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A WEIGHTED-PATH FOLLOWING INTERIOR-POINT ALGORITHM FOR CARTESIAN $P_*(\kappa)$-LCP OVER SYMMETRIC CONES

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Abstract. Finding an initial feasible solution on the central path is the main difficulty of feasible interior-point methods. Although, some algorithms have been suggested to remedy this difficulty, many practical implementations often do not use perfectly centered starting points. Therefore, it is worth to analyze the case that the starting point is not exactly on the central path. In this paper, we propose a weighted-path following interior-point algorithm for solving the Cartesian $P_*(\kappa)$-linear complementarity problems (LCPs) over symmetric cones. The convergence analysis of the algorithm is shown and it is proved that the algorithm terminates after at most $O\left((1+4\kappa)\sqrt{r}\log\frac{\mathbf{Tr}(x^0\diamond s^0)}{\varepsilon}\right)$ iterations.

1. Introduction

The symmetric cone linear complementarity problem (SCLCP) in the standard form is the problem of finding $(x,s)\in\mathcal{K}\times\mathcal{K}$ such that

$$s=\mathcal{A}x+q,\ \langle x,s\rangle=0, \tag{1}$$

where $\mathcal{A}:\mathcal{V}\longrightarrow\mathcal{V}$ is a linear operator, $q\in\mathcal{V}$ and $\mathcal{K}$ is the symmetric cone related to the Euclidean Jordan algebra $(\mathcal{V},\circ)$ which is equipped with the standard inner product $\langle x,s\rangle:=\mathbf{Tr}(x\circ s)$. The SCLCP is called monotone if $\mathcal{A}$ is a positive semidefinite operator, i.e., for each $x\in\mathcal{V}$, $\langle x,\mathcal{A}(x)\rangle\geq 0$.

The SCLCPs are a general class of mathematical problems which include symmetric cone optimization (SO) problems, convex quadratic symmetric cone optimization (CQSCO) problems, second-order cone linear complementarity problems (SOLCPs), semidefinite optimization (SDO) problems and semidefinite linear complementarity problems (SDLCPs). So, this is an interesting research to investigate and analyze the algorithms of solving SCLCPs. There are many approaches for solving this class of problems. Among them, the path

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following interior-point methods (IPMs) are the most efficient and fundamental methods which obtain the best complexity bounds.

There are two types of IPMs based on choosing the starting point. Feasible IPMs, when the initial point and the subsequent iterates are in the interior of the feasible region and infeasible IPMs (IIPMs) when the initial point and the subsequent iterates are not necessary feasible. For monotone LCPs (LCPs over non-negative orthant with positive semidefinite operator), several interior-point algorithms have been proposed by researchers (for example [11, 12]). In conic optimization, Faybusovich [5] made the first attempt to generalize IPMs to SO and SCLCP by using Euclidean Jordan algebras (EJAs). Rangarajan [14] proved the polynomial time convergence of IPMs for SO. Potra [13] proposed an infeasible corrector-predictor IPM for monotone SCLCPs.

Gu et al. [7] extended Roos's algorithm [15] for LO to SO by using the Nesterov-Todd (NT) search directions. Yoshise [24] analyzed some IPMs for symmetric cone nonlinear complementarity problem (SCNCP) and proposed a homogeneous interior-point algorithm for solving this class of mathematical problems. A theoretical framework of path-following IPMs for Cartesian $P_*(\kappa)$-SCLCP has been established by Luo and Xiu [10]. Wang and Lesaja [23] proposed a feasible IPM for Cartesian $P_*(\kappa)$-SCLCPs and proved that the complexity bound of their algorithm is $O\left((1+4\kappa)\sqrt{r}\log\frac{1}{\varepsilon}\right)$ which coincides with the currently best-known iteration bound for feasible IPMs. Wang and Bai [22] presented a class of polynomial interior-point algorithms for Cartesian P-matrix SCLCPs.

The most of above mentioned algorithms are feasible and therefore they need to start from an initial feasible solution on the central path. Some difficulties and shortages with these methods can be expressed as follow.

Sometimes, finding an initial feasible solution of the underlying problem is arduous. In some other times, the starting point is feasible but it is not exactly on the central path. The first difficulty has been solved by suggesting the infeasible IPMs while the weighted-path following methods have been proposed for the remedy of second difficulty. About the monoton complementarity problems, although we can find a starting feasible solution on the central path by using the embedding model introduced by Kojima et al. [9], practical implementations often do not use perfectly centered starting points. Therefore, it is worth to study and analyze the case when the starting point is not exactly on the central path.

As it is usual for the algorithms following the central path, we can associate a target sequence on the central path. This observation lead to the concept of target-following methods introduced by Jansen et al. [8]. After the initial weighted-path following algorithm for linear optimization presented by Darvay [3], some authors generalized this algorithm to various classes of optimization problems. Achache [1] and Wang and Bai [19–21] respectively extended Darvay's algorithm [3] for LO to CQO, second order cone optimization (SOCO), SDO problems and SO problems. Motivated by these works, the main goal of this paper is to generalize the Darvay's weighted-path following algorithm [3] for LO to the Cartesian $P_*(\kappa)$-SCLCPs. The convergence analysis of the algorithm is proved and it is shown that the proposed algorithm has quadratically convergent with polynomial-time complexity.

The paper is organized as follows. In Section 2, we state some concepts and definitions in Cartesian EJAs and the related symmetric cones. We also recall some well-known and key lemmas which are required in our analysis. Section 3 describes the idea of path following IPMs and presents the well-known NT-search directions. The generic path following interior-point algorithm for the Cartesian $P_*(\kappa)$-SCLCP is presented in Section 4. Section 5 is devoted to prove the convergence analysis of the algorithm. Finally, the paper ends with some concluding remarks in Section 6.

2. Preliminaries

In this section, we review some basic definitions and imperative lemmas which will be used in convergence analysis of the algorithm. The classical EJA $(\mathcal{V}, \circ)$ is a finite dimensional vector space over $\mathbb{R}$ equipped with the bilinear map $\text{o} : (x, y) \longrightarrow x \circ y \in \mathcal{V}$ and the standard inner product $\langle x, s\rangle = \mathbf{Tr}(x \circ s)$. The related cone of squares corresponding with $(\mathcal{V}, \circ)$ is called the classical symmetric cone $\mathcal{K}$. For each $x \in \mathcal{V}$, $L(x)y := x \circ y$ and $P(x) := 2L(x)^2 - L(x^2)$, where $L(x)^2 := L(x)L(x)$, denote the linear and quadratic representation of $\mathcal{V}$, respectively.

The general state of these definitions known as the Cartesian EJA $(\mathcal{V}, \diamond)$ and the Cartesian symmetric cone $\mathcal{K}$ can be defined as the product of a finite number of the classical EJAs $(\mathcal{V}_j, \circ)$ and the classical symmetric cones $\mathcal{K}_j$ for $j = 1, 2, \ldots, N$. That is, $\mathcal{V} := \mathcal{V}_1 \times \mathcal{V}_2 \times \cdots \times \mathcal{V}_N$ and $\mathcal{K} = \mathcal{K}_1 \times \mathcal{K}_2 \times \cdots \times \mathcal{K}_N$. According to these definitions, the Cartesian EJA $(\mathcal{V}, \diamond)$ has the dimension $n = \sum_{j=1}^{N} n_j$ and the rank $r = \sum_{j=1}^{N} r_j$ where n_j and r_j are the dimension and rank of the classical EJAs $(\mathcal{V}_j, \circ)$. Some other basic definitions in Cartesian EJA are listed as follow:

• **Bilinear map** $\diamond$: For any

$$x := \left(x^{(1)}, x^{(2)}, \ldots, x^{(N)}\right)^T, \; s := \left(s^{(1)}, s^{(2)}, \ldots, s^{(N)}\right)^T,$$

in Cartesian EJA $(\mathcal{V}, \diamond)$, we define

$$x \diamond s := \left(x^{(1)} \circ s^{(1)}, x^{(2)} \circ s^{(2)}, \ldots, x^{(N)} \circ s^{(N)}\right)^T.$$

• **Identity element**: Let $e^{(j)}$ be the identity element of the classical EJA $(\mathcal{V}_j, \circ)$ then $e := \left(e^{(1)}, e^{(2)}, \ldots, e^{(N)}\right)^T$ is the identity element of the Cartesian EJA $(\mathcal{V}, \diamond)$.

• **Trace**($\mathbf{x}$): Let $\lambda_i\left(x^{(j)}\right)$ be the i-th eigenvalue of the $x^{(j)}$, then

$$\mathbf{Tr}(x) := \sum_{j=1}^{N} \mathbf{Tr}\left(x^{(j)}\right),$$

where $\mathbf{Tr}\left(x^{(j)}\right) := \sum_{i=1}^{r_j} \lambda_i\left(x^{(j)}\right)$.
• **Determinate**($\mathbf{x}$): For any $x \in (\mathcal{V}, \diamond)$, we define $\det(x) := \prod_{j=1}^{N} \det\left(x^{(j)}\right)$ where $\det\left(x^{(j)}\right) := \prod_{i=1}^{r_j} \lambda_i\left(x^{(j)}\right)$.
• **Inner product**: While $\langle x^{(j)}, s^{(j)}\rangle := \mathbf{Tr}\left(x^{(j)} \circ s^{(j)}\right)$ is the inner product related to the classical EJAs $(\mathcal{V}_j, \circ)$ for $j = 1, 2, \ldots, N$, the inner product related to the Cartesian EJA $(\mathcal{V}, \diamond)$ will be defined as follows:

$$\langle x, s\rangle := \mathbf{Tr}(x \diamond s) = \sum_{j=1}^{N} \mathbf{Tr}\left(x^{(j)} \circ s^{(j)}\right).$$

• **Min**(**Max**) **eigenvalue**: Considering

$$\lambda_{\min}(x^{(j)}) := \min\{\lambda_i(x^{(j)}) \,|\, i = 1, 2, \ldots, r_j\},\ j = 1, 2, \ldots, N,$$
$$\lambda_{\max}(x^{(j)}) := \max\{\lambda_i(x^{(j)}) \,|\, i = 1, 2, \ldots, r_j\},\ j = 1, 2, \ldots, N,$$

we define for any $x \in (\mathcal{V}, \diamond)$

$$\lambda_{\min}(x) := \min\{\lambda_{\min}(x^{(j)}) \,|\, j = 1, 2, \ldots, N\},$$
$$\lambda_{\max}(x) := \max\{\lambda_{\max}(x^{(j)}) \,|\, j = 1, 2, \ldots, N\}.$$

• **Frobenius norm**: For any $x \in (\mathcal{V}, \diamond)$, the Frobenius norm of x will be defined as follows:

$$\|x\|_F^2 := \sum_{j=1}^{N} \left\|x^{(j)}\right\|_F^2,$$

where

$$\left\|x^{(j)}\right\|_F^2 := \sum_{i=1}^{r} \lambda_i^2(x^{(j)}).$$

• **Infinty norm**: Let $x \in (\mathcal{V}, \diamond)$. The infinity norm of x, denoted by $\|x\|_2$, will be defined as follows:

$$\|x\|_2 := \max_j \left\|x^{(j)}\right\|_2,$$

where

$$\left\|x^{(j)}\right\|_2 := \max_i\{\lambda_i(x^{(j)})\}.$$

• **Spectral decomposition**: Let $\sum_{i=1}^{r_j} \lambda_i(x^{(j)}) c_i^j$ be the spectral decomposition of the vector $x^{(j)} \in \mathcal{V}_j$, then

$$x := \Big(\sum_{i=1}^{r_1} \lambda_i(x^{(1)}) c_i^1, \sum_{i=1}^{r_2} \lambda_i(x^{(2)}) c_i^2, \ldots, \sum_{i=1}^{r_N} \lambda_i(x^{(N)}) c_i^N\Big)^T,$$

is the spectral decomposition of $x \in (\mathcal{V}, \diamond)$ where the set $\{c_1^j, c_2^j, \dots, c_k^j\}$ is the Jordan frame related to the classical EJA $(\mathcal{V}_j, \circ)$.

Let $x \in \mathcal{V}$. The vector-valued function $\psi(x)$ is defined as

$$\psi(x) := \left(\psi(x^{(1)}), \psi(x^{(2)}), \dots, \psi(x^{(N)})\right)^T,$$

in which

$$\psi(x^{(j)}) := \psi\left(\lambda_1\left(x^{(j)}\right)\right) c_1^j + \dots + \psi\left(\lambda_{r_j}\left(x^{(j)}\right)\right) c_{r_j}^j,\ j = 1, 2, \dots, N.$$

Definition 2.1. The linear operator $\mathcal{H} : \mathcal{V} \longrightarrow \mathcal{V}$ has the Cartesian $P_*(\kappa)$ property, if for some nonnegative constant κ

$$(1 + 4\kappa) \sum_{j \in I^+(x)} \langle x^{(j)}, (\mathcal{H}x)^{(j)} \rangle + \sum_{j \in I^-(x)} \langle x^{(j)}, (\mathcal{H}x)^{(j)} \rangle \geq 0,$$

or equivalently

$$\langle x, \mathcal{H}(x) \rangle \geq -4\kappa \sum_{j \in I^+(x)} \langle x^{(j)}, (\mathcal{H}x)^{(j)} \rangle$$

for all $x \in (\mathcal{V}, \diamond)$, where

$$I^+(x) := \{j \mid \langle x^{(j)}, (\mathcal{H}x)^{(j)} \rangle > 0\},\ I^-(x) := \{j \mid \langle x^{(j)}, (\mathcal{H}x)^{(j)} \rangle \leq 0\}.$$

Here, we recall some lemmas which are required in our analysis.

Lemma 2.2 ([6, Lemma 3.2])**.** *Let int$\mathcal{K}$, be the interior of $\mathcal{K}$. For $x, s \in \mathrm{int}\mathcal{K}$ there exists a unique $\bar{w} \in \mathrm{int}\mathcal{K}$ such that*

$$x = P(\bar{w})s,$$

where

$$\bar{w} := P(x^{\frac{1}{2}})\big(P(x^{\frac{1}{2}})s\big)^{\frac{-1}{2}} \left[:= P(s^{\frac{-1}{2}})\big(P(s^{\frac{1}{2}})x\big)^{\frac{1}{2}} \right].$$

The point $\bar{w}$ is called the scaling point related to x and s.

Lemma 2.3 ([16, Lemma 28])**.** *Let $u \in \mathrm{int}\mathcal{K}$. Then*

$$x \diamond s = \mu e \Leftrightarrow P(u)x \diamond P(u)^{-1}s = \mu e.$$

Lemma 2.4 ([16, Lemma 30])**.** *Let $x, s \in \mathrm{int}\mathcal{K}$. Then*

$$\left\| P(x)^{\frac{1}{2}} s - \mu e \right\|_F \leq \| x \diamond s - \mu e \|_F.$$

Lemma 2.5 ([17, Theorem 4])**.** *Let $x, s \in \mathrm{int}\mathcal{K}$. Then*

$$\lambda_{\min}\big(P(x)^{\frac{1}{2}} s\big) \geq \lambda_{\min}(x \diamond s).$$

3. The central path and feasible weighted-IPM

Let $(\mathcal{V}, \diamond)$ be an n-dimensional Cartesian EJA with rank r equipped with the standard inner product $\langle x, s\rangle := \mathbf{Tr}(x \diamond s)$ and $\mathcal{K}$ be the Cartesian symmetric cone related to $(\mathcal{V}, \diamond)$. For the $P_*(\kappa)$-linear transformation $\mathcal{A} : \mathcal{V} \longrightarrow \mathcal{V}$ and a vector $q \in \mathcal{V}$, the Cartesian $P_*(\kappa)$-SCLCP is to find a pair $(x, s) \in \mathcal{K} \times \mathcal{K}$ such that

$$s = \mathcal{A}x + q, \; x \diamond s = 0. \tag{P}$$

The basic idea of the primal-dual interior-point algorithm is to replace the so-called complementarity equation $x \diamond s = 0$ by the parameterized equation $x \diamond s = \mu e$, with $\mu > 0$. Thus, we consider the following equivalent system

$$\begin{aligned} s = \mathcal{A}x + q, \; x \in \mathcal{K}, \\ x \diamond s = \mu e, \; s \in \mathcal{K}. \end{aligned} \tag{2}$$

Throughout this paper, we assume that (P) satisfies the interior point condition (IPC), i.e., there exists $(x^0, s^0) \in \mathrm{int}\mathcal{K} \times \mathrm{int}\mathcal{K}$ such that $s^0 = \mathcal{A}x^0 + q$. The solution of (2) is denoted by $(x(\mu), s(\mu))$, and called the μ-center of the Cartesian $P_*(\kappa)$-SCLCP. The set of all μ-centers, is called the central path and it is used as a guide line to solution of the Cartesian $P_*(\kappa)$-SCLCP. As μ tends to zero, $x(\mu)$ and $s(\mu)$ converge to an ε-approximate solution of (P). For more details see [10].

The target-following approach starts from the observation that system (2) can be generalized by replacing the vector μe with an arbitrary positive vector $w^2 = w \diamond w$. Thus, we obtain the following system

$$\begin{aligned} s = \mathcal{A}x + q, \; x \in \mathcal{K}, \\ x \diamond s = w^2, \;\; s \in \mathcal{K}, \end{aligned} \tag{3}$$

where $w \in \mathrm{int}\mathcal{K}$. Following Darvay's strategy in [2] for LO, we replace the standard centering equation $x \diamond s = w^2$ by $\varphi(x \diamond s) = \varphi(w^2)$, where $\varphi(t)$ is a real valued function on $[0, +\infty)$ and differentiable on $(0, +\infty)$ such that $\varphi'(t) > 0$ for all $t > 0$. Then, system (3) can be rewritten as follows:

$$\begin{aligned} s = \mathcal{A}x + q, \; x \in \mathcal{K}, \\ \varphi(x \diamond s) = \varphi(w^2), \;\; s \in \mathcal{K}. \end{aligned} \tag{4}$$

Applying Newton's method to system (4), we proceed to find its approximate solutions. After using Newton's method and neglecting the quadratic term $\Delta x \diamond \Delta s$, we obtain the following search direction system:

$$\begin{aligned} &\mathcal{A}(\Delta x) - \Delta s = 0, \;\; x \in \mathcal{K}, \\ &x \diamond \Delta s + s \diamond \Delta x = \Big(\varphi'(x \diamond s)\Big)^{-1} \diamond \Big(\varphi(w^2) - \varphi(x \diamond s)\Big), \qquad s \in \mathcal{K}. \end{aligned} \tag{5}$$

Note that linearizing the second equation in (5) may not lead to an element in $(\mathcal{V}, \diamond)$. Thus, it is necessary to symmetrize that equation before linearizing it.

To this end, replacing the second equation in (2) with $P(u)x \diamond P(u)^{-1}s = \mu e$ and applying Newton's method, we obtain the following system

$$\begin{aligned} &\mathcal{A}(\Delta x) - \Delta s = 0, \\ &P(u)x \diamond P(u)^{-1}\Delta s + P(u)^{-1}s \diamond P(u)\Delta x \\ &= \Big(\varphi'\big(P(u)x \diamond P(u)^{-1}s\big)\Big)^{-1} \diamond \Big(\varphi(w^2) - \varphi\big(P(u)x \diamond P(u)^{-1}s\big)\Big). \end{aligned} \tag{6}$$

Let $u := \bar{w}^{\frac{-1}{2}}$, where $\bar{w}$ is the NT-scaling point of x and s defined in Lemma 2.2. Since $\mathcal{A}$ is a $P_*(\kappa)$-mapping, then due to Proposition 2.1 in [25] and Lemma 4.1 in [9] system (6) uniquely defines the search directions Δx and Δs. In analysis of IPMs, it is convenient to associate to any $(x, s) \in \mathrm{int}\mathcal{K} \times \mathrm{int}\mathcal{K}$, the scaled vector υ and the scaled search directions d_x and d_s according to

$$\upsilon := P(\bar{w})^{\frac{1}{2}} s \ \big[:= P(\bar{w})^{\frac{-1}{2}} x \big],\ d_x := P(\bar{w})^{\frac{-1}{2}} \Delta x,\ d_s := P(\bar{w})^{\frac{1}{2}} \Delta s. \tag{7}$$

Using these definitions, the linear system (6) can be rewritten in the following form

$$\begin{aligned} \bar{\mathcal{A}}(d_x) - d_s &= 0, \\ d_x + d_s &= p_\upsilon, \end{aligned} \tag{8}$$

where $\bar{\mathcal{A}} := P(\bar{w})^{\frac{1}{2}} \mathcal{A} P(\bar{w})^{\frac{1}{2}}$ and

$$p_\upsilon = \upsilon^{-1} \diamond \left({\varphi'}^{-1}(\upsilon^2) \diamond \big(\varphi(w^2) - \varphi(\upsilon^2)\big) \right). \tag{9}$$

Since $\mathcal{A}$ is a $P_*(\kappa)$-operator, it follows from Proposition 3.4 in [10], $\bar{\mathcal{A}}$ also has the Cartesian $P_*(\kappa)$-property.

4. The algorithm

In this section, let $\varphi(x) := \sqrt{x}$. Then, we present a weighted-path following interior point algorithm based on the appropriate search directions. Making the substitution $\varphi(x) := \sqrt{x}$ in (9), we get

$$p_\upsilon = 2(w - \upsilon). \tag{10}$$

In the analysis of our algorithm, we define a norm based proximity measure $\delta(x, s; w)$ as follows:

$$\delta(x, s; w) := \frac{\|p_\upsilon\|_F}{2\lambda_{\min}(w)} = \frac{\|w - \upsilon\|_F}{\lambda_{\min}(w)}. \tag{11}$$

We also define another proximity measure to measure the distance of the target point w^2 to the central path. To this end, we define $\delta_c(w)$ as follows:

$$\delta_c(w) := \frac{\lambda_{\max}(w^2)}{\lambda_{\min}(w^2)}. \tag{12}$$

In order to have an easy understanding of our analysis, let us introduce

$$q_\upsilon := d_x - d_s, \tag{13}$$

then, one may easily obtain

$$d_x = \frac{p_v + q_v}{2} \text{ and } d_s = \frac{p_v - q_v}{2},$$

which implies,

$$d_x \diamond d_s = \frac{p_v^2 - q_v^2}{4}. \tag{14}$$

The new scaled search directions d_x and d_s are obtained by solving (8) while Δx and Δs are computed via (6). If $(x, s) \neq (x(\mu), s(\mu))$, then $(\Delta x, \Delta s)$ is nonzero. Applying (7), the new iterates are given by

$$x^+ = x + \Delta x = P(\bar{w})^{\frac{1}{2}}(v + d_x), \; s^+ = s + \Delta s = P(\bar{w})^{-\frac{1}{2}}(v + d_s). \tag{15}$$

The generic form of the weighted-path following interior-point algorithm is described bellow.

Algorithm 1: The Full NT-step weighted-path following algorithm
Step 0 (Initialize): Choose an accuracy parameter $\varepsilon > 0$, a barrier update parameter θ, $0 < \theta < 1$ and an initial feasible solution $\left(x^0, s^0\right)$, such that $\left\|w^0\right\|_F = \sqrt{\mathbf{Tr}(x^0 \diamond s^0)}$ and $\delta\left(x^0, s^0; w^0\right) \leq \tau = \frac{1}{1+4\kappa}$. Set $(x, s) = \left(x^0, s^0\right)$.
Step1 (Test convergence): If $\mathbf{Tr}(x \diamond s) \leq \varepsilon$, declare convergence and stop. Otherwise, proceed to the next step.
Step2 (Computation): Compute the Newton search directions Δx and Δs by solving system (6) and compute (x^+, s^+) by using (15). Update the vector w by the factor $1 - \theta$ and go to the next step.
Step3 (Update iterate): Set $(x, s) = (x^+, s^+)$ and go to the **Step 1**.

5. Convergence analysis

In this section, we prove that Algorithm **1** obtains an ε-approximate solution of the Cartesian $P_*(\kappa)$-SCLCP in polynomial time complexity. To this end, we first need to obtain an upper and lower bound for the term $\langle d_x, d_s\rangle$ which plays a key role in our analysis.

Lemma 5.1. *Let $\delta := \delta(x, s; w)$. Then*

$$-4\kappa\lambda_{\min}(w^2)\delta^2 \leq \langle d_x, d_s\rangle \leq \lambda_{\min}(w^2)\delta^2. \tag{16}$$

Proof. The most right hand side inequality in (16) can be concluded by using (11) and (13) as follows:

$$\|p_v\|_F^2 = \|q_v\|_F^2 + 4\langle d_x, d_s\rangle \geq 4\langle d_x, d_s\rangle. \tag{17}$$

Using the first equation of (8), (17) and the fact that $\bar{\mathcal{A}}$ has the Cartesian $P_*(\kappa)$-property, the most left hand side inequality in (16) can be proved as follows:

$$\langle d_x, d_s\rangle = (1 + 4\kappa) \sum_{j\in I^+(d_x)} \langle d_x^j, d_s^j\rangle + \sum_{j\in I^-(d_x)} \langle d_x^j, d_s^j\rangle - 4\kappa \sum_{j\in I^+(d_x)} \langle d_x^j, d_s^j\rangle$$

$$\geq -4\kappa \sum_{j\in I^+(d_x)} \langle d_x^j, d_s^j\rangle \geq -\kappa \left\|p_v\right\|_F^2 = -4\kappa\lambda_{\min}(w^2)\delta^2.$$

This completes the proof. ☐

Corollary 5.2. *Let $\delta := \delta(x, s; w)$. Then*

$$\|q_v\|_F^2 \leq 4(1+4\kappa)\lambda_{\min}(w^2)\delta^2. \tag{18}$$

Lemma 5.3. *Let $\delta := \delta(x, s; w)$ and $x, s \in (\mathcal{V}, \diamond)$. Then*

$$|\lambda_j(d_x \diamond d_s)| \leq (1+4\kappa)\lambda_{\min}(w^2)\delta^2. \tag{19}$$

Proof. From $d_x \diamond d_s = \frac{1}{4}\left(p_v^2 - q_v^2\right)$, the elementary relations of norms, (16) and (18), we obtain

$$\begin{aligned}\|d_x \diamond d_s\|_2 &\leq \frac{1}{4}\max\left\{\|p_v\|_2^2,\, \|q_v\|_2^2\right\} \leq \frac{1}{4}\max\left\{\|p_v\|_F^2,\, \|q_v\|_F^2\right\}\\ &= (1+4\kappa)\lambda_{\min}(w^2)\delta^2.\end{aligned}$$

The proof is completed. ☐

5.1. Convergence analysis

In this section, we investigate the convergence analysis of Algorithm **1**. To this end, we need to prove that the generated points by the algorithm are strictly feasible and show that the algorithm is quadratic convergent. The following lemma states a necessary condition which guaranties the strictly feasibility of the iterates x^+ and s^+.

Lemma 5.4. *Let $\delta := \delta(x, s; w)$ and $\delta < \frac{1}{\sqrt{1+4\kappa}}$. Then the iterates x^+ and s^+ are strictly feasible.*

Proof. We first define $v_x(\alpha) := v + \alpha d_x$ and $v_s(\alpha) := v + \alpha d_s$ for all $\alpha \in [0,\, 1]$. Thus,

$$\begin{aligned} v_x(\alpha) \diamond v_s(\alpha) &= v^2 + \alpha v \diamond (d_x + d_s) + \alpha^2 d_x \diamond d_s\\ &= v^2 + \alpha v \diamond p_v + \frac{1}{4}\alpha^2\left(p_v^2 - q_v^2\right)\\ &= (1-\alpha)v^2 + \alpha(v^2 + v \diamond p_v) + \alpha^2\Big(\frac{p_v^2}{4} - \frac{q_v^2}{4}\Big)\\ &= (1-\alpha)v^2 + \alpha\left(w^2 - (1-\alpha)\frac{p_v^2}{4} - \alpha\frac{q_v^2}{4}\right), \end{aligned} \tag{20}$$

where the last equality is due to

$$v^2 + v \diamond p_v = w^2 - (w-v)^2 = w^2 - \frac{p_v^2}{4}.$$

Clearly, $v_x(\alpha) \diamond v_s(\alpha) \in \operatorname{int}\mathcal{K}$, if

$$\left\|(1-\alpha)\frac{{p_v}^2}{4} + \alpha\frac{{q_v}^2}{4}\right\|_2 < \lambda_{\min}(w^2).$$

Using (11) and (18), we have

$$\begin{aligned}\left\|(1-\alpha)\frac{p_v^2}{4}+\alpha\frac{q_v^2}{4}\right\|_2 &\leq (1-\alpha)\left\|\frac{p_v^2}{4}\right\|_F+\alpha\left\|\frac{q_v^2}{4}\right\|_F\\ &\leq (1-\alpha)\frac{\|p_v\|_F^2}{4}+\alpha\frac{\|q_v\|_F^2}{4}\\ &\leq (1-\alpha)\delta^2\lambda_{\min}(w^2)+\alpha\delta^2(1+4\kappa)\lambda_{\min}(w^2)\\ &\leq (1+4\kappa)\delta^2\Big[(1-\alpha)\lambda_{\min}(w^2)+\alpha\lambda_{\min}(w^2)\Big]\\ &< \lambda_{\min}(w^2),\end{aligned}$$

where the last inequality is due to $\delta < \frac{1}{\sqrt{1+4\kappa}}$. This implies

$$w^2-(1-\alpha)\frac{p_v^2}{4}-\alpha\frac{q_v^2}{4}\in \mathrm{int}\mathcal{K}.$$

Thus, for all $\alpha \in [0,1]$

$$(1-\alpha)v^2+\alpha\Big(w^2-(1-\alpha)\frac{p_v^2}{4}-\alpha\frac{q_v^2}{4}\Big)\in \mathrm{int}\mathcal{K}.$$

This implies $v_x(\alpha)\diamond v_s(\alpha)\in \mathrm{int}\mathcal{K}$ for all $\alpha\in[0,\,1]$. Using Lemma 2.15 in [7], we conclude the terms $\det(v_x(\alpha))$ and $\det(v_s(\alpha))$ cannot vanish when $\alpha\in[0,\,1]$. Hence, since $\det(v_x(0))=\det(v_s(0))=\det(v)>0$, by continuity, $\det(v_x(\alpha))$ and $\det(v_s(\alpha))$ stay positive for any such α, especially for $\alpha=1$. Hence, all the eigenvalues of $v_x(1)$ and $v_s(1)$ are positive. Therefore, $v+d_x\in\mathrm{int}\mathcal{K}$ and $v+d_s\in\mathrm{int}\mathcal{K}$. Since $P(\bar{w})^{\frac{1}{2}}$ and its inverse $P(\bar{w})^{\frac{-1}{2}}$ are automorphisms of $\mathcal{K}$, Proposition 2.2 in [4] guarantees the iterates x^+ and s^+ belong to $\mathrm{int}\mathcal{K}$. This completes the proof. □

Considering $\bar{w}^+$ as the scaling point of the new iterates x^+ and s^+, we define the v-vector at the new iterates x^+ and s^+, as follows:

$$v^+ := P(\bar{w}^+)^{\frac{1}{2}}s^+\big[:= P(\bar{w}^+)^{\frac{-1}{2}}x^+\big]. \tag{21}$$

Lemma 5.5 (Proposition 5.9.3 in [18]). *One has*

$$(v^+)^2 \sim P(v+d_x)^{\frac{1}{2}}(v+d_s). \tag{22}$$

In next lemmas, we proceed to prove the local quadratic convergence of the full NT-step.

Lemma 5.6. *After a new iteration*

$$\lambda_{\min}(v^+)\geq \lambda_{\min}(w)\sqrt{1-(1+4\kappa)\delta^2}. \tag{23}$$

Proof. Using Lemmas 5.5 and 2.5 and substituting $\alpha=1$ in (20), we have

$$\begin{aligned}|\lambda_{\min}((v^+)^2) &= \lambda_{\min}\big(P(v+d_x)^{\frac{1}{2}}(v+d_s)\big)\\ &\geq \lambda_{\min}\big((v+d_x)\diamond(v+d_s)\big)\end{aligned}$$

$$= \lambda_{\min}\big(w^2 - \frac{q_v^2}{4}\big)$$
$$\geq \lambda_{\min}^2(w) - \frac{\|q_v\|_F^2}{4} = \lambda_{\min}^2(w)\big(1 - (1+4\kappa)\delta^2\big),$$

where the last equality follows from Corollary 5.2. This completes the proof. □

The following lemma investigates the quadratic convergence of the algorithm.

Lemma 5.7. *Let* $\delta := \delta(x, s; w) < \frac{1}{\sqrt{1+4\kappa}}$. *Then*

$$\delta(x^+, s^+; w) < \frac{(1+4\kappa)\delta^2}{1 + \sqrt{1 - (1+4\kappa)\delta^2}}.$$

Proof. From (20) with $\alpha = 1$, we have

$$(v + d_x) \diamond (v + d_s) = w \diamond w - \tfrac{q_v \diamond q_v}{4}. \tag{24}$$

On the other hand, due to (18), (23), (24) and Lemma 14 in [16], we have

$$\begin{aligned}
\|w - v^+\|_F &= \left\| \frac{w \diamond w - v^+ \diamond v^+}{w + v^+} \right\|_F \\
&\leq \frac{\|w \diamond w - (v + d_x) \diamond (v + d_s)\|_F}{\lambda_{\min}(w) + \lambda_{\min}(v^+)} \\
&\leq \frac{1}{\lambda_{\min}(w) + \lambda_{\min}(w)\sqrt{1 - (1+4\kappa)\delta^2}} \left\| \frac{q_v \diamond q_v}{4} \right\|_F \\
&\leq \frac{1}{\lambda_{\min}(w) + \lambda_{\min}(w)\sqrt{1 - (1+4\kappa)\delta^2}} \frac{\|q_v\|_F^2}{4} \\
&\leq \frac{(1+4\kappa)\lambda_{\min}(w)^2\delta^2}{\lambda_{\min}(w) + \lambda_{\min}(w)\sqrt{1 - (1+4\kappa)\delta^2}}.
\end{aligned}$$

Substituting the last inequality in (11), we conclude the lemma. □

The following lemma gives an upper bound of the duality gap after a full NT-step.

Lemma 5.8. *After a full NT-step,*

$$\langle x^+, s^+ \rangle = \|w^2\|_F - \frac{\|q_v^2\|_F}{4},$$

hence, $\langle x^+, s^+ \rangle \leq \|w\|_F^2$.

Proof. From (24), we have

$$\begin{aligned}
\langle x^+, s^+ \rangle = \mathbf{Tr}(x^+ \diamond s^+) = \mathbf{Tr}\left((v + d_x) \diamond (v + d_s)\right) &= \mathbf{Tr}\Big(w \diamond w - \frac{q_v \diamond q_v}{4}\Big) \\
&= \|w^2\|_F - \frac{\|q_v^2\|_F}{4},
\end{aligned}$$

which implies the result and ends the proof. □

5.2. Updating the target parameter w

In this subsection, we discuss the influence of the Newton process on the proximity measure. We assume that vector w will be updated by the constant factor $1-\theta$. It should be noted $\delta_c(w) = \delta_c(w^0)$ for all iterates produced by the algorithm.

Lemma 5.9. *Let $\delta := \delta(x, s; w) < \frac{1}{\sqrt{1+4\kappa}}$ and r be the rank of Cartesian EJA $(\mathcal{V}, \diamond)$ and $w^+ = (1-\theta)w$, where $0 \le \theta \le 1$. Then*

$$\delta(x^+, s^+; w^+) \le \frac{1}{1-\theta}\Big(\theta\sqrt{r}\delta_c(w) + \delta(x^+, s^+; w)\Big).$$

Proof. Due to the definition of δ, we have

$$\begin{aligned}
\delta(x^+, s^+; w^+) &= \frac{1}{\lambda_{\min}(w^+)} \left\|w^+ - v^+\right\|_F \\
&\le \frac{1}{\lambda_{\min}(w^+)}\Big(\left\|w^+ - w\right\|_F + \left\|w - v^+\right\|_F \Big) \\
&= \frac{1}{(1-\theta)}\Big(\frac{1}{\lambda_{\min}(w)}\|\theta w\|_F + \delta(x^+, s^+;\ w)\Big) \\
&\le \frac{1}{1-\theta}\Big(\theta\sqrt{r}\delta_c(w) + \delta(x^+, s^+; w)\Big),
\end{aligned}$$

where the last inequality follows from $\|w\|_F \le \sqrt{r}\lambda_{\max}(w)$. This ends the proof. □

5.3. Complexity analysis

We are ready to state the main result of the paper. First, let to investigate the proposed algorithm is well-defined. The following lemma tasks this goal.

Lemma 5.10. *Let $\delta := \delta(x, s; w) \le \frac{1}{2(1+4\kappa)}$, $\theta = \frac{1}{5(1+4\kappa)\sqrt{r\delta_c(w)}}$ and $r \ge 4$. Then, after a w-update, we have*

$$\delta(x^+, s^+; w^+) \le \frac{1}{2(1+4\kappa)}.$$

Proof. Let $\theta = \frac{1}{5(1+4\kappa)\sqrt{r\delta_c(w)}}$. Clearly, $\delta_c(w) \ge 1$ and for $r \ge 4$ we have $\theta \le \frac{1}{10(1+4\kappa)}$. If $\delta \le \frac{1}{2(1+4\kappa)}$, due to Lemma 5.7, we deduce $\delta(x^+, s^+; w) \le \frac{1}{4(1+4\kappa)}$. Finally, Lemma 5.9 yields $\delta(x^+, s^+; w^+) \le \frac{1}{2(1+4\kappa)}$. The proof is completed. □

Theorem 5.11. *Suppose that the pair $(x^0,\ s^0)$ is strictly feasible and let $w^0 = \big(x^0,\ s^0\big)$. If $\theta = \frac{1}{5(1+4\kappa)\sqrt{r\delta_c(w^0)}}$, then the algorithm requires at most*

$$\left\lceil 5(1+4\kappa)\sqrt{r\delta_c(w^0)}\log\frac{\boldsymbol{Tr}\langle x^0\diamond, s^0\rangle}{\varepsilon}\right\rceil,$$

iterations.

6. Concluding remarks

In this paper, we developed the proposed weighted-path following interior-point algorithm for LO by Darvay [2] to Cartesian $P_*(\kappa)$-SCLCP. The algorithm uses the NT-direction as the search directions and takes the full-Newton steps to obtain an ε-solution of the underlying problem. We showed that the proposed algorithm is well-defined and derived the currently best known iteration bound for feasible IPMs with the small-update method, namely,

$$O\left((1+4\kappa)\sqrt{r}\log\frac{\mathbf{Tr}(x^0\diamond s^0)}{\varepsilon}\right).$$

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References

[1] M. Achache, *A new primal-dual path-following method for convex quadratic programming*, Comput. Appl. Math, **25** (2006), no. 1, 97–110.

[2] Z. Darvay, *A weighted-path-following method for linear optimization*, Stud. Univ. Babes-Bolyai Inform. **47** (2002), no. 2, 3–12.

[3] ———, *New interior point algorithms in linear programming*, Adv. Model. Optim. **5** (2003), no. 1, 51–92.

[4] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford University Press, New York, 1994.

[5] L. Faybusovich, *Euclidean Jordan algebras and interior-point algorithms*, Positivity **1** (1997), no. 4, 331–357.

[6] ———, *A Jordan-algebraic approach to potential-reduction algorithms*, Math. Z. **239** (2002), no. 1, 117–129.

[7] G. Gu, M. Zangiabadi, and C. Roos, *Full Nesterov-Todd step interior-point methods for symmetric optimization*, European J. Oper. Res. **214** (2011), no. 3, 473–484.

[8] B. Jansen, C. Roos, T. Terlaky, and J. Vial, *Long-step primal-dual target-following algorithms for linear programming*, Math. Methods Oper. Res. **44** (1996), no. 1, 11–30.

[9] M. Kojima, N. Megiddo, T. Noma, and A. Yoshise, *A Unified Approach to Interior Point Algorithms for Linear Complementarity Problems*, Lecture Notes in Computer Science, vol. 538. Springer, New York, 1991.

[10] Z. Y. Luo and N. H. Xiu, *Path-following interior-point algorithms for the Cartesian $P_*(\kappa)$-LCP over symmetric cones*, Sci. China Math. **52** (2009), no. 8, 1769–1784.

[11] H. Mansouri and M. Pirhaji, *A polynomial interior-point algorithm for linear complementarity problems*, J. Optim. Theory Appl. **157** (2013), no. 2, 451–461.

[12] H. Mansouri, M. Zangiabadi, and M. Pirhaji, *A full-Newton step $O(n)$ infeasible interior-point algorithm for linear complementarity problems*, Nonlinear Anal. Real World Appl. **12** (2011), no. 1, 545–561.

[13] F. A. Potra, *An infeasible interior point method for linear complementarity problems over symmetric cones*, Proceedings of the 7th International Conference of Numerical

Analysis and Applied Mathematics, Rethymno, Crete, Greece, 18–22 September 2009, pp. 1403–1406. Am. Inst. of Phys., New York, 2009.

[14] B. K. Rangarajan, *Polynomial convergence of infeasible-interior-point methods over symmetric cones*, SIAM J. Optim. **16** (2006), no. 4, 1211–1229.

[15] C. Roos, *A full-Newton step $O(n)$ infeasible interior-point algorithm for linear optimization*, SIAM J. Optim. **16** (2006), no. 4, 1110–1136.

[16] S. H. Schmieta and F. Alizadeh, *Extension of primal-dual interior-point algorithms to symmetric cones*, Math. Program. **96** (2003), no. 3, 409–438.

[17] J. F. Sturm, *Similarity and other spectral relations for symmetric cones*, Linear Algebra Appl. **312** (2000), no. 1-3, 135–154.

[18] M. V. C. Vieira, *Jordan Algebraic Approach to Symetric Optimization*, PhD thesis, Delft University of Thecnology, 2007.

[19] G. Q. Wang and Y. Q. Bai, *A primal-dual interior-point algorithm for second-order cone optimization with full Nesterov-Todd step*, Appl. Math. Comput. **215** (2009), no. 3, 1047–1061.

[20] ______, *A new primal-dual path-following interior-point algorithm for semidefinite optimization*, J. Math. Anal. Appl. **353** (2009), no. 1, 339–349.

[21] ______, *A new full Nesterov-Todd step primal-dual path-following interior-point algorithm for symmetric optimization*, J. Optim. Theory Appl. **154** (2012), no. 3, 966–985.

[22] ______, *A class of polynomial interior-point algorithms for the Cartesian P-matrix linear complementarity problem over symmetric cones*, J. Optim. Theory Appl. **152** (2012), no. 3, 739–772.

[23] G. Q. Wang and G. Lesaja, *Full Nesterov-Todd step feasible interior-point method for the Cartesian $P_*(\kappa)$-SCLCP*, Optim. Methods Softw. **28** (2013), no. 3, 600–618.

[24] A. Yoshise, *Interior point trajectories and a homogeneous model for nonlinear complementarity problems over symmetric cones*, SIAM J. Optim. **17** (2006), no. 4, 1129–1153.

[25] Y. B. Zhao and J. Han, *Two interior-point methods for nonlinear $P_*(\kappa)$-complementarity problems*, J. Optim. Theory Appl. **102** (1999), no. 3, 659–679.

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