Commun. Korean Math. Soc. **32** (2017), No. 3, pp. 779–787
https://doi.org/10.4134/CKMS.c160158
pISSN: 1225-1763 / eISSN: 2234-3024

SOME PROPERTIES OF THE BEREZIN TRANSFORM IN THE BIDISC

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Abstract. Let m be the Lebesgue measure on $\mathbb{C}$ normalized to $m(D) = 1$, μ be an invariant measure on D defined by $d\mu(z) = (1-|z|^2)^{-2}\, dm(z)$. For $f \in L^1(D^n, m \times \cdots \times m)$, Bf the Berezin transform of f is defined by,

$$(Bf)(z_1, \ldots, z_n) = \int_D \cdots \int_D f\big(\varphi_{z_1}(x_1), \ldots, \varphi_{z_n}(x_n)\big) dm(x_1) \cdots dm(x_n).$$

We prove that if $f \in L^1(D^2, \mu \times \mu)$ is radial and satisfies $\int\int_{D^2} f d\mu \times d\mu = 0$, then for every bounded radial function ℓ on D^2 we have

$$\lim_{n\to\infty} \int\int_{D^2} (B^n f)(z, w)\ell(z, w) d\mu(z) d\mu(w) = 0.$$

Then, using the above property we prove n-harmonicity of bounded function which is invariant under the Berezin transform. And we show the same results for the weighted the Berezin transform in the polydisc.

1. Introduction

Let D be the unit disc of $\mathbb{C}$ and let m be the Lebesgue measure on $\mathbb{C}$ normalized to $m(D) = 1$. For $u \in L^1(D, m)$, the Berezin transform Tu on D is defined by,

$$(Tu)(z) = \int_D u\big(\varphi_z(x)\big) dm(x) \quad \text{for } z \in D,$$

where φ_z is the canonical automorphism given by

$$\varphi_z(x) = \frac{z - x}{1 - \bar{z}x}.$$

Equivalently we can write

$$(Tu)(z) = \int_D u(x) \frac{(1 - |z|^2)^2}{|1 - \bar{z}x|^4} dm(x). \tag{1.1}$$

Received July 27, 2016; Accepted December 9, 2016.
2010 *Mathematics Subject Classification.* Primary 32A70; Secondary 47G10.
Key words and phrases. Berezin transform, bidisc, harmonic function.
Supported by 2016 Sogang University Research Grant.

Let μ be a measure on D defined by $d\mu(z) = (1-|z|^2)^{-2}dm(z)$, then we have

$$\int_D u d\mu = \int_D Tu d\mu = \int_D u \circ \psi d\mu$$

for $u \in L^1(D,\mu)$ and $\psi \in \mathrm{Aut}(D)$. It is known that (Lemma 3.3 of [4]), for $1 \le p \le \infty$, T is a self-adjoint bounded operator on $L^p(D,\mu)$ with $\|T\|_p \le 1$. The advantage of using the invariant measure μ is that even though μ is not a finite measure on D, the space $L^\infty(D,m)$ is the dual space of $L^1(D,d\mu)$ on which the operator T has a nice behavior.

If $u \in L^1(D,m)$ is harmonic, then it is obvious that u satisfies $Tu = u$. In 1993, Ahern, Flores and Rudin ([1]) showed that if $u \in L^1(D,m)$ satisfies $Tu = u$, then it is harmonic.

We can define the Berezin transform of a function on the polydisc. For $f \in L^1(D^n)$, Bf the Berezin transform of f is defined by,

$$(Bf)(z_1,\dots,z_n) = \int_D \cdots \int_D f\big(\varphi_{z_1}(x_1),\dots,\varphi_{z_n}(x_n)\big) dm(x_1)\cdots dm(x_n).$$

If $f \in L^1(D^n, m \times \cdots \times m)$ is n-harmonic, then it is obvious that f satisfies $Bf = f$. In 1998 the author ([4]) showed that, for $1 \le p < \infty$, a function $f \in L^p(D^2, m\times m)$ satisfying $Bf = f$ needs not be 2-harmonic while $f \in L^\infty(D^n)$ satisfying $Bf = f$ has to be n-harmonic. Earlier, Furstenberg ([2], [3]) proved that, on any dimensional symmetric domain, a bounded function which satisfies an invariant mean value property is harmonic with respect to the intrinsic metric. The proof of n-harmonicity in [4], which is completely analytic, is based on Lemma 3.5 stated as follows: If $u \in L^1(D,\mu)$ is radial and $\int_D u d\mu = 0$, then

$$\lim_{n\to\infty} \int_D \left|T^n u\right| d\mu = 0.$$

In this paper, we extend Lemma 3.5 of [4] to functions on the polydisc. We denote the space $L^p_R(D^n)$ as the subspace of $L^p(D^n)$ which consists of all radial functions, i.e.,

$$L^p_R(D^n) = \{f \in L^p(D^n) \mid f(z_1,\dots,z_n) = f(|z_1|,\dots,|z_n|) \text{ for every } (z_1,\dots,z_n) \in D^n\}.$$

We also denote for $f \in L^p(D^2,\mu\times\mu)$ and $q \in L^q(D^2,\mu\times\mu)$ $(1 \le p \le \infty,\ 1/p + 1/q = 1)$,

$$\langle f, g\rangle = \iint_{D^2} f \cdot g d\mu \times d\mu.$$

Then by self-adjointness of the Berezin operator B, we get $\langle Bf, g\rangle = \langle f, Bg\rangle$.

In Section 2, we extend Lemma 3.5 of [4] to functions on D^n so that the same statement is true in the polydisc. And then we give more straightforward proof of the n-harmonicity of functions which are invariant under the Berezin transform. Proposition 2.1 is the main result of the paper. And for notational simplicity, we will prove our main result in the bidisc D^2. In Section 3, we do the same thing as in Section 2 for a weighted Berezin transform of a function in the polydisc.

2. Main result

For $u \in L^p(D, \mu)$ with $1 \le p \le \infty$, let T^n be the iteration of T for n times, then by induction we can write

$$(T^n u)(z) = \int_D u(x) K_n(z, x) dm(x), \tag{2.1}$$

where for every $z \in D$ and $n \in \mathbb{N}$ satisfying

$$\int_D K_n(z, x) dm(x) = 1. \tag{2.2}$$

The next proposition is the main result of this paper.

Proposition 2.1. *If $f \in L^1_R(D^2, \mu \times \mu)$ satisfies $\langle f, 1 \rangle = 0$, then for every $\ell \in L^\infty_R(D^2)$*

$$\lim_{n \to \infty} \iint_{D^2} (B^n f)(z, w) \ell(z, w) d\mu(z) d\mu(w) = 0.$$

Proof. The proof consists of three steps.

Step (i). Basic notations and preliminaries.

Given $f \in L^1_R(D^2, \mu \times \mu)$ such that

$$\langle f, 1 \rangle = \iint_{D^2} f d\mu \times d\mu = 0,$$

we define u on D by

$$u(x) = \int_D f(x, y) d\mu(y). \tag{2.1}$$

Then $u \in L^1_R(D, \mu)$ with

$$\int_D u d\mu = 0.$$

We also define g on $D \times D$ by

$$g(x, y) = (1 - |y|^2)^2 u(x) = (1 - |y|^2)^2 \int_D f(x, z) d\mu(z). \tag{2.2}$$

Then, we can write for $v \in L^1_R(D, \mu)$,

$$(T^n v)(x) = \int_D v(y) K_n(x, y) dm(y), \tag{2.3}$$

where

$$\int_D K_n(x, y) dm(y) = 1.$$

Then we can express

$$(B^n f)(x, y) = \iint_{D^2} f(t, s) K_n(x, t) K_n(y, s) dm(t) dm(s). \tag{2.4}$$

Now choose $\ell \in L^\infty_R(D^2)$, then define ℓ_n on D by

$$\ell_n(x) = \iint_{D^2} \ell(x,s)K_n(y,s)dm(s)dm(y), \tag{2.5}$$

to get $\|\ell_n\|_\infty \leq \|\ell\|_\infty$.

To prove the proposition is to show $\langle B^n f, \ell\rangle \to 0$ as $n \to \infty$. And we have

$$\langle B^n f, \ell\rangle = \langle f, B^n \ell\rangle = \langle g, B^n \ell\rangle + \langle f - g, B^n \ell\rangle. \tag{2.6}$$

Hence to complete the proof , we will show

$$\langle g, B^n \ell\rangle \to 0 \qquad \text{in step (ii)}$$

and

$$\langle f - g, B^n \ell\rangle \to 0 \qquad \text{in step (iii).}$$

Step (ii). Proof of $\langle g, B^n \ell\rangle \to 0$.

By using Fubini's theorem, we get

$$\begin{aligned}
&\langle g, B^n \ell\rangle \\
&= \int_D (1-|y|^2)^2 u(x)\Big(\int_D B^n \ell(x,y)d\mu(y)\Big)d\mu(x) \\
&= \int_D u(x)\Big(\int_D B^n \ell(x,y)dm(y)\Big)d\mu(x) \\
&= \int_D u(x)\cdot\Big(\int_D \{\int\int_{D^2} \ell(t,s)K_n(x,t)K_n(y,s)dm(s)dm(t)\}dm(y)\Big)d\mu(x) \\
&= \int_D u(x)\cdot\Big(\int_D \{\int\int_{D^2} \ell(t,s)K_n(y,s)dm(s)dm(y)\}K_n(x,t)dm(t)\Big)d\mu(x) \\
&= \int_D u(x)\Big(\int_D \ell_n(t)K_n(x,t)dm(t)\Big)d\mu(x) \\
&= \int_D u(x)(T^n \ell_n)(x)d\mu(x) \\
&= \int_D (T^n u)(x)\ell_n(x)d\mu(x).
\end{aligned}$$

Hence we have

$$\begin{aligned}
\big|\langle g, B^n \ell\rangle\big| &= \Big|\int_D (T^n u)(x)\ell_n(x)d\mu(x)\Big| \\
&\leq \|T^n u\|_1 \cdot \|\ell_n\|_\infty \\
&\leq \|T^n u\|_1 \cdot \|\ell\|_\infty.
\end{aligned}$$

Since $u \in L^1_R(D,\mu)$ and $\int_D u d\mu = 0$, Lemma 3.5 of [4] implies that

$$\lim_{n\to\infty} \|T^n u\|_1 = 0.$$

This completes step (ii).

Step (iii). Proof of $\langle f-g, B^n\ell\rangle \to 0$.
Now fix $x \in D$, then define $(\ell_x)_n$ on D by

$$(\ell_x)_n(y) = \int_D \ell(t,y)K_n(x,t)dm(t) \tag{2.7}$$

then $\|(\ell_x)_n(y)\|_\infty \le \|\ell\|_\infty$. And we get

$$\begin{aligned} T^n(\ell_x)_n(y) &= \iint_{D^2} \ell(t,s)K_n(x,t)K_n(y,s)dm(t)dm(s) \\ &= (B^n\ell)(x,y). \end{aligned} \tag{2.8}$$

We define V_x on D by $V_x(y) = f(x,y) - g(x,y)$. Then $V_x \in L^1_R(D,\mu)$ and

$$\begin{aligned} \int_D V_x(y)d\mu(y) &= \int_D f(x,y)d\mu(y) - u(x)\int_D (1-|y|^2)^2 d\mu(y) \\ &= u(x) - u(x) = 0. \end{aligned}$$

Now, we have

$$\begin{aligned} &\left|\int_D \big(f(x,y)-g(x,y)\big)(B^n\ell)(x,y)d\mu(y)\right| \\ &= \left|\int_D V_x(y)T^n(\ell_x)_n(y)d\mu(y)\right| \\ &= \left|\int_D (T^nV_x)(y)(\ell_x)_n(y)d\mu(y)\right| \\ &\le \|(\ell_x)_n\|_\infty \cdot \|T^nV_x\|_1 \\ &\le \|\ell\|_\infty \cdot \|T^nV_x\|_1. \end{aligned}$$

Once again by Lemma 3.5 of [4], we get

$$\lim_{n\to\infty}\left|\int_D \big(f(x,y)-g(x,y)\big)(B^n\ell)(x,y)d\mu(y)\right| = 0 \tag{2.9}$$

for fixed $x \in D$. But

$$\begin{aligned} &\left|\int_D V_x(y)T^n(\ell_x)_n(y)d\mu(y)\right| \\ &\le \|\ell\|_\infty \int_D |V_x(y)|d\mu(y) \\ &\le \|\ell\|_\infty \int_D \Big(|f(x,y)| + |g(x,y)|\Big)d\mu(y) \end{aligned}$$

and the function

$$\int_D \Big(|f(\cdot,y)| + |g(\cdot,y)\ |\Big)d\mu(y) \in L^1(D,d\mu).$$

Hence by dominated convergence theorem,

$$\lim_{n\to\infty}\Big|\langle f-g, B^n\ell\rangle\Big| \le \int_D \lim_{n\to\infty}\left|\int_D V_x(y)T^n(\ell_x)_n(y)d\mu(y)\right|d\mu(x) = 0.$$

This completes step (iii). And the proof of the proposition is complete if we combine steps (i), (ii) and (iii). □

The next corollary, which comes directly from Proposition 2.1, is the bidisc version of Lemma 3.5 of [4].

Corollary 2.2. *Let* $f \in L^1_R(D^2, \mu \times \mu)$. *Then*

$$\lim_{n\to\infty} \iint_{D^2} |B^n f| d\mu \times d\mu = 0$$

if and only if $\langle f, 1\rangle = 0$.

Proof. For $f \in L^1_R(D^2, \mu \times \mu)$, we have

$$\|B^n f\|_1 = \sup_{\ell \in L^\infty_R(D^2)} \{|\langle\, B^n f, \ell\rangle| : \|\ell\|_\infty \le 1\}. \tag{2.1}$$

But we showed in the proof of the proposition that for $f \in L^1_R(D^2)$ and $\ell \in L^\infty_R(D^2)$

$$\langle B^n f, \ell\rangle = \langle g, B^n \ell\rangle + \langle f - g, B^n \ell\rangle,$$

where

$$|\langle g, B^n \ell\rangle| \le \|T^n u\|_1 \cdot \|\ell\|_\infty$$

and

$$|\langle f - g, B^n \ell\rangle| \le \left(\int_D \|T^n V_x\|_1 d\mu(x)\right) \|\ell\|_\infty.$$

Thus

$$|\langle B^n f, \ell\rangle| \le \left(\|T^n u\|_1 + \int_D \|T^n V_x\|_1 d\mu(x)\right) \cdot \|\ell\|_\infty. \tag{2.2}$$

Since $\int_D u d\mu = \int_D V_x d\mu = 0$, by dominated convergence theorem

$$\begin{aligned}\lim_{n\to\infty} \|B^n f\|_1 &= \lim_{n\to\infty} \left(\|T^n u\|_1 + \int_D \|T^n V_x\|_1 d\mu(x)\right)\\ &= 0.\end{aligned}$$

This completes the proof. □

Now by using Proposition 2.1, we give more straightforward proof of the 2-harmonicity of functions which are invariant under the Berezin transform, which was given as Theorem 3.1 of [4].

Corollary 2.3. *Let* $f \in L^\infty(D^2)$ *satisfy* $Bf = f$. *Then* f *is* 2*-harmonic.*

Proof. First, we assume that $f \in L^\infty_R(D^2)$. Then for every $g \in L^1_R(D^2, \mu \times \mu)$,

$$\lim_{n\to\infty} \iint_{D^2} (B^n g)(z, w) f(z, w) d\mu(z) d\mu(w) = 0$$

by Proposition 2.1. Since B is self-adjoint and $Bf = f$, we have

$$\langle B^n g, f\rangle = \langle g, B^n f\rangle = \langle g, f\rangle.$$

Hence

$$\iint_{D^2} gf d\mu \times d\mu = 0$$

for every $g \in L^1_R(D^2, d\mu \times d\mu)$ satisfying

$$\iint_{D^2} g d\mu \times d\mu = 0.$$

Therefore, f is a constant.

The rest of the proof is identical to the step 2 of the proof of Theorem 3.1 of [4]. □

3. Weighted Berezin transform

For $c > -1$, we define a finite measure ν_c on $\mathbb{C}$ by $d\nu_c(z) = (c+1)(1 - |z|^2)^c \, dm(z)$ so that $\nu_c(D) = 1$. If a function $u \in L^1(D, \nu_c)$ is harmonic, then $u \circ \psi$ is also harmonic for every $\psi \in \mathrm{Aut}(D)$. Thus u satisfies a mean value property

$$\int_D (u \circ \psi) \, d\nu_c = f(\psi(0)) \quad \text{for every } \psi \in \mathrm{Aut}(D), \tag{3.1}$$

which is equivalent to saying that $\int_D (u \circ \varphi_z) \, d\nu_c = u(z)$ for every $z \in D$. Now for $c > -1$, $u \in L^1(D, \nu_c)$ and $z \in D$, we define $T_c u$ the weighted Berezin transform of u by

$$\big(T_c u\big)(z) = \int_D (u \circ \varphi_z) d\nu_c.$$

We can easily see that, for $1 \le p \le \infty$, T_c is a self-adjoint bounded operator on $L^p(D, \mu)$ with $\|T\|_p \le 1$. For $c_1, c_2 > -1$ and $f \in L^1(D^2, \nu_{c_1} \times \nu_{c_2})$, we define the weighted Berezin transform $B_{c_1,c_2} f$ on D^2 by

$$\big(B_{c_1,c_2}\big) f(z, w) = \int_D \int_D f\big(\varphi_z(x), \varphi_w(y)\big) d\nu_{c_1}(x) \, d\nu_{c_2}(y).$$

If f is 2-harmonic, then just as the case in one complex variable we see that $B_{c_1,c_2} f = f$ for every $c_1, c_2 > -1$ and conversely, from the theorem of Furstenberg, it is already known that if $f \in L^\infty(D^2)$ satisfies $B_{c_1,c_2} f = f$ for some $c_1, c_2 > -1$, then it 2-harmonic. The author ([5]) proved that for every $1 \le p < \infty$ and $c_1, c_2 > -1$, a function $f \in L^p(D^2, \nu_{c_1} \times \nu_{c_2})$ which is invariant under the weighted Berezin transform; $B_{c_1,c_2} f = f$ needs not be 2-harmonic. In Lemma 2.1 of [6] the author showed that

$$\lim_{n\to\infty} \big\| T_c^n (I - T_c) \big\| = 0 \quad \text{on } L^1_R(D, \mu).$$

By using this, we can prove the following lemma.

Lemma 3.1. *If $u \in L^1(D, \mu)$ is radial and $\int_D u d\mu = 0$, then for $c > -1$*

$$\lim_{n\to\infty} \int_D \big| T_c^n u \big| d\mu = 0.$$

Proof. By Lemma 2.1 of [6], we have

$$\lim_{n\to\infty}\int_D \left|T_c^n u\right| d\mu = 0 \quad \text{for all} \;\; u \in (I - T_c)\ L_R^1(\mu).$$

Now let X be the subspace of $L_R^1(\mu)$ defined by

$$X = \left\{ u \in L_R^1(\mu) \mid \int_D u\ d\mu = 0 \right\}.$$

Then, since $\int_D u\ d\mu = \int_D T_c u\ d\mu$ for $u \in L_R^1(\mu)$, we get

$$(I - T_c)\ L_R^1 \ \subset\ X.$$

Hence the proof is complete when we show that $(I - T_c)L_R^1$ is dense in X.

Now let $w \in L_R^\infty(D)$ satisfy that

$$\int_D (v - T_c v) \cdot w d\mu = 0 \quad \text{for every } v \in L_R^1(D, \mu).$$

Then by self-adjointness we get

$$\int_D v \cdot (w - T_c w) d\mu = 0 \quad \text{for every } v \in L_R^1(D, \mu).$$

Thus $w = T_c w$ and by Furstenberg ([2], [3]) or Theorem 1.1 of [6] w is radial and harmonic, which means w is a constant. Therefore we get

$$\int_D u \cdot w d\mu = 0 \quad \text{for every } u \in X.$$

Thus Hahn-Banach Theorem implies that $(I - T_c)L_R^1$ is dense in X. This proves the lemma. □

Using Lemma 3.1 we can prove the following proposition. Since the proof is literally line by line identical to that of Proposition 2.1, we omit the proof of Proposition 3.2.

Proposition 3.2. *If $f \in L_R^1(D^2, \mu \times \mu)$ satisfies $\langle f, 1\rangle = 0$, then for every $\ell \in L_R^\infty(D^2)$ and $c_1, c_2 > -1$*

$$\lim_{n\to\infty} \iint_{D^2} (B_{c_1,c_2}^n f)(z, w)\ell(z, w) d\mu(z) d\mu(w) = 0.$$

Once again, by using Proposition 3.2, we can provide a straightforward proof of the 2-harmonicity of functions which are invariant under the weighted Berezin transform.

Corollary 3.3. *If $f \in L^\infty(D^2)$ satisfies $B_{c_1,c_2} f = f$ for some $c_1, c_2 > -1$, then f is 2-harmonic.*

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