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Bayesian estimation for Rayleigh models

Ji Eun Oh[1] · Joon Jin Song[2] · Joong Kweon Sohn[3]

[13]Department of Statistics, Kyungpook National University
[2]Department of Statistical Science, Baylor University
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Abstract

The Rayleigh distribution has been commonly used in life time testing studies of the probability of surviving until mission time. We focus on a reliability function of the Rayleigh distribution and deal with prior distribution on $R(t)$. This paper is an effort to obtain Bayes estimators of rayleigh distribution with three different prior distribution on the reliability function; a noninformative prior, uniform prior and inverse gamma prior. We have found the Bayes estimator and predictive density function of a future observation y with each prior distribution. We compare the performance of the Bayes estimators under different sample size and in simulation study. We also derive the most plausible region, prediction intervals for a future observation.

Keywords: Bayes estimator, inversed gamma prior, predictive distribution, predict intervals.

1. Introduction

The Rayleigh distribution is a suitable model for life testing studies. Polovko (1968), Dyer and Whisenand (1973), demonstrated the importance of this distribution in electro vacuum devices and communication engineering. Howlader and Hossian (1995) obtained Bayes estimators for the scale parameter and the reliability function, $R(t)$, in the case of type-II censored sampling. Lalitha and Anand (1996) used the modified maximum likelihood to estimate the scale parameter of the Rayleigh distribution. Mazloum (1997) concerned with the problem of estimating the scale parameter and the reliability under type-II censoring. Meintanis and Iliopoulos (2003) proposed a class of goodness of fit tests for the Rayleigh distribution. Abd Elfattah *et al.* (2006) studied the efficiency of maximum likelihood estimate of the parameter of Rayleigh distribution under three cases, type-I, type-II and progressive type-II censored sampling schemes.

[1] Graduate student, Department of Statistics, Kyungpook National University, 80 Daehak-ro, Buk-gu, Daegu 41566, Korea.
[2] Associate professor, Department of Statistical Science, Baylor University, One Bear Place, Waco, Texas 76798 USA.
[3] Corresponding Author: Professor, Department of Statistics, Kyungpook National University, 80 Daehak-ro, Buk-gu, Daegu 41566, Korea, E-mail: jsohn@knu.ac.kr

Dey and Dey (2012) obtained Bayes estimators of Rayleigh parameter and its associated risk based on the extended jeffrey's prior and conjugate prior (square root inverted gamma prior) with respect to both symmetric loss function (squared error loss), and asymmetric loss function (precautionary loss function). They also derive the highest posterior density (HPD) interval for the Rayleigh parameter as well as the HPD prediction intervals for a future observation from this distribution. An illustrative example to test how the Rayleigh distribution fits a real data set is presented.

Ahmed *et al.* (2013) consider the estimation of the parameter of Rayleigh distribution. Bayes estimator is obtained by using Jeffreys and extension of Jeffreys prior under squared error loss function and Al-Bayyatis loss function.

We focus on Bayesian estimation of the parameter of the Rayleigh model using three different prior which is a noninformative prior distribution, uniform prior distribution and inverse gamma prior distribution on the reliability function $R(t) = P(X > t)$. We also interested in evaluating the each predictive Bayes estimator with their MSE.

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be an observed random sample of size n from the Rayleigh distribution, where σ is a scale parameter. Let Y be a future observation from the same distribution function. Let $\Pi(\sigma)$ be a prior distribution of σ and also $\Pi(\sigma|\underline{X})$ be the posterior distribution of σ given $\underline{X} = \underline{x}$. Then the predictive density function, $f(y|\underline{x})$, for a future observation y will be obtained by

$$f(y|\underline{x}) = \Psi \int_{\sum} f(y|\sigma)\Pi(\sigma|\underline{x})d\sigma.$$

where $\sum$ is the range space of σ and Ψ is the normalizing constant.

So the idea of Bayesian predictive inference is the average of the likelihood of a future observation based on updated posterior density of σ given $\underline{X} = \underline{x}$.

The $100(1-\alpha)\%$ equal-tail most plausible region (c_L, c_U) for a future observation y is obtained by solving the following equation.

$$\int_{-\infty}^{c_L} f(y|\underline{x})dy = \frac{\alpha}{2}.$$

The $100(1-\alpha)\%$ most plausible region (c_L, c_U) is said to be a Bayesian most plausible region of cover κ if (c_L, c_U) has the form

$$(c_L, c_U) = y \ : \ f(y|\underline{x}) \geq \alpha,$$

where α is determined by

$$f((c_L, c_U)|\underline{x}) = \kappa.$$

In section2, one derive the predictive density, Bayes predictive estimator and most plausible region of a future observation. As a prior distribution on the reliability function $R(t) = P(X > t)$, one consider a non-informative prior, a locally uniform prior and a inversed gamma prior distribution.

In section3, using some Monte Carlo simulation results we consider the performances of Bayes estimator and study the prior-robustness.

In section4, we conclude the paper.

2. Bayesian prediction analysis under several priors

2.1. Non-informative prior case

For the Bayesian setting, two major components are required. First, the prior distribution of the unknown parameter of the model and second, the risk function to estimate the risk associated with the estimation of the parameter. The hazard function of this distribution is an increasing function in x, which is interest in the life testing problem. Thus this could be suitable for life testing experiments on components which age with time in that way.

For the Rayleigh model, the reliability function θ at a specified 'mission' time $t > 0$ is given by

$$\theta = P(X > t) = \exp(-\frac{t^2}{2}), \quad t > 0.$$

Let $\underline{X} = (X_1, X_2, \cdots, X_n)$ be a random sample from the Rayleigh model with probability density function given. Then the likelihood function is given by

$$L(\sigma|\bar{X}) = \frac{1}{\sigma^{2n}} \left(\prod_{i=1}^{n} X_i \right) \exp\left(-\sum_{i=1}^{n} \frac{X_i^2}{2\sigma^2} \right), \quad 0 < \sigma, \quad 0 < X_i < \infty, \quad i = 1, 2, \cdots, n. \quad (2.1)$$

The likelihood function of θ, which can be obtained from (2.1) by letting $\sigma^2 = -t^2/(2\log\theta)$ is

$$\begin{aligned} L(\theta|\underline{X}) &= \left(-\frac{2\log\theta}{t^2} \right)^n \left(\prod_{i=1}^{n} X_i \right) \exp\left(\sum_{i=1}^{n} \frac{X_i^2}{t^2} \log\theta \right) \\ &\propto (-\log\theta)^n \exp\left(\sum_{i=1}^{n} \frac{X_i^2}{t^2} \log\theta \right), \quad 0 < \theta < 1. \end{aligned}$$

Here one consider the Jefferey's noninformative prior distribution for θ. When the mission time t is given, the noninformative prior distribution $\pi(\theta)$ for θ is

$$\Pi(\theta) \propto I(\theta)^{\frac{1}{2}} = -\frac{1}{(\theta\log\theta)},$$

where

$$I(\theta) = -E\left[\frac{\partial^2}{\partial\theta^2} \log L(\theta|\underline{X}) \right] = -\frac{1}{\theta^2(\log\theta)^2}, \quad 0 < \theta < 1$$

and $I(\theta)$ is the Fisher's information.

By the properties of the invariance under parametric transformations, the Bayesian predictive densities under noninformative prior distribution for θ and σ are equivalent.

Now one derive predictive density distribution which is given in the following theorem.

Theorem 2.1 Under the noninformative prior distribution for θ, the Bayesian predictive density of future observation y is given by

$$f(y|\underline{x}) = \frac{2ny\left(\sum_{i=1}^n x_i^2\right)^n}{\left(\sum_{i=1}^n x_i^2 + y^2\right)^{n+1}}, \quad 0 < y < \infty. \tag{2.2}$$

Proof: The posterior density of θ given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{1}{\Gamma(n)}\left(\sum_{i=1}^n \frac{x_i^2}{t^2}\right)^n (-\log\theta)^{n-1}\theta^{\left(\sum_{i=1}^n \frac{x_i^2}{t^2} - 1\right)}, \quad 0 < \theta < 1.$$

Then one have

$$f(y|\underline{x}) = \int_0^1 \Pi(\theta|\underline{x}) f(y|\theta) d\theta.$$

Letting $\Phi = \frac{1}{\Gamma(n)}\left(\sum_{i=1}^n \frac{x_i^2}{t^2}\right)^n\left(\frac{2y}{t^2}\right)$ and $Z = -\log\theta$,

$$f(y|\underline{x}) = \Phi\int_0^\infty z^n e^{-z\left(\frac{\sum_{i=1}^n x_i^2 + y^2}{t^2}\right)} dz, \quad 0 < z < \infty. \tag{2.3}$$

With the aid of Gamma kernel,

$$\int_0^\infty z^n e^{-z\left(\frac{\sum_{i=1}^n x_i^2 + y^2}{t^2}\right)} dz = \Gamma(n+1)\left(\frac{t^2}{\sum_{i=1}^n x_i^2 + y^2}\right)^{n+1},$$

the equation (2.3) becomes

$$n\left(\sum_{i=1}^n \frac{x_i^2}{t^2}\right)^n\left(\frac{2y}{t^2}\right)\left(\frac{t^2}{\sum_{i=1}^n x_i^2 + y^2}\right)^{n+1} = \frac{2ny(\sum_{i=1}^n x_i^2)^n}{(\sum_{i=1}^n x_i^2 + y^2)^{n+1}}.$$

The proof is completed. □

Now consider the Bayes predictive estimator and most plausible region for a future observation of the Rayleigh model under a noninformative prior distribution. In equation (2.2), the Y^2 has an inverted beta distribution, denoted by $InBe(1, n, \sum_{i=1}^n x_i^2)$, where the inverted beta distribution with parameter α, β and γ is defined by

$$f(z|\alpha,\beta,\gamma) = \frac{z^{\alpha-1}\gamma^\beta}{B(\alpha,\beta)(z+\gamma)^{\alpha+\beta}}, \quad z \geq 0, \quad \alpha,\beta,\gamma > 0,$$

where $B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$.

The inverted beta distribution is also known as the distribution of a beta random variable of the second kind, which is related to the F distribution.

Now let us consider to find the Bayes estimator and the most plausible region.

Theorem 2.2 Under the squared-error loss, the Bayes predictive estimator of a future observation Y is given by

$$\hat{Y}_{NI} = \frac{\Gamma(\frac{3}{2})\Gamma(n-\frac{1}{2})(\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}}{\Gamma(n)}. \tag{2.4}$$

Proof: Since the Bayes predictive estimator of Y is the predictive density mean with squared error loss, it can be obtained as

$$\begin{aligned}\hat{Y}_{NI} =& E(y|\underline{x}) \\ =& \int_0^\infty \frac{2ny(\sum_{i=1}^{n} x_i^2)^n}{(y^2+\sum_{i=1}^{n} x_i^2)^{n+1}}.\end{aligned}$$

Letting $Z = Y^2$,

$$\begin{aligned}\hat{Y}_{NI} =& \int_0^\infty \frac{n(\sum_{i=1}^{n} x_i^2)^n \sqrt{z}}{(z+\sum_{i=1}^{n} x_i^2)^{n+1}} dz \\ =& n(\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}} B\left(\frac{3}{2}, n-\frac{1}{2}\right) \int_0^\infty \frac{\sqrt{z}(\sum_{i=1}^{n} x_i^2)^{n-\frac{1}{2}}}{B\left(\frac{3}{2}, n-\frac{1}{2}\right)(z+\sum_{i=1}^{n} x_i^2)^{n+1}} dz,\end{aligned}$$

where the Z has an inverted beta distribution, $InBe(\frac{3}{2}, n-\frac{1}{2}, \sum_{i=1}^{n} x_i^2)$ and one can get the result. This complete the proof. □

Now one consider the $100(1-\alpha)\%$ most plausible region (c_L, c_U) for a future observation Y. The predictive density $f(y|\underline{x})$ is

$$f(y|\underline{x}) = \frac{2y(\sum x_i^2)^n}{B(1,n)(y^2+\sum x_i^2)^{n+1}},$$

where $Z = Y^2 \sim InBe(1, n, \sum x_i^2)$ and $B(1,n) = \Gamma(1)\Gamma(n)/\Gamma(1+n)$, one can see that

$$\int_{c_L^2}^\infty \frac{\left(1+\frac{z}{\sum x_i^2}\right)^{-(n+1)}}{\sum x_i^2 B(1,n)} dz - \int_{c_U^2}^\infty \frac{\left(1+\frac{z}{\sum x_i^2}\right)^{-(n+1)}}{\sum x_i^2 B(1,n)} dz = 1-\alpha. \tag{2.5}$$

By substituting $\frac{1}{w} = 1 + \frac{Z}{\sum_{i=1}^{n} x_i^2}$, equation (2.5) becomes

$$I_{p1}(n,1) - I_{p2}(n,1) = 1-\alpha,$$

where $I_x(a,b)$ is incomplete beta function

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)} = \frac{1}{B(a,b)} \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad a,b>0,$$

$$p_1 \equiv \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + c_L^2}$$

and

$$p_2 \equiv \frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + c_U^2}.$$

The relationship between the incomplete beta function and the binomial distribution results in

$$\left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + c_L^2}\right)^n - \left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + c_L^2}\right)^n = 1 - \alpha.$$

Thus the $100(1-\alpha)\%$ most plausible region (c_L, c_U) can be obtained by solving the following equations :

$$\left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + c_L^2}\right)^n - \left(\frac{\sum_{i=1}^n x_i^2}{\sum_{i=1}^n x_i^2 + c_L^2}\right)^n = 1 - \alpha$$

and

$$\left(\frac{c_U^2 + \sum_{i=1}^n x_i^2}{c_L^2 + \sum_{i=1}^n x_i^2}\right)^{n+1} = \frac{c_U}{c_L}.$$

2.2. Locally uniform prior case

In this section, we consider a locally uniform prior distribution, $U^\theta(0,1)$ for θ with the probability density function

$$\Pi(\theta) = 1, \quad 0 < \theta < 1.$$

Then the posterior density of θ, given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \frac{1}{\Gamma(n+1)}\left(1 + \frac{\sum_{i=1}^n x_i^2}{t^2}\right)^{n+1}(-\log\theta)^n \exp\left(\frac{\sum_{i=1}^n x_i^2}{t^2}\log\theta\right), \quad 0 < \theta < 1.$$

The following theorem can be obtained.

Theorem 2.3 Under the locally uniform prior distribution for , the Bayesian predictive density of future observation Y is given by

$$f(y|\underline{x}) = \frac{2y(n+1)(t^2 + \sum_{i=1}^n x_i^2)^{n+1}}{(\sum_{i=1}^n x_i^2 + y^2 + t^2)^{n+2}}, \quad 0 < y < \infty.$$

Proof: The predictive density of Y is obtained from

$$f(y|\underline{x}) = \int_0^1 \Pi(\theta|\underline{x}) f(y|\theta) d\theta$$

and

$$f(y|\underline{x}) = \Psi \int_0^1 (-\log\theta)^{n+1} \theta^{-\left(\frac{\sum_{i=1}^n x_i^2 + y^2}{t^2}\right)} d\theta,$$

where Ψ is the normalizing constant. Thus by transforming $Z = -\log\theta$, one can obtain the following results :

$$f(y|\underline{x}) = \frac{1}{\Gamma(n+1)} \left(\frac{1 + \sum_{i=1}^n x_i^2}{t^2} \right) \left(\frac{2y}{t^2} \right) \int_0^\infty z^{n+1} e^{-z\left(\frac{\sum_{i=1}^n x_i^2 + y^2 + t^2}{t^2}\right)} dz. \quad (2.6)$$

By the Gamma kernel, the equation (2.6) is equal to

$$\frac{\Gamma(n+2)}{\Gamma(n+1)} \left(\frac{t^2 + sum_{i=1}^n x_i^2}{t^2} \right)^{n+1} \left(\frac{2y}{t^2} \right) \left(\frac{t^2}{t^2 + \sum_{i=1}^n x_i^2 + y^2} \right)^{n+2}$$
$$= \frac{2y(n+1)(t^2 + \sum_{i=1}^n x_i^2)^{n+1}}{(\sum_{i=1}^n x_i^2 + y^2 + t^2)^{n+2}}.$$

This completes the proof. □

One consider the Bayes predictive estimator and the most plausible region from a future observation of the Rayleigh model with locally uniform prior distribution. Note that the distribution of Y^2 has an inverted beta distribution, $InBe(1, n+1, t^2 + \sum_{i=1}^n x_i^2)$.

Theorem 2.4 Under the squared error loss and a locally uniform prior distribution, the Bayes predictive estimator of future observation Y is

$$\hat{Y}_{LU} = \frac{\Gamma(\frac{3}{2})\Gamma(n+\frac{1}{2})}{\Gamma(n+1)} \left(t^2 + \sum_{i=1}^n x_i^2 \right)^{\frac{1}{2}}.$$

Proof: By transforming $Z = Y^2$,

$$\begin{aligned} \hat{Y}_{LU} &= E(y|\underline{x}) \\ &= \int_0^\infty \frac{2y^2(n+1)(t^2 + \sum_{i=1}^n x_i^2)^{n+1}}{(\sum_{i=1}^n x_i^2 + t^2 + y^2)^{n+2}} dy \\ &= \int_0^\infty \frac{\sqrt{z}(n+1)(t^2 + \sum_{i=1}^n x_i^2)^{n+1}}{(\sum_{i=1}^n x_i^2 + t^2 + z)^{n+2}} dz. \end{aligned}$$

The Z has an inverted beta distribution, $InBe(\frac{3}{2}, n+\frac{1}{2}, t^2+\sum_{i=1}^n x_i^2)$.

$$\begin{aligned}\hat{Y}_{LU} =&(n+1)\left(t^2+\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} B\left(\frac{3}{2}, n+\frac{1}{2}\right)\\ &\times\int_0^\infty \frac{\sqrt{z}(t^2+\sum_{i=1}^n x_i^2)^{n+1}}{B\left(\frac{3}{2}, n+\frac{1}{2}\right)(t^2+\sum_{i=1}^n x_i^2+z)^{n+2}}dz\\ =&\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}{\Gamma(n+2)(n+1)(t^2+\sum_{i=1}^n x_i^2)^{\frac{1}{2}}}.\end{aligned}$$

Thus the theorem holds. □

Now consider the $100(1-\alpha)\%$ most plausible region (c_L, c_U) for Y under the locally uniform prior distribution. The predictive density $f(y|\underline{x})$ is

$$f(y|\underline{x}) = \frac{2y(t^2+\sum_{i=1}^n x_i^2)^{n+1}}{B(1,n+1)(\sum_{i=1}^n x_i^2+t^2+y^2)^{n+2}}.$$

Since $Z=Y^2$ has a inverted beta distribution, $InBe(1, n+1, t^2+\sum_{i=1}^n x_i^2)$, $P(c_L<Y<c_U) = P(c_L^2<Z<c_U^2) = 1-\alpha$.

By the substitution of $\frac{1}{w} = 1+\frac{Z}{\sum_{i=1}^n x_i^2}$,

$$P(c_L^2<Z<c_U^2) = I_{p1}(n+1,1) - I_{p2}(n+1,1),$$

where $I_{pi}(n+1,1)$ is the incomplete beta function and

$$\begin{aligned}p_1 \equiv&\frac{t^2+\sum_{i=1}^n x_i^2}{t^2+\sum_{i=1}^n x_i^2+c_L^2},\\ p_2 \equiv&\frac{t^2+\sum_{i=1}^n x_i^2}{t^2+\sum_{i=1}^n x_i^2+c_U^2}.\end{aligned}$$

From the relation between the incomplete beta function and the binomial distribution. The $100(1-\alpha)\%$ most plausible region (c_L, c_U) is the simultaneous solution of the following two equations :

$$\left(\frac{t^2+\sum_{i=1}^n x_i^2}{t^2+\sum_{i=1}^n x_i^2+c_L^2}\right)^n - \left(\frac{t^2+\sum_{i=1}^n x_i^2}{t^2+\sum_{i=1}^n x_i^2+c_U^2}\right)^n = 1-\alpha \tag{2.7}$$

and

$$\left(\frac{t^2+\sum_{i=1}^n x_i^2+c_U^2}{t^2+\sum_{i=1}^n x_i^2+c_L^2}\right)^{n+2} = \frac{c_U}{c_L}. \tag{2.8}$$

2.3. Inverse gamma prior case

In this section, we consider the inversed gamma prior distribution for θ with probability density function. First the original inversed gamma prior distribution for σ is

$$\begin{aligned}\pi(\sigma) =& \frac{a^b}{2^{b-1}\Gamma(b)}\sigma^{-(2b+1)}\exp\left(\frac{-a}{2\sigma^2}\right), \quad a,b>0, \quad \sigma>0\\ &\propto \sigma^{-(2b+1)}\exp\left(\frac{-a}{2\sigma^2}\right)\end{aligned}$$

and substituting $\sigma^2 = -\frac{t^2}{2\log\theta}$,

$$\begin{aligned}\Pi(\theta) &= \left(-\frac{2\log\theta}{t^2}\right)^{b+1}\exp\left(\frac{a\log\theta}{t^2}\right)\\ &= \left(\frac{2}{t^2}\right)^{b+1}(-\log\theta)\exp\left(\frac{a}{t^2}\log\theta\right).\end{aligned}$$

Then the posterior density of θ, given $\underline{X} = \underline{x}$ is

$$\Pi(\theta|\underline{x}) = \left(\frac{2}{t^2}\right)^{b+1}(-\log\theta)^{n+b+1}\exp\left(\frac{\sum_{i=1}^n x_i^2 + a}{t^2}\log\theta\right).$$

Thus the following theorem can be obtained.

Theorem 2.5 Under the inversed gamma prior distribution for , the Bayesian predictive density function of future observation Y is given by

$$f(y|\underline{x}) = \left(\frac{2}{t^2}\right)^{b+1}\frac{2ny(\sum_{i=1}^n x_i^2 + a)^n}{(\sum_{i=1}^n x_i^2 + y^2 + a)^{n+1}}.$$

Proof: The predictive density of Y is obtained from

$$f(y|\underline{x}) = \int_0^1 \Pi(\theta|\underline{x})f(y|\theta)d\theta$$

and

$$f(y|\underline{x}) = \Psi\int_0^1 (-\log\theta)^{n+b+2}\exp\left(\frac{\sum_{i=1}^n x_i^2 + y^2 + a}{t^2}\log\theta\right)d\theta,$$

where Ψ is the normalizing constant. Thus by transforming $Z = -\log\theta$, one can obtained the following results :

$$f(y|\underline{x}) = \Psi\int_0^\infty (z)^{n+b+2}e^{-z\left(\frac{\sum_{i=1}^n x_i^2+y^2+a}{t^2}\right)}dz. \tag{2.9}$$

By the Gamma kernel, the equation (2.9) is equal to

$$\left(\frac{2y}{t^2}\right)\left(\frac{2}{t^2}\right)^{b+1}\Gamma(n+1)\left(\frac{t^2}{\sum_{i=1}^n x_i^2+y^2+a}\right)^{n+1}$$
$$=\left(\frac{2}{t^2}\right)^{b+1}\frac{2ny(\sum_{i=1}^n x_i^2+a)^n}{(\sum_{i=1}^n x_i^2+y^2+a)^{n+1}}.$$

This completes the proof. □

One consider the Bayes predictive estimator and the most plausible region from a future observation of the Rayleigh model with Inversed gamma prior distribution. Note that the distribution of Y^2 has an inverted beta distribution $InBe(1, n, \sum_{i=1}^n x_i^2)$.

Theorem 2.6 Under the squared error loss and inversed gamma prior distribution, the Bayes predictive estimator of future observation Y is

$$\hat{Y}_{IG}=\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)}\left(\frac{2}{t^2}\right)^{b+1}\left(\sum_{i=1}^n x_i^2+a\right)^{\frac{1}{2}}.$$

Proof: By transforming $Z=Y^2$,

$$\begin{aligned}\hat{Y}_{LU} =& E(y|\underline{x})\\ =&\left(\frac{2}{t^2}\right)^{b+1}\int_0^\infty \frac{2yn(a+\sum_{i=1}^n x_i^2)}{(\sum_{i=1}^n x_i^2+a+y^2)^{n+1}}dy\\ =&\left(\frac{2}{t^2}\right)^{b+1}\int_0^\infty \frac{n(a+\sum_{i=1}^n x_i^2)^n\sqrt{z}}{(\sum_{i=1}^n x_i^2+a+z)^{n+1}}dz\\ =&\left(\frac{2}{t^2}\right)^{b+1}B\left(\frac{3}{2},n-\frac{1}{2}\right)\int_0^\infty \frac{n(a+\sum_{i=1}^n x_i^2)^{n-\frac{1}{2}}\sqrt{z}}{B\left(\frac{3}{2},n-\frac{1}{2}\right)(\sum_{i=1}^n x_i^2+a+z)^{n+1}}dz.\end{aligned}$$

Thus the theorem holds. □

Now consider the $100(1-\alpha)\%$ most plausible region (c_L, c_U) for Y under the inversed gamma prior distribution. The predictive density $f(y|\underline{x})$ is

$$f(y|\underline{x})=\left(\frac{2}{t^2}\right)^{b+1}\frac{2ny(\sum_{i=1}^n x_i^2+a)^n}{(\sum_{i=1}^n x_i^2+y^2+a)^{n+1}}.$$

Since $Z=Y^2$ has a inverted beta distribution, $InBe(1, n, a+\sum_{i=1}^n x_i^2)$, $P(c_L<Y<c_U)=P(c_L^2<Z<c_U^2)=1-\alpha$.
By the substitution of $\frac{1}{w}=1+\frac{Z}{\sum_{i=1}^n x_i^2}$,

$$P(c_L^2 < Z < c_U^2) = I_{p1}(n+1,1) - I_{p2}(n+1,1),$$

where $I_{pi}(n+1,1)$ is the incomplete beta function,

$$p_1 \equiv \frac{a + \sum_{i=1}^n x_i^2}{a + \sum_{i=1}^n x_i^2 + c_L^2}$$

and

$$p_2 \equiv \frac{a + \sum_{i=1}^n x_i^2}{a + \sum_{i=1}^n x_i^2 + c_U^2}.$$

From the relation between the incomplete beta function and the binomial distribution. The $100(1-\alpha)\%$ most plausible region (c_L, c_U) is the simultaneous solution of the following two equations :

$$\left(\frac{a + \sum_{i=1}^n x_i^2}{a + \sum_{i=1}^n x_i^2 + c_L^2}\right)^n - \left(\frac{a + \sum_{i=1}^n x_i^2}{a + \sum_{i=1}^n x_i^2 + c_U^2}\right)^n = 1 - \alpha \tag{2.10}$$

and

$$\left(\frac{a + \sum_{i=1}^n x_i^2 + c_U^2}{a + \sum_{i=1}^n x_i^2 + c_L^2}\right)^{n+2} = \frac{c_U}{c_L}. \tag{2.11}$$

3. Simulation study and discussion

In this section, we conduct a simulation experiments in order to assess the performances of Bayes estimator of θ using the three different kinds of prior. The behavior of the Bayes predictive estimator of future observation Y can evaluated by the MSE.

We simulated samples from the Rayleigh model with $\sigma^2 = 1$ and $\sigma^2 = 2$ using four different sample sizes (n = 10, 30, 50, 100). We give the 'mission time' $t = 1$. All results are based on 1000 repetitions. In the Table 3.1 the Bayes estimators for the parameter is averaged over the total number of repetitions with the specified time $t = 2$.

Table 3.1 shows that the Bayes estimator with non-informative prior case provides little underestimated both case which is $\sigma^2 = 1$ and $\sigma^2 = 2$, even if sample size is quite large. The Bayes estimator of two kinds of prior, uniform prior and inverse gamma prior are almost identical results. For the inverse gamma prior case any dramatic difference is not detected but the performance with the inverse gamma prior are generally better than those of noninformative prior and uniform prior case.

Table 3.1 Bayes predictive estimator of a future observation of y

	$\sigma^2 = 1$			
	$n = 10$	$n = 30$	$n = 50$	$n = 100$
$\hat{Y}_{NI}$	1.820	1.802	1.785	1.774
$\hat{Y}_U$	3.635	3.584	3.566	3.559
$\hat{Y}_{IG}$	3.794	3.634	3.604	3.560
	$\sigma^2 = 2$			
	$n = 10$	$n = 30$	$n = 50$	$n = 100$
$\hat{Y}_{NI}$	2.238	2.190	2.181	2.179
$\hat{Y}_U$	4.360	4.386	4.360	4.351
$\hat{Y}_{IG}$	4.630	4.416	4.390	4.368

* NI: Noninformative prior, U: Uniform prior
IG : Inverse gamma prior

In Table 3.2, the most plausible region for σ^2 have been reported. We have analyzed these interval for three different prior. As expected, it is observed that as the sample size increases, the most plausible region becomes narrower. There is a little difference in the behavior of the most plausible region depends on their Bayes estimator.

The intervals are more or less same for large sample except the noninformative case. There is not much difference in the behavior of most plausible region when comparing Uniform prior over Inverse gamma prior. One cannot say that the most plausible region are changed much with respect to the significant change of the prior from a noninformative prior distribution to a uniform prior distribution for σ, one can say that the most plausible regions are relatively related to the change of the prior distribution.

Table 3.2 Most plausible region (c_L, c_U)

	$\sigma^2 = 1$			
	$n = 10$	$n = 30$	$n = 50$	$n = 100$
$\hat{Y}_{NI}$	(1.801, 1.838)	(1.792, 1.812)	(1.777, 1.793)	(1.769, 1.778)
$\hat{Y}_U$	(3.599, 3.671)	(3.564, 3.604)	(3.550, 3.580)	(3.548, 3.570)
$\hat{Y}_{IG}$	(3.761, 3.837)	(3.614, 3.654)	(3.589, 3.619)	(3.549, 3.571)
	$\sigma^2 = 2$			
	$n = 10$	$n = 30$	$n = 50$	$n = 100$
$\hat{Y}_{NI}$	(2.221, 2.254)	(2.181, 2.199)	(2.174, 2.188)	(2.174, 2.184)
$\hat{Y}_U$	(4.350, 4.370)	(4.368, 4.404)	(4.345, 4.373)	(4.340, 4.361)
$\hat{Y}_{IG}$	(4.599, 4.661)	(4.397, 4.434)	(4.375, 4.403)	(4.357, 4.378)

* NI: Noninformative prior, U: Uniform prior
IG : Inverse gamma prior

In Table 3.3, MSE of the Bayes predictive estimator of a future observation of y is not much variation in behavior of MSE of the Bayes estimator and when sample size increase, MSE is decrease.

Table 3.3 MSE of the Bayes estimator

	$\sigma^2 = 1$			
	$n = 10$	$n = 30$	$n = 50$	$n = 100$
$\hat{Y}_{NI}$	.2936	.1600	.1272	.0848
$\hat{Y}_{U}$	.5791	.3236	.2456	.1767
$\hat{Y}_{IG}$	.5310	.3243	.2450	.1730
	$\sigma^2 = 2$			
	$n = 10$	$n = 30$	$n = 50$	$n = 100$
$\hat{Y}_{NI}$	.2651	.1498	.1131	.0825
$\hat{Y}_{U}$	.3624	.3030	.2297	.1674
$\hat{Y}_{IG}$	.5026	.3021	.2262	.1652

* NI: Noninformative prior, U: Uniform prior
IG : Inverse gamma prior

4. Conclusion

In this paper, we have primarily studied the Bayes estimator of the parameter of the Rayleigh distribution under three different prior. The performance of each prior distribution with the Rayliegh distribution has a little different among those estimators. In real life time data, we recommend using the inverse gamma prior to estimate the Bayes estimator of future observation. The inverse gamma prior distribution showed the more stable Bayesian estimator and most plausible region from the simulation results. In almost all case expect noninformative prior distribution case, the predicted Bayese estimators are extremely similar to each other.

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