ARL and ATS were approximately equal to 800 and the sample size was 2, 4 for $p = 2$, and we used $d_0 = 1.0, d_1 = 0.1, d_2 = 1.9$. Also simulation with 10,000 runs was used.

3. Numerical results

The ability of a control chart to detect any shifts in the production process is determined by the length of time required to signal. In evaluating the switching properties of the VSI control chart, we compare the performance of the VSI control charts to FSI control charts.

In order to compare the performances of the proposed multivariate Shewhart control charts fairly, some kinds of standards for comparison are necessary. In our computation, each control chart was callibrated so that on-target ARL and ATS were approximately equal to 800 and the sample size for each control chart was 2, 4 for $p = 2$. And we used $d_0 = 0, d_1 = 1.9, d_2 = 0.1$.

When the production process changes, suppose that shifts in the mean vector μ and covariance matrix Σ are simultaneously changed. The control limits h and g for multivariate Shewhart control charts for monitoring μ and Σ were obtained by using $\chi^2(np)$. Table 3.1 gives the values of h and g for $p = 2, n = 2, 4$ when the in-control ARL is 800.

Table 3.1 Values of h and g when the in-control ARL is approximately 800

$n = 2$	$n = 4$
$p = 2$	$p = 2$
$h = 17.9715$	$h = 25.5573$
$g = 3.3527$	$g = 7.3382$

3.1. Shifts in mean vectors and variance-covariance matrix

ANSW, P(switch), ANNSW and ASI of control charts are obtained by simulation with 10,000 runs.

For $n = 2, 4$ and $p = 2$, Tables 3.2-3.3 give the numerical results for FSI and VSI multivariate Shewhart control charts when mean vectors, covariances are changed and variances are not changed. Here the changed values of τ ($\tau^2 = n(\mu - \mu_0)'\Sigma_0^{-1}(\mu - \mu_0)$ is noncentrality parameter) are those of increased from 0.1 to 1.0, and the changed values of correlation coefficient ρ are decreased from 0.81 to 0.09, respectively.

As shown in Tables 3.2-3.3, the VSI multivariate Shewhart control charts for monitoring mean vectors and covariances are effective in detecting simultaneously changes in mean vectors and covariances in terms of ATS.

From the results in Tables 3.2-3.3, when large changes in τ and smaller value of ρ, we found that multivariate Shewhart control chart gives fewer switches between two different sampling intervals. In addition, when the shift in mean vector is large or the changed values of ρ are decreased, ANSW, P(switch) and ASI of Shewhart control chart become remarkably smaller.

For $n = 2, 4$ and $p = 2$, Tables 3.4-3.5 give ARL, ATS, ANSW, ANSSW, ASI, P(switch) in each cell when mean vectors, 1 or 2 variances are changed and covariances are not changed respectively for FSI and VSI multivariate Shewhart control charts. Here the changed values of τ are those of increased from 0.1 to 1.0, and standard deviations are changed from σ_0 to $\sigma = \sqrt{c}\sigma_0$, for $c = 1.21, 1.44, 1.69, 4.00$.

As shown in Tables 3.4-3.5, the VSI multivariate Shewhart control charts for monitoring mean vectors and variance-covariance matrix are also effective in detecting simultaneously changes in mean vectors and variances in terms of ATS.

From the results in Tables 3.4-3.5, when large changes in τ and larger value of σ, we found that multivariate Shewhart control chart gives fewer switches between two different sampling intervals. In addition, when the shift in mean vector is large or the changed values of σ are increased, ANSW, P(switch) and ASI of Shewhart control chart become remarkably smaller.

For $n = 2, 4$ and $p = 2$, Tables 3.6-3.9 give ARL, ATS, ANSW, ANSSW, ASI, P(switch) in each cell when mean vectors, 1 or 2 variances are changed and covariances are simultaneously changed respectively for FSI and VSI multivariate Shewhart control charts. Here the changed values of τ in Tables 3.6-3.9 are those of increased from 0.1 to 1.0, standard deviations are changed from σ_0 to $\sigma = \sqrt{c}\sigma_0$, for $c = 1.21, 1.44, 1.69, 4.00$ and covariances are changed from $\rho_0 = 0.9$ to $\rho = 0.72, 0.54$.

As shown in Tables 3.6-3.9, the VSI multivariate Shewhart control charts for monitoring mean vectors and covariance matrix are very effective in detecting simultaneously changes in mean vectors, variances and covariances in terms of ATS.

From the results in Tables 3.2-3.9, when large changes in τ and c, we found that multivariate Shewhart control chart gives fewer switches between two different sampling intervals. In addition, when the shift in mean vector and standard deviation are large, ANSW, P(switch) and ASI of Shewhart control chart become remarkably smaller.

Table 3.2 ARL and ATS, ANSW, P(switch), ANSSW, ASI values when mean vectors and covariances are changed ($n = 2, p = 2, \rho_0 = 0.9$)

		FSI			VSI		
		ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.1$	$\rho = 0.81$	52.17	37.56	22.15	0.42	2.35	0.70
	$\rho = 0.45$	4.36	2.61	0.70	0.16	6.26	0.39
	$\rho = 0.09$	2.50	1.67	0.20	0.08	12.49	0.31
$\tau = 0.4$	$\rho = 0.81$	19.04	11.26	6.23	0.33	3.06	0.54
	$\rho = 0.45$	3.64	2.18	0.47	0.13	7.74	0.35
	$\rho = 0.09$	2.32	1.57	0.16	0.07	14.58	0.29
$\tau = 0.7$	$\rho = 0.81$	6.45	2.98	1.00	0.15	6.47	0.32
	$\rho = 0.45$	2.68	1.67	0.21	0.08	12.65	0.29
	$\rho = 0.09$	2.02	1.41	0.10	0.05	20.81	0.25
$\tau = 1.0$	$\rho = 0.81$	2.77	1.38	0.10	0.04	28.35	0.17
	$\rho = 0.45$	1.97	1.32	0.07	0.04	28.16	0.21
	$\rho = 0.09$	1.72	1.27	0.05	0.03	36.81	0.21

Table 3.3 ARL and ATS, ANSW, P(switch), ANSSW, ASI values when mean vectors and covariances are changed ($n = 4, p = 2, \rho_0 = 0.9$)

		FSI			VSI		
		ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.1$	$\rho = 0.81$	32.35	19.50	11.67	0.36	2.77	0.57
	$\rho = 0.45$	2.51	1.46	0.14	0.05	18.53	0.23
	$\rho = 0.09$	1.56	1.16	0.02	0.01	74.28	0.16
$\tau = 0.4$	$\rho = 0.81$	10.63	4.96	2.29	0.22	4.65	0.38
	$\rho = 0.45$	2.13	1.32	0.07	0.04	28.43	0.19
	$\rho = 0.09$	1.48	1.12	0.02	0.01	94.57	0.15
$\tau = 0.7$	$\rho = 0.81$	3.29	1.52	0.16	0.05	20.44	0.19
	$\rho = 0.45$	1.65	1.15	0.02	0.01	79.30	0.15
	$\rho = 0.09$	1.34	1.08	0.01	0.01	179.82	0.13
$\tau = 1.0$	$\rho = 0.81$	1.54	1.07	0.01	0.00	310.07	0.11
	$\rho = 0.45$	1.31	1.06	0.00	0.00	355.06	0.12
	$\rho = 0.09$	1.21	1.04	0.00	0.00	580.42	0.12

Table 3.4 ARL and ATS, ANSW, P(switch), ANSSW, ASI values when mean vectors and variances are changed ($n = 2, p = 2, \rho_0 = 0.9$)

		FSI	VSI				
		ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.1$	$c = 1.21$	215.88	190.78	104.72	0.49	2.06	0.88
		173.09	140.47	81.37	0.47	2.13	0.81
	$c = 1.44$	71.46	56.84	32.43	0.45	2.20	0.78
		63.83	43.58	26.57	0.42	2.40	0.67
	$c = 1.69$	29.50	21.29	11.97	0.41	2.47	0.69
		30.23	17.74	10.58	0.35	2.86	0.56
	$c = 4.00$	2.63	1.76	0.23	0.09	11.30	0.33
		2.90	1.54	0.17	0.06	17.12	0.22
$\tau = 0.4$	$c = 1.21$	49.27	31.80	19.41	0.39	2.54	0.63
		43.57	25.95	16.00	0.37	2.72	0.57
	$c = 1.44$	24.63	15.30	8.79	0.36	2.80	0.58
		23.15	12.40	7.18	0.31	3.22	0.50
	$c = 1.69$	13.81	8.36	4.30	0.31	3.21	0.54
		13.95	6.89	3.55	0.25	3.93	0.43
	$c = 4.00$	2.44	1.64	0.18	0.08	13.30	0.30
		2.51	1.40	0.11	0.04	22.72	0.20
$\tau = 0.7$	$c = 1.21$	10.34	4.18	1.78	0.17	5.83	0.32
		9.66	3.71	1.47	0.15	6.59	0.29
	$c = 1.44$	7.40	3.31	1.21	0.16	6.11	0.33
		6.97	2.80	0.90	0.13	7.78	0.27
	$c = 1.69$	5.69	2.77	0.85	0.15	6.67	0.33
		5.29	2.29	0.58	0.11	9.33	0.26
	$c = 4.00$	2.08	1.43	0.11	0.05	19.74	0.26
		2.00	1.23	0.05	0.02	44.09	0.17
$\tau = 1.0$	$c = 1.21$	3.26	1.38	0.09	0.03	37.93	0.15
		3.10	1.33	0.07	0.02	46.40	0.14
	$c = 1.44$	2.94	1.37	0.09	0.03	31.25	0.16
		2.73	1.37	0.06	0.02	47.95	0.14
	$c = 1.69$	2.70	1.37	0.10	0.04	27.60	0.18
		2.46	1.26	0.05	0.02	50.43	0.14
	$c = 4.00$	1.76	1.26	0.05	0.03	35.35	0.21
		1.59	1.12	0.01	0.01	124.58	0.14

Table 3.5 ARL and ATS, ANSW, P(switch), ANSSW, ASI values when mean vectors and variances are changed $(n = 4, p = 2, \rho_0 = 0.9)$

		FSI	VSI				
		ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.1$	$c = 1.21$	174.53	143.94	82.58	0.47	2.11	0.82
		123.83	90.20	54.87	0.44	2.26	0.72
	$c = 1.44$	49.47	34.26	20.45	0.41	2.42	0.67
		38.48	21.53	13.31	0.35	2.89	0.54
	$c = 1.69$	18.57	11.02	6.10	0.33	3.04	0.54
		16.38	7.48	3.99	0.24	4.11	0.40
	$c = 4.00$	1.64	1.19	0.03	0.02	51.38	0.18
		1.65	1.11	0.01	0.01	136.62	0.13
$\tau = 0.4$	$c = 1.21$	28.26	14.62	8.78	0.31	3.22	0.49
		23.50	10.87	6.31	0.27	3.73	0.42
	$c = 1.44$	14.14	6.93	3.58	0.25	3.95	0.43
		11.88	4.89	2.25	0.19	5.28	0.34
	$c = 1.69$	7.93	3.84	1.52	0.19	5.22	0.37
		7.02	2.82	0.89	0.13	7.88	0.27
	$c = 4.00$	1.54	1.15	0.02	0.01	73.54	0.16
		1.50	1.08	0.01	0.00	298.06	0.12
$\tau = 0.7$	$c = 1.21$	4.74	1.77	0.27	0.06	17.49	0.18
		4.28	1.62	0.19	0.04	22.40	0.17
	$c = 1.44$	3.68	1.59	0.20	0.05	18.84	0.19
		3.30	1.42	0.11	0.03	29.60	0.16
	$c = 1.69$	2.97	1.45	0.13	0.04	22.38	0.19
		2.65	1.29	0.06	0.02	43.31	0.15
	$c = 4.00$	1.37	1.09	0.01	0.01	145.60	0.14
		1.28	1.04	0.00	0.00	992.83	0.11
$\tau = 1.0$	$c = 1.21$	1.64	1.07	0.00	0.00	694.74	0.11
		1.56	1.06	0.00	0.00	1068.25	0.10
	$c = 1.44$	1.58	1.07	0.00	0.00	516.74	0.11
		1.47	1.05	0.00	0.00	1370.89	0.10
	$c = 1.69$	1.53	1.07	0.00	0.00	382.83	0.11
		1.39	1.05	0.00	0.00	1408.53	0.10
	$c = 4.00$	1.22	1.04	0.00	0.00	3568.85	0.12
		1.13	1.02	0.00	0.00	10629.50	0.10

Table 3.6 ARL and ATS, ANSW, P(switch), ANSSW, ASI values when mean vectors, variances and covariances are changed ($n = 2, p = 2, \rho_0 = 0.9$)

			FSI		VSI			
			ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.1$	$c = 1.21$	$\rho = 0.72$	11.41	6.65	3.23	0.28	3.54	0.50
			9.37	5.19	2.33	0.25	4.02	0.46
		$\rho = 0.54$	4.74	2.74	0.79	0.17	6.04	0.39
			4.10	2.36	0.57	0.14	7.17	0.36
	$c = 1.44$	$\rho = 0.72$	8.18	4.61	1.95	0.24	4.19	0.45
			6.21	3.27	1.14	0.18	5.45	0.38
		$\rho = 0.54$	3.97	2.30	0.54	0.14	7.40	0.35
			3.19	1.85	0.31	0.10	10.44	0.30
	$c = 1.69$	$\rho = 0.72$	6.12	3.42	1.21	0.20	5.06	0.41
			4.48	2.34	0.59	0.13	7.65	0.32
		$\rho = 0.54$	3.36	1.97	0.37	0.11	9.15	0.32
			2.58	1.55	0.16	0.06	15.98	0.25
	$c = 4.00$	$\rho = 0.72$	1.98	1.36	0.08	0.04	24.22	0.23
			1.60	1.14	0.02	0.01	77.28	0.15
		$\rho = 0.54$	1.69	1.24	0.04	0.02	41.06	0.20
			1.37	1.08	0.01	0.01	164.25	0.13
$\tau = 0.4$	$c = 1.21$	$\rho = 0.72$	7.24	4.03	1.60	0.22	4.53	0.43
			6.29	3.39	1.22	0.19	5.17	0.40
		$\rho = 0.54$	3.84	2.24	0.50	0.13	7.60	0.35
			3.45	2.01	0.39	0.11	8.89	0.32
	$c = 1.44$	$\rho = 0.72$	5.78	3.20	1.08	0.19	5.35	0.40
			4.65	2.46	0.66	0.14	7.01	0.34
		$\rho = 0.54$	3.36	1.99	0.37	0.11	9.06	0.33
			2.80	1.67	0.22	0.08	12.60	0.28
	$c = 1.69$	$\rho = 0.72$	4.64	2.60	0.72	0.16	6.42	0.37
			3.65	1.96	0.38	0.10	9.68	0.29
		$\rho = 0.54$	2.93	1.78	0.27	0.09	11.02	0.30
			2.38	1.47	0.13	0.05	18.34	0.24
	$c = 4.00$	$\rho = 0.72$	1.91	1.33	0.07	0.04	26.95	0.22
			1.56	1.13	0.02	0.01	92.74	0.15
		$\rho = 0.54$	1.65	1.22	0.04	0.02	44.75	0.20
			1.35	1.08	0.01	0.01	197.64	0.13

Table 3.7 (continued)

			FSI		VSI			
			ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.7$	$c = 1.21$	$\rho = 0.72$	3.92	2.11	0.45	0.12	8.64	0.31
			3.61	1.94	0.36	0.10	9.92	0.29
		$\rho = 0.54$	2.77	1.68	0.22	0.08	12.52	0.28
			2.58	1.58	0.17	0.07	14.95	0.26
	$c = 1.44$	$\rho = 0.72$	3.46	1.94	0.36	0.10	9.58	0.30
			3.02	1.68	0.23	0.08	13.19	0.26
		$\rho = 0.54$	2.55	1.59	0.18	0.07	14.24	0.27
			2.25	1.43	0.11	0.05	20.09	0.24
	$c = 1.69$	$\rho = 0.72$	3.05	1.78	0.27	0.09	11.10	0.29
			2.62	1.52	0.15	0.06	17.39	0.23
		$\rho = 0.54$	2.34	1.50	0.14	0.06	17.05	0.26
			2.02	1.34	0.07	0.04	27.08	0.22
	$c = 4.00$	$\rho = 0.72$	1.75	1.27	0.05	0.03	35.71	0.21
			1.47	1.11	0.01	0.01	122.78	0.14
		$\rho = 0.54$	1.56	1.19	0.03	0.02	56.17	0.19
			1.32	1.07	0.01	0.00	248.99	0.13
$\tau = 1.0$	$c = 1.21$	$\rho = 0.72$	2.30	1.36	0.09	0.04	26.31	0.20
			2.20	1.32	0.07	0.03	30.77	0.19
		$\rho = 0.54$	1.99	1.32	0.07	0.04	27.99	0.21
			1.93	1.29	0.06	0.03	32.34	0.20
	$c = 1.44$	$\rho = 0.72$	2.19	1.35	0.08	0.04	25.95	0.21
			2.04	1.28	0.06	0.03	36.59	0.19
		$\rho = 0.54$	1.93	1.31	0.06	0.03	29.85	0.21
			1.80	1.24	0.05	0.03	39.96	0.19
	$c = 1.69$	$\rho = 0.72$	2.09	1.33	0.08	0.04	27.10	0.21
			1.91	1.24	0.04	0.02	42.82	0.18
		$\rho = 0.54$	1.86	1.29	0.06	0.03	32.73	0.21
			1.69	1.21	0.03	0.02	50.18	0.18
	$c = 4.00$	$\rho = 0.72$	1.57	1.19	0.03	0.02	58.90	0.18
			1.36	1.08	0.01	0.01	183.11	0.13
		$\rho = 0.54$	1.47	1.15	0.02	0.01	81.18	0.17
			1.27	1.06	0.00	0.00	315.01	0.13

Table 3.8 ARL and ATS, ANSW, P(switch), ANSSW, ASI values when mean vectors, variances and covariances are changed ($n = 4, p = 2, \rho_0 = 0.9$)

			FSI		VSI			
			ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.1$	$c = 1.21$	$\rho = 0.72$	6.34	3.04	1.02	0.16	6.22	0.34
			5.06	2.36	0.62	0.12	8.17	0.29
		$\rho = 0.54$	2.68	1.50	0.15	0.06	17.42	0.22
			2.32	1.36	0.09	0.04	24.58	0.20
	$c = 1.44$	$\rho = 0.72$	4.49	2.19	0.51	0.11	8.76	0.29
			3.33	1.64	0.22	0.07	15.21	0.22
		$\rho = 0.54$	2.26	1.35	0.09	0.04	25.11	0.20
			1.85	1.20	0.04	0.02	48.46	0.16
	$c = 1.69$	$\rho = 0.72$	3.33	1.73	0.26	0.08	12.92	0.25
			2.47	1.35	0.09	0.04	27.27	0.18
		$\rho = 0.54$	1.96	1.25	0.05	0.03	37.48	0.18
			1.57	1.12	0.01	0.01	106.78	0.14
	$c = 4.00$	$\rho = 0.72$	1.32	1.07	0.01	0.00	206.94	0.13
			1.13	1.02	0.00	0.00	7119.00	0.10
		$\rho = 0.54$	1.19	1.04	0.00	0.00	534.51	0.12
			1.06	1.01	0.00	0.00	∞	0.10
$\tau = 0.4$	$c = 1.21$	$\rho = 0.72$	3.95	1.95	0.38	0.10	10.39	0.27
			3.41	1.69	0.25	0.07	13.49	0.23
		$\rho = 0.54$	2.22	1.33	0.08	0.04	26.72	0.20
			2.02	1.25	0.05	0.03	38.06	0.18
	$c = 1.44$	$\rho = 0.72$	3.19	1.65	0.23	0.07	13.92	0.24
			2.57	1.39	0.11	0.04	24.10	0.19
		$\rho = 0.54$	1.98	1.26	0.05	0.03	36.64	0.18
			1.69	1.15	0.02	0.01	80.23	0.15
	$c = 1.69$	$\rho = 0.72$	2.60	1.45	0.13	0.05	19.65	0.21
			2.06	1.23	0.04	0.02	46.25	0.16
		$\rho = 0.54$	1.76	1.19	0.03	0.02	55.46	0.16
			1.48	1.09	0.01	0.01	155.12	0.13
	$c = 4.00$	$\rho = 0.72$	1.28	1.06	0.01	0.00	240.48	0.12
			1.12	1.02	0.00	0.00	3535.67	0.10
		$\rho = 0.54$	1.18	1.03	0.00	0.00	587.84	0.11
			1.06	1.01	0.00	0.00	10302.50	0.10

Table 3.9 (continued)

			FSI		VSI			
			ARL	ATS	ANSW	P(switch)	ANNSW	ASI
$\tau = 0.7$	$c = 1.21$	$\rho = 0.72$	2.21	1.28	0.06	0.03	35.27	0.17
$\tau = 0.7$	$c = 1.21$	$\rho = 0.72$	2.05	1.23	0.04	0.02	46.52	0.16
$\tau = 0.7$	$c = 1.21$	$\rho = 0.54$	1.69	1.16	0.02	0.01	74.65	0.15
$\tau = 0.7$	$c = 1.21$	$\rho = 0.54$	1.59	1.13	0.01	0.01	114.11	0.14
$\tau = 0.7$	$c = 1.44$	$\rho = 0.72$	2.00	1.23	0.05	0.02	42.50	0.17
$\tau = 0.7$	$c = 1.44$	$\rho = 0.72$	1.78	1.15	0.02	0.01	86.07	0.14
$\tau = 0.7$	$c = 1.44$	$\rho = 0.54$	1.58	1.13	0.02	0.01	95.76	0.15
$\tau = 0.7$	$c = 1.44$	$\rho = 0.54$	1.44	1.09	0.01	0.01	198.02	0.13
$\tau = 0.7$	$c = 1.69$	$\rho = 0.72$	1.83	1.19	0.03	0.02	57.05	0.16
$\tau = 0.7$	$c = 1.69$	$\rho = 0.72$	1.59	1.11	0.01	0.01	156.79	0.13
$\tau = 0.7$	$c = 1.69$	$\rho = 0.54$	1.48	1.11	0.01	0.01	125.88	0.14
$\tau = 0.7$	$c = 1.69$	$\rho = 0.54$	1.32	1.06	0.00	0.00	430.48	0.12
$\tau = 0.7$	$c = 4.00$	$\rho = 0.72$	1.22	1.04	0.00	0.00	382.19	0.12
$\tau = 0.7$	$c = 4.00$	$\rho = 0.72$	1.10	1.01	0.00	0.00	10489.00	0.10
$\tau = 0.7$	$c = 4.00$	$\rho = 0.54$	1.15	1.03	0.00	0.00	766.43	0.11
$\tau = 0.7$	$c = 4.00$	$\rho = 0.54$	1.05	1.01	0.00	0.00	∞	0.10
$\tau = 1.0$	$c = 1.21$	$\rho = 0.72$	1.42	1.07	0.01	0.00	263.11	0.12
$\tau = 1.0$	$c = 1.21$	$\rho = 0.72$	1.37	1.06	0.00	0.00	365.37	0.12
$\tau = 1.0$	$c = 1.21$	$\rho = 0.54$	1.32	1.06	0.00	0.00	330.96	0.12
$\tau = 1.0$	$c = 1.21$	$\rho = 0.54$	1.28	1.05	0.00	0.00	518.91	0.12
$\tau = 1.0$	$c = 1.44$	$\rho = 0.72$	1.39	1.07	0.01	0.00	268.56	0.12
$\tau = 1.0$	$c = 1.44$	$\rho = 0.72$	1.32	1.05	0.00	0.00	609.29	0.11
$\tau = 1.0$	$c = 1.44$	$\rho = 0.54$	1.29	1.05	0.00	0.00	362.97	0.12
$\tau = 1.0$	$c = 1.44$	$\rho = 0.54$	1.23	1.04	0.00	0.00	675.09	0.11
$\tau = 1.0$	$c = 1.69$	$\rho = 0.72$	1.35	1.06	0.00	0.00	326.49	0.12
$\tau = 1.0$	$c = 1.69$	$\rho = 0.72$	1.26	1.04	0.00	0.00	1027.32	0.11
$\tau = 1.0$	$c = 1.69$	$\rho = 0.54$	1.26	1.05	0.00	0.00	402.73	0.12
$\tau = 1.0$	$c = 1.69$	$\rho = 0.54$	1.18	1.03	0.00	0.00	992.41	0.11
$\tau = 1.0$	$c = 4.00$	$\rho = 0.72$	1.14	1.03	0.00	0.00	1072.35	0.11
$\tau = 1.0$	$c = 4.00$	$\rho = 0.72$	1.06	1.01	0.00	0.00	20644.00	0.10
$\tau = 1.0$	$c = 4.00$	$\rho = 0.54$	1.10	1.02	0.00	0.00	1618.23	0.11
$\tau = 1.0$	$c = 4.00$	$\rho = 0.54$	1.04	1.01	0.00	0.00	∞	0.10

4. Summary and concluding remarks

This paper has investigated the performance of the VSI feature used in multivariate Shewhart control charts for the simultaneous monitoring mean vector and covariance matrix.

We consider multivariate Shewhart control charts for $n = 2, 4, p = 2$ and $\rho_0 = 0.9$. We can get numerical results of the proposed FSI and VSI multivariate Shewhart control charts by using simulations. The performance of these proposed multivariate Shewhart control charts is compared with a VSI scheme and a FSI scheme.

The objective of monitoring process is to detect small as well as large changes in μ, or small or large increases in σ, where there is no particular shift direction that is of primary interest. The conclusions from this study can be summarized as follows.

The numerical results presented here show that using a VSI scheme instead of a FSI scheme give very significant improvements in performance for most shifts in mean vector μ and variance-covariance matrix Σ. Using the VSI feature seems to give the biggest improvements in performance for the Shewhart control charts.

As shown in the Tables 3.2-3.7, the VSI control charts are far superior to the corresponding FSI control charts in terms of ATS. And, it can be also seen that the VSI procedures have substantially fewer switches for small or moderate changes of the mean vector and variances.

Using the VSI feature is extremely helpful to a Shewhart control charts for shifts of mean vectors and variance-covariance matrix because, when there is some indications of a change in the process, the VSI feature allows additional information to be obtained quickly.

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