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COFINITE PROPER CLASSIFYING SPACES FOR LATTICES IN SEMISIMPLE LIE GROUPS OF $\mathbb{R}$-RANK 1

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Abstract. The Borel–Serre partial compactification gives cofinite models for the proper classifying space for arithmetic lattices. Non-arithmetic lattices arise only in semisimple Lie groups of $\mathbb{R}$-rank one. The author generalizes the Borel–Serre partial compactification to construct cofinite models for the proper classifying space for lattices in semisimple Lie groups of $\mathbb{R}$-rank one by using the reduction theory of Garland and Raghunathan.

1. Introduction

Let Γ be a topological group. The classifying space for Γ, denoted by $B\Gamma$, is a CW-complex such that $\pi_1(B\Gamma) \cong \Gamma$ and $\pi_i(B\Gamma) = \{e\}$ for all $i \geq 2$. The universal cover $E\Gamma$ of $B\Gamma$ and the universal bundle $E\Gamma \to B\Gamma$ classify Γ-principal bundles (cf. [18, §6, §7]).

Classifying spaces appear in topological invariants such as higher signatures in algebraic K-theory (cf. [25]). To compute these invariants, explicit models for the universal cover $E\Gamma$ are necessary. In [34,35] Milnor explicitly constructed a general model for $E\Gamma$. Although Milnor's model is always infinite dimensional, finite dimensional model does exist for certain cases (cf. [1, Corollary 4.14], [42, Theorem 0.1], [43, Theorem B], [46, §1]).

The *proper classifying spaces*, denoted by $\underline{E}\Gamma$ (cf. Definition 3), are also used in formulations of algebraic K-theoretic conjectures such as the Baum–Connes conjecture (cf. [5,17,44]) and the integral Novikov conjecture (cf. [3,4,20,22,29]). Conjectures such as strong Novikov conjecture (cf. [17, Conjecture 4.1]) and the generalized integral Novikov conjecture (cf. [20, Equation (1.3)]) are known to be true for groups admitting a *cofinite* (cf. Definition 1) model of $\underline{E}\Gamma$ with finite asymtotic dimensions (cf. [4], [48]).

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There are explicit models for the proper classifying spaces for mapping class groups (cf. [21, Theorem 1.3], [23, Theorem 2]), outer automorhpism groups of free groups (cf. [26, Theorem 8.1], [47, Theorem 5.1]), p-adic algebraic groups (cf. [27, Theorem 4.13]), discrete groups acting on trees (cf. [27, Theorem 4.7]), discrete groups acting isometrically on $CAT(0)$-space (cf. [28, Theorem 1.1(1)]), and δ-hyperbolic groups (cf. [33, Theorem 1]).

For *arithmetic groups* (cf. Definition 6), Borel and Serre constructed a partial compactification of symmetric spaces in [9], called the *Borel–Serre partial compactification* (cf. §2.4). For torsion-free arithmetic group, it is a cofinite model for the classifying space. Moreover, it is known by Borel and Prasad (cf. [2, Remark 5.8]) that this compactification is also a cofinite model for the proper classifying spaces for arbitrary arithmetic groups (cf. [20, Theorem 3.2]).

Arithmetic groups in semisimple algebraic groups are lattices (cf. [6, Lemma 9.2]). There are non-arithmetic lattices in semisimple Lie groups of $\mathbb{R}$-rank 1 (cf. [11, 12, 14, 30, 36, 37, 41]). In this paper, we construct a cofinite classifying space for general lattices in semisimple Lie groups of $\mathbb{R}$-rank 1.

Theorem. *Let G be a non-compact connected semisimple Lie group of $\mathbb{R}$-rank 1 and Γ a non-uniform lattice in G. There exists a cofinite model for the proper classifying space for Γ.*

We provide preliminary backgrounds in §2, and a proof in §3. The proof uses the ideas of the Borel–Serre partial compactifications (cf. §2.4) and the reduction theory of Garland and Raghunathan (cf. §2.5). To the best of author's knowledge, good references are: [27] for classifying spaces; [24], [38], [40], [45] for lattices in Lie groups; [8] for the Borel–Serre partial compactification.

2. Preliminaries

In §2.1-2.2, we overview the concepts of the classifying spaces and lattices in Lie groups. In §2.3, we describe the horospherical decompositions of symmetric spaces and the natural actions of lattices on symmetric spaces. In §2.4, we review the construction of the Borel–Serre partial compactification by using horospherical decompositions and the reduction theory of arithmetic lattices. In §2.5, we explain the reduction theory of Garland and Raghunathan on lattices in semisimple Lie groups of $\mathbb{R}$-rank 1.

2.1. Classifying spaces

Definition 1 ([27, Definition 1.1]). Let Γ be a topological group. A Γ-*CW-complex* X is a Γ-space with a Γ-fibration

$$X_0 \subset \cdots \subset X_n \subset \cdots \subset X = \bigcup_{n=0}^{\infty} X_n$$

and a Γ-pushout

$$\begin{array}{ccc} \coprod_{\alpha\in I_n}\left(\Gamma/H_\alpha \times S^{n-1}_\alpha\right) & \xrightarrow{q^n_\alpha} & X_{n-1} \\ \downarrow & & \downarrow \\ \coprod_{\alpha\in I_n}\left(\Gamma/H_\alpha \times D^{n}_\alpha\right) & \xrightarrow{Q^n_\alpha} & X_{n} \end{array}$$

for each $n \geq 1$. Each map Q^n_α is a natural extension of the map q^n_α, and all maps q^n_α and Q^n_α are Γ-equivariant homeomorphisms. Each subgroup $H_\alpha \subset \Gamma$ is called an *isotropy group* of n-cell D^n_α. The image of each orbit $\Gamma/H_\alpha \times D^n_\alpha$ in X_n is called a *Γ-equivariant n-cell.* A Γ-CW-complex is called *cofinite* if it has only finitely many Γ-equivariant cells, that is, the quotient space is a finite CW-complex.

Recall that a CW-complex X is a model for the classifying space $B\Gamma$ if the fundamental group $\pi(X)$ is isomorphic to Γ and all higher homotopy groups are trivial. Thus the universal cover $E\Gamma$ of the classifying space $B\Gamma$ is a contractible Γ-CW-complex on which Γ acts freely. In other words, a Γ-CW-complex is a model for $E\Gamma$ if and only if the space is contractible and all isotopy group are trivial. The definition of the classifying space generalizes to various types of families of isotropy groups, such as the families of compact subgroups $\mathcal{COM}$, and virtually cyclic subgroups $\mathcal{VCYC}$.

Definition 2 ([27, Definition 1.8, Theorem 1.9])**.** A Γ-CW-complex X is called *the classifying space for* $\mathcal{F}$, and denoted by $E_{\mathcal{F}}\Gamma$, if (1) all isotropy groups belong to $\mathcal{F}$, and (2) for any subgroup $H \in \mathcal{F}$, the fixed point set X^H is non-empty and contractible.

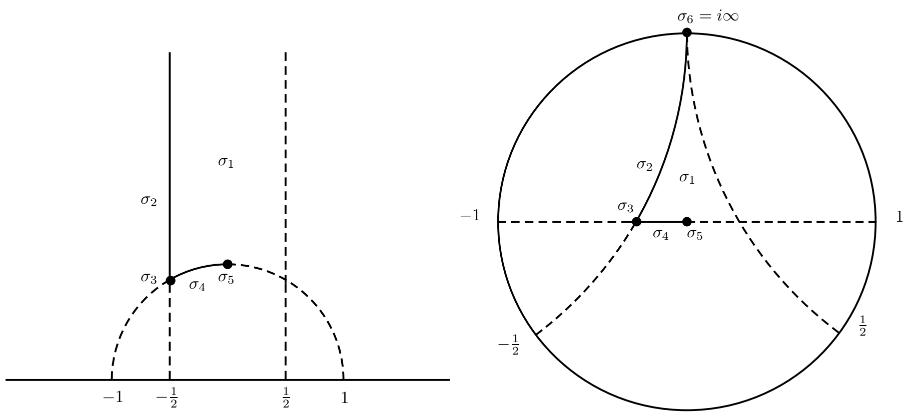
The classifying space for $E_{\mathcal{COM}}\Gamma$ is called the *proper classifying space* for Γ, and denoted by $\underline{E}\Gamma$.

Definition 3. Let Γ be a discrete group. A Γ-CW-complex X is a model for $\underline{E}\Gamma$ if (1) all isotopy groups are finite, and (2) for every finite subgroup H of Γ the fixed point set X^H is non-empty and contractible.

Let G be a Lie group with finitely many path components and K be a maximal compact subgroup of G. Abel showed in [1] that for any discrete subgroup Γ in G, the group G and the space G/K are models for $\underline{E}\Gamma$ under the canonical action of Γ (cf. [27, Theorem 4.4]).

Example 4. The group $\Gamma = PSL_2(\mathbb{Z})$ acts on the upper half-plane $\mathbf{H} = \{x + yi \in \mathbb{C} \mid y > 0\}$ by the Möbius transformation: $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \cdot z = \frac{az+b}{cz+d}$. The space $\mathbf{H}$ is a finite-dimensional (but not cofinite) model for the classifying space $\underline{E}\Gamma$. Figure 1(A) shows a fundamental domain of Γ in $\mathbf{H}$. The union of all Γ-translates of σ_i ($1 \leq i \leq 5$) is the upper half-plane $\mathbf{H}$, which is a Γ-CW complex. The isotropy groups of 0-dimensional simplices σ_3, σ_5 are isomorphic to $\mathbb{Z}_3$, $\mathbb{Z}_2$ respectively. The isotropy groups of all other simplices are trivial.

Let $\overline{\mathbf{H}}^{\mathbb{Q}}$ be the union of the upper half-plane $\mathbf{H}$ and the set of all rational boundary points $\mathbb{Q} \cup \{i\infty\}$. The action of $PSL_2(\mathbb{R})$ extends naturally to $\overline{\mathbf{H}}^{\mathbb{Q}}$. With additional simplex $\sigma_6 = i\infty$, we get a fundamental domain for Γ in $\overline{\mathbf{H}}^{\mathbb{Q}}$ (cf. Figure 1(B)). Thus $\overline{\mathbf{H}}^{\mathbb{Q}}$ is a cofinite Γ-CW-complex. However, it is not a model for $\underline{E}\Gamma$ because the isotropy group of the point $i\infty$ is an infinite group: $\left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \cdot i\infty = i\infty$ for all $n \in \mathbb{Z}$.



(A) A fundamental domain in $\mathbf{H}$ (B) A fundamental domain in $\overline{\mathbf{H}}^{\mathbb{Q}}$

FIGURE 1. Fundamental domains for $SL(2, \mathbb{Z})$

2.2. Lattices in Lie groups

A *lattice* Γ is a discrete subgroup of a (connected) Lie group G such that the volume of the quotient $\Gamma \backslash G$ is finite. A lattice is called *uniform* if the quotient $\Gamma \backslash G$ is compact. Arithmetic lattices are typical examples of lattices in Lie groups. To define a notion of arithmeticity, we first define *algebraic groups.*

Definition 5. Let P_α, $\alpha \in I$, be a collection of polynomials defined on $GL_N(\mathbb{C})$. The group $\mathbf{G} = \{g \in GL_N(\mathbb{C}) \mid P_\alpha(g) = 0 \text{ for all } \alpha \in I\}$ is called a (*linear*) *algebraic group.* For a field k, the group $\mathbf{G}$ is said to be *defined over* k if all polynomials P_α have coefficients only in k. An algebraic subgroup $\mathbf{T}$ of $\mathbf{G}$ is called a *torus* if there is an isomorphism $\mathbf{T} \cong GL_1(\mathbb{C})^m$ for some integer $m \geq 1$. If this isomorphism is defined over k, then $\mathbf{T}$ is said to be k*-split.* The k*-rank* of $\mathbf{G}$ is the common dimension of the maximal k-split tori.

Definition 6. Two subgroups Γ_1, Γ_2 of a group G are said to be *commensurable* if their intersection $\Gamma_1 \cap \Gamma_2$ has finite indices in Γ_1 and Γ_2. A discrete subgroup Γ of a semisimple Lie group G is called *arithmetic* if there exists a linear algebraic group $\mathbf{H}$ defined over $\mathbb{Q}$ and a surjective morphism $\varphi : \mathbf{H}(\mathbb{R}) \to G$ with compact kernel such that the image $\varphi(\mathbf{H}(\mathbb{Z}))$ is commensurable with Γ.

A lattice Γ in a Lie group G is said to be *reducible* if there is an isomorphism $\varphi : G \overset{\cong}{\to} G_1 \times G_2$ and a subgroup $\Gamma' \subset \Gamma$ of finite index such that $\varphi(\Gamma') = \Gamma_1 \times \Gamma_2$. A lattice which is not reducible is called *irreducible*. One of the most remarkable results on lattices in semisimple Lie groups is Margulis's superrigidity (cf. [32]).

Theorem 7 (Margulis [31]). *Every irreducible lattice in semisimple Lie group of $\mathbb{R}$-rank greater than one is arithmetic.*

From this theorem, we conclude that there is no arithmetic lattice in semisimple Lie group except for the case of $\mathbb{R}$-rank 1: $SO(1,n)$, $SU(1,n)$, $Sp(1,n)$, and the Cayley surface F^4_{-20}. Gromov–Schoen [15] and Corlette [10] showed that superrigidity also holds for lattices in $Sp(1,n)$ and F_4^{-20}. However, there are examples of non-arithmetic lattices in $SO(1,n)$, $SU(1,n)$ (cf. [11], [14], [39]).

2.3. Horospherical decompositions of symmetric spaces

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{R}$. Let $G = \mathbf{G}(\mathbb{R})$ be the real locus of $\mathbf{G}$ and K a maximal compact subgroup of G. Since the algebraic group $\mathbf{G}$ is semisimple, the Lie group G is semisimple. Thus we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of the Lie algebra $\mathfrak{g}$ of G with respect to the Lie algebra $\mathfrak{k}$ of K. For an abelian subalgebra $\mathfrak{a}$ of $\mathfrak{p}$, a linear map $\alpha : \mathfrak{a} \to \mathbb{R}$ is called a *restricted root* with respect to $\mathfrak{a}$ if the subalgebra $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid \mathrm{Ad}_H(X) = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}$ is non-zero. Let $\Phi(\mathfrak{g},\mathfrak{a})$ be the set of all restricted roots with respect to $\mathfrak{a}$. Let $\Phi^+(\mathfrak{g},\mathfrak{a})$ ($\Delta(\mathfrak{g},\mathfrak{a})$, respectively) be a subset of $\Phi(\mathfrak{g},\mathfrak{a})$ consists of all positive (simple, respectively) restricted roots. When there is no ambiguity, we simply write them as Φ, Φ^+, and Δ.

For each subset $I \subset \Delta$, let Φ^I be the set of restricted roots generated by I, and $\mathfrak{n}_I = \sum_{\alpha\in\Phi^+-\Phi^I} \mathfrak{g}_\alpha$, $\mathfrak{a}_I = \bigcap_{\alpha\in I} \ker\alpha$. Let $\mathfrak{z}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$, and $\mathfrak{a}^I$ is the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{a}$. Then for $\mathfrak{m}_I = \mathfrak{k} \cap \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{a}^I \oplus \sum_{\alpha\in\Phi^I} \mathfrak{g}_\alpha$, the Lie subalgebra $\mathfrak{p}_I = \mathfrak{n}_I + \mathfrak{a}_I + \mathfrak{m}_I$ is called *the standard parabolic subalgebra* of $\mathfrak{g}$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called *parabolic* if $\mathfrak{p}$ is Ad_G-conjugate to a standard parabolic subgroup. Thus any parabolic subalgebra $\mathfrak{p}$ admits a decomposition $\mathfrak{p} = \mathfrak{n} + \mathfrak{a} + \mathfrak{m}$.

A subgroup $P \subset G$ is called a *parabolic subgroup* if it is the normalizer of a parabolic subalgebra under the adjoint representation of G. Every maximal parabolic subgroup is conjugate to a unique maximal standard parabolic subgroup (cf. [7, §I.1]).

Definition 8. Let P be a parabolic subgroup of G with respect to the parabolic subalgebra $\mathfrak{p} = \mathfrak{n}+\mathfrak{a}+\mathfrak{m}$. Let N, A, M be the Lie groups of $\mathfrak{n}$, $\mathfrak{a}$, $\mathfrak{m}$, respectively. The decomposition

$$\begin{aligned} N \times A \times M &\to P \\ (n,a,m) &\mapsto nam \end{aligned}$$

is called *the real Langlands decomposition* of P. The self-action of P_I is given as follows: for $n, n' \in N_I$, $a, a' \in A_I$, and $m, m' \in M_I$,

$$nam \cdot (n', a', m') = (n({}^{am}n'), aa', mm'), \tag{2.1}$$

where ${}^{am}n' = (am)n'(am)^{-1}$.

For any parabolic subgroup $P \subset G$, the Iwasawa decomposition (cf. [24, Theorem 6.46]) $G \cong N_P \times A_P \times K$ induces $G = P \cdot K$ (cf. [16, Theorem IX.1.3]). Since $P \cap K = M_P \cap K$, X decomposes into

$$\begin{aligned} N_P \times A_P \times X_P &\cong X \\ (n, a, mK) &\mapsto nam \cdot K, \end{aligned} \tag{2.2}$$

where $X_P = M_P/(M_P \cap K)$. This is called the *horospherical decomposition* of X with respect to P. The component X_P is called *the boundary symmetric space*. Let $g = p \cdot k$ be an element in G such that $p \in P$ and $k \in K$. In terms of the horospherical decomposition with respect to P, the element k acts as

$$k \cdot z = ({}^k n', {}^k a', kz') \in N_{{}^kP} \times A_{{}^kP} \times X_{{}^kP}. \tag{2.3}$$

(2.2) and (2.3) define the action of G on X. If $k \notin M_P$, then the coordinate system of X changes to the horospherical decomposition of X with respect to ${}^kP$. Thus we have:

Remark 9. If the $\mathbb{R}$-rank of G is strictly greater than 1, then for two parabolic subgroups $P \subsetneq Q$, there exists a parabolic subgroup $P' \subset M_Q$ satisfying

$$N_P = N_Q N_{P'},\ A_P = A_Q A_{P'},\ M_{P'} = M_P. \tag{2.4}$$

Thus $X_Q = N_{P'} \times A_{P'} \times X_P$. The horospherical decomposition (2.2) further decomposes into

$$X \cong N_Q \times A_Q \times (N_{P'} \times A_{P'} \times X_P), \tag{2.5}$$

which is called the *relative horospherical decomposition* of X (cf. [7, §I.1.11]). For example, let $\mathbf{G} = SL_2(\mathbb{C}) \times SL_2(\mathbb{C})$, $K = SO(2) \times SO(2)$, $P = P_0 \times P_0$, and $Q = P_0 \times SL(2, \mathbb{R})$. Then $N_Q = N_{P_0} \times \{I\}$, $A_Q = A_{P_0} \times \{I\}$, $M_Q = \{\pm I\} \times SL_2(\mathbb{R})$, $N_P = N_{P_0} \times N_{P_0}$, $A_P = A_{P_0} \times A_{P_0}$, $M_P = \{\pm I\} \times \{\pm I\}$, and $P' = \{\pm I\} \times P_0$ satisfy (2.4) and (2.5).

Let $\mathbf{G}$ be a semisimple linear algebraic group defined over $\mathbb{Q}$. An algebraic subgroup $\mathbf{P} \subset \mathbf{G}$ is called a *rational parabolic subgroup* if it is defined over $\mathbb{Q}$ and the real locus $P = \mathbf{P}(\mathbb{R})$ is a parabolic subgroup of the semisimple Lie group $G = \mathbf{G}(\mathbb{R})$. Let $\mathbf{N_P}$ be the unipotent radical of $\mathbf{P}$ and $\mathbf{L_P} = \mathbf{N_P} \backslash \mathbf{P}$ be the Levi quotient. In order to write $\mathbf{P}$ as a product of $\mathbf{N_P}$ and $\mathbf{L_P}$, we first lift $\mathbf{L_P}$ into $\mathbf{G}$ whose image is invariant under the extended Cartan involution on $\mathbf{G}$ (cf. [9, §1.9]). For simplicity, let us denote $\mathbf{L_P}$ be the image of this lift. The subgroups $\mathbf{N_P}$ and $\mathbf{L_P}$ are also defined over $\mathbb{Q}$ (cf. [7, Proposition §III.1.11]). Let $\mathbf{S_P}$ be the maximal $\mathbb{Q}$-split center of $\mathbf{L_P}$. For the set of all $\mathbb{Q}$-characters $X_{\mathbb{Q}}(\mathbf{L_P})$ on $\mathbf{L_P}$, define $\mathbf{M_P} = \bigcap_{\xi \in X_{\mathbb{Q}}(\mathbf{L_P})} \ker \xi^2$. For $N_{\mathbf{P}} = \mathbf{N_P}(\mathbb{R})$,

$A_{\mathbf{P}} = \mathbf{S}_{\mathbf{P}}(\mathbb{R})^0$, $M_{\mathbf{P}} = \mathbf{M}_{\mathbf{P}}(\mathbb{R})$, decomposition $P \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times M_{\mathbf{P}}$ is called the *rational Langland decomposition* of P. It induces the *rational horospherical decomposition* $X \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}$.

Example 10. Let $\mathbf{G} = SL_2(\mathbb{C})$ and $K = SO(2)$. Every parabolic subgroup of $G = SL_2(\mathbb{R})$ is minimal, hence conjugate to the standard parabolic subgroup $P_0 = \left\{\left(\begin{smallmatrix} a & 0 \\ c & a^{-1}\end{smallmatrix}\right) \in G \mid a \neq 0\right\}$. The symmetric space $X = G/K$ is identified with the upper half-plane $\mathbf{H}$. For $N_{P_0} = \{\left(\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix}\right) \in P \mid c \in \mathbb{R}\}$, $A_{P_0} = \left\{\left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right) \in P \mid a > 0\right\}$, $M_{P_0} = \{\pm I\}$, the horospherical decomposition is $N_{P_0} \times A_{P_0} \cong \mathbf{H}$: $\left(\left(\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix}\right), \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)\right) \to \left(\begin{smallmatrix} 1 & 0 \\ c & 1\end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1}\end{smallmatrix}\right)$ (The one-pointed set X_{P_0} is omitted).

Remark 11. Let $\mathbf{P} \subsetneq \mathbf{Q}$ be two rational parabolic subgroups of $\mathbf{G}$. If the $\mathbb{Q}$-rank of $\mathbf{G}$ is greater than 1, then there exists a parabolic subgroup $\mathbf{P}'$ in $\mathbf{M}_{\mathbb{Q}}$ satisfying $N_{\mathbf{P}} = N_{\mathbf{Q}} N_{\mathbf{P}'}$, $A_{\mathbf{P}} = A_{\mathbf{Q}} A_{\mathbf{P}'}$, $M_{\mathbf{P}} = M_{\mathbf{P}}$ (cf. [7, §III.1.16]). Thus

$$X_{\mathbf{Q}} \cong N_{\mathbf{P}'} \times A_{\mathbf{P}'} \times X_{\mathbf{P}}. \tag{2.6}$$

For example, $\mathbf{Q} = \mathbf{P}_0 \times \mathbf{P}_0$, $\mathbf{P}' = \{I\} \times \mathbf{P}_0$ satisfies (2.6).

2.4. The Borel–Serre partial compactification

Let $\mathbf{G}$ be a semisimple algebraic group defined over $\mathbb{Q}$, K a maximal compact subgroup of $G = \mathbf{G}(\mathbb{R})$, and $X = G/K$ the corresponding symmetric space. Let $\mathbf{P}$ be a rational parabolic subgroup of $\mathbf{G}$ and $X \cong N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ be the rational horospherical decomposition of X with respect to $\mathbf{P}$. Let $\Phi(P, A_{\mathbf{P}})$ be the set of all restricted root of the Lie algebra of $P = \mathbf{P}(\mathbb{R})$ with respect to the Lie algebra of $A_{\mathbf{P}}$. For any element $a \in A_{\mathbf{P}}$ and a restricted root $\alpha \in \Delta(P, A_{\mathbf{P}})$, we define $a^\alpha = \exp \alpha(\log a)$. Let $\Delta(P, A_{\mathbf{P}})$ be the subset of $\Phi(P, A_{\mathbf{P}})$ consists of all simple restricted roots.

Let $A_{\mathbf{P},t} = \{a \in A_{\mathbf{P}} \mid a^\alpha \geq t \text{ for all } \alpha \in \Phi^+\}$ for some $t > 0$. Let U and V be open subsets of $N_{\mathbf{P}}$ and $M_{\mathbf{P}}$, respectively. The subset $\mathfrak{S}_{\mathbf{P},U,t,V} = U \times A_{\mathbf{P},t} \times V \subset N_{\mathbf{P}} \times A_{\mathbf{P}} \times X_{\mathbf{P}}$ is called a *Siegel set*. For any arithmetic subgroup Γ of G, Borel and Harish-Chandra showed in [6] that there are only finitely many rational parabolic subgroups $\mathbf{P}_1, \ldots, \mathbf{P}_n$ with distinct Γ-conjugacy classes such that

$$\Omega = \bigcup_{i=1}^{n} \mathfrak{S}_{\mathbf{P}_i, U_i, t_i, V_i} \tag{2.7}$$

is a fundamental set for Γ.

For rational parabolic subgroup $\mathbf{P} \subset \mathbf{G}$, define $e(\mathbf{P}) = N_{\mathbf{P}} \times X_{\mathbf{P}}$. The union

$$\overline{X}^{BS} = X \cup \bigcup_{\mathbf{P}\text{: rational}} e(\mathbf{P})$$

with the following convergence class of sequences $\mathcal{C}$ (cf. [7, §I.8.9]) is called the *Borel–Serre partial compactification* of X.

(1) If $y_j \to y_\infty$ in X, then $y_j \xrightarrow{\mathcal{C}} y_\infty$.

(2) If $y_j \to y_\infty$ in $e(\mathbf{P})$, then $y_j \xrightarrow{\mathcal{C}} y_\infty$.
(3) If $y_j = (n_j, a_j, m_j) \in N_\mathbf{P} \times A_\mathbf{P} \times X_\mathbf{P} \cong X$ is a unbounded sequence such that (a) $n_j \to n_\infty$, (b) $m_j \to m_\infty$, and (c) $a_j^\alpha \to \infty$ for every $\alpha \in \Phi^+(P, A_P)$, then $y_j \xrightarrow{\mathcal{C}} n_\infty$.
(4) Let $\mathbf{P} \subsetneq \mathbf{Q}$ be two rational parabolic subgroups in $\mathbf{G}$. From Remark 11, $e(\mathbf{Q}) = N_\mathbf{Q} \times X_\mathbf{Q} \cong N_\mathbf{P} \times A_{\mathbf{P}'} \times X_\mathbf{P}$. If a sequence $y_j = (n_j, a_j, w_j) \in N_\mathbf{P} \times A_{\mathbf{P}'} \times X_\mathbf{P}$ satisfies (a) $n_j \to n_\infty$ in $N_\mathbf{P}$, (b) $w_j \to w_\infty$ in $X_\mathbf{P}$, and (c) $a_j^\alpha \to \infty$ for every positive restricted root $\alpha \in \Phi^+(P', A_{\mathbf{P}'})$, then $y_j \xrightarrow{\mathcal{C}} (n_\infty, w_\infty) \in e(\mathbf{P})$.

The conditions (1) and (2) give the canonical topologies on X and $e(\mathbf{P})$. If Γ is torsion-free, then the space $\overline{X}^{BS}$ is a cofinite model for the classifying space (cf. [9, §11.1]). Since every arithmetic group contains a torsion-free subgroup of finite index, the space $\overline{X}^{BS}$ is a proper Γ-space for any arithmetic group Γ in G (cf. [20, Theorem 3.2]).

Let us visualize the condition (3) in $\overline{\mathbf{H}}^{BS}$. Note that each parabolic subgroup of $SL_2(\mathbb{R})$ is the stabilizer of a point in the geodesic boundary $\mathbf{H}(\infty) = \mathbb{R} \cup \{i\infty\}$. Thus the collection of rational parabolic subgroups in SL_2 is in one-to-one correspondence with the set $\mathbb{Q} \cup \{i\infty\}$. For example, real locus P_∞ of $\mathbf{P}_\infty = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \in SL_2(\mathbb{C}) \mid a \neq 0 \right\}$ fixes $i\infty$, and $e(\mathbf{P}_\infty) \cong N_\infty = \{ \left(\begin{smallmatrix} 1 & b \\ 0 & 1 \end{smallmatrix} \right) \in P \}$. For sequences $a_j \to \infty$ and $b_j \to b_\infty \in \mathbb{R}$, the sequence $y_j = b_j + a_j i \in \mathbf{H}$ converges to $e(\mathbf{P}_\infty)$ (cf. Figure 2(A)):

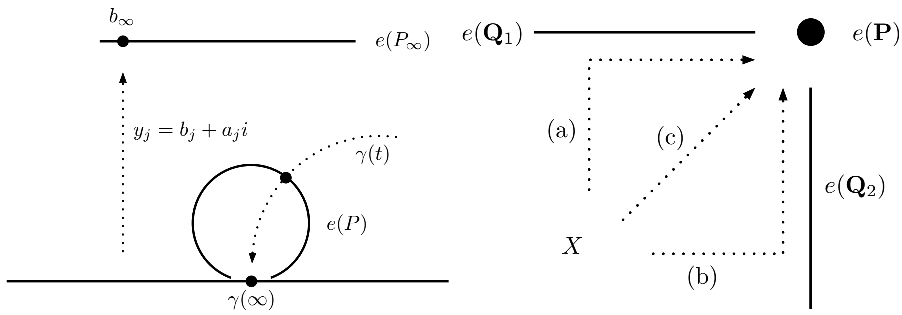
$$b_j + a_j i \overset{\cong}{\longmapsto} \left(\begin{pmatrix} 1 & b_j \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \sqrt{a_j} & 0 \\ 0 & 1/\sqrt{a_j} \end{pmatrix} \right) \longrightarrow \begin{pmatrix} 1 & b_\infty \\ 0 & 1 \end{pmatrix} \in e(\mathbf{P}_\infty).$$

In general, if a unbounded sequence $y_j \in \mathbf{H}$ converges to a geodesic line $\gamma(t)$ in $\mathbf{H}$, and P stabilizes the point at infinity $\gamma(\infty) \in \mathbf{H}(\infty)$, then y_j converges to a point in $e(P)$.

The condition (4) is equivalent to saying that $e(\mathbf{P})$ is a subspace of $\overline{e(\mathbf{Q})}$. For example, let $X = \mathbf{H} \times \mathbf{H}$, $\mathbf{Q}_1 = \mathbf{P}_\infty \times SL_2$, $\mathbf{Q}_2 = SL_2 \times \mathbf{P}_\infty$, and $\mathbf{P} = \mathbf{P}_\infty \times \mathbf{P}_\infty$. Let $a_j > 0$, $a'_k > 0$, b_j, b'_k be sequences of real numbers such that $a_j, a'_k \to \infty$ and $b_j \to b_\infty$, $b'_k \to b'_\infty$ as $j, k \to \infty$. The double sequence $y_{j,k} = (b_j + a_j i, b'_k + a'_k i) \in \mathbf{H} \times \mathbf{H}$ is unbounded and converges to $\left(\left(\begin{smallmatrix} 1 & b_\infty \\ 0 & 1 \end{smallmatrix} \right), \left(\begin{smallmatrix} 1 & b'_\infty \\ 0 & 1 \end{smallmatrix} \right) \right) \in e(\mathbf{P}) \in \overline{\mathbf{H} \times \mathbf{H}}^{BS}$. The path (c) in Figure 2(B) illustrates this convergence. There are other convergent paths too: (a) $y_{j,k} \to y_{\infty,k} \to y_{\infty,\infty}$ and (b) $y_{j,k} \to y_{j,\infty} \to y_{\infty,\infty}$.

2.5. Reduction theory of Garland and Raghunathan

Let $\mathbf{G}$ be a semisimple algebraic group of the $\mathbb{R}$-rank 1, and $G = \mathbf{G}(\mathbb{R})$ its real locus. Let K be a maximal compact subgroup of G, and $X = G/K$ the corresponding symmetric space. Since the $\mathbb{R}$-rank of G is 1, the boundary



(A) Convergence sequences in $\overline{\mathbf{H}}^{BS}$

(B) Convergence sequences in $\overline{\mathbf{H} \times \mathbf{H}}^{BS}$

FIGURE 2. The topology of the Borel–Serre partial compactifications

symmetric space X_P is a one-point space. Thus $X \cong N_P \times A_P$ is the real horospherical decomposition of X.

Two geodesics γ_1, γ_2 in X are said to be *equivalent* if

$$\limsup_{t \to \infty} d(\gamma_1(t), \gamma_2(t)) < \infty.$$

The set $X(\infty) = \{\text{all geodesics in } X\}/\sim$ is called the *geodesic boundary* of X. The subalgebra $\mathfrak{p}$ in the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is canonically identified with the tangent space $T_{x_0}X$ at the point $x_0 = e \cdot K$. The Killing form on $\mathfrak{g}$ induces a Riemannian metric d on X and thus a norm $|| \cdot ||$ on $\mathfrak{p}$. Then the subset $\mathfrak{p}^1 = \{Y \in \mathfrak{p} \mid ||Y|| = 1\}$ is identified with the geodesic boundary $X(\infty)$ (cf. [7, Proposition I.2.3]).

For each vector $Y \in \mathfrak{p}^1$, algebra $\mathfrak{a}_Y = \mathbb{R}Y$ is a maximal because the $\mathbb{R}$-rank of G is 1. Thus we obtain a unique parabolic subgroup P_Y. Conversely, let P be a parabolic subgroup in G, and $\mathfrak{p}$ its Lie algebra. Again, since the $\mathbb{R}$-rank of G is 1, there is a unique vector $Y \in \mathfrak{p}^1$ which generates the maximal abelian subalgebra of $\mathfrak{p}$.

In summary, there is a one-to-one correspondence between the set $\mathcal{P}(G)$ of all real parabolic subgroups of G, $\mathfrak{p}^1$, and $X(\infty)$: $\mathcal{P}(G) \longleftrightarrow \mathfrak{p}^1 \longleftrightarrow X(\infty)$.

Garland and Raghunathan formulated the reduction theory for non-uniform lattice Γ in G in [13]. Let $D(x_0, \Gamma)$ be the *Dirichlet fundamental domain* for Γ: for $x_0 = e \cdot K$,

$$D(x_0, \Gamma) = \{x \in X \mid d(x, x_0) \leq d(x, \gamma \cdot x_0) \text{ for all } \gamma \in \Gamma\}.$$

For $t > 0$, let $A_{P,t} = \{a \in A_P \mid a^\alpha \geq t \text{ for all } \alpha \in \Phi^+(P, A_P)\}$. For an open subsets $U \subset N_P$, the set $\mathfrak{S}_{P,U,t} = U \times A_{P,t}$ is called a *Siegel set* in the symmetric space X. The below summarizes the main theorems of [13].

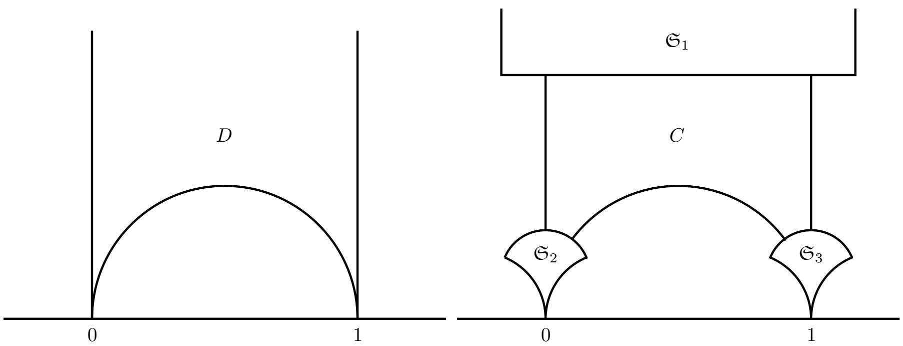
Theorem 12 ([13]). *Under the notions, the following holds.*

(1) *There are only finitely many geodesic rays in* $D(x_0, \Gamma)$ *eminating from* x_0 (cf. [13, Theorem 4.6]).
(2) *For each parabolic subgroup* P *corresponding to a ray in* $D(x_0, \Gamma)$, *the subgroup* $\Gamma \cap N_P \subset N_P$ *cocompact* (cf. [13, Theorem 0.7]).
(3) *Let* $P_1, \ldots, P_n$ *be the real parabolic subgroups of* G *corresponding to geodesic rays in* $D(\Gamma, x_0)$. *There exist open subsets* $U_i \subset N_{P_i}$, *real numbers* $t_i > 0$, *mutually disjoint Siegel sets* $\mathfrak{S}_{P_i,U_i,t_i}$, *and a compact subset* $C \subset X$ *such that*

$$\Omega = C \cup \bigcup_{i=1}^{n} \mathfrak{S}_{P_i,U_i,t_i} \tag{2.8}$$

is a fundamental set for Γ (cf. [13, Theorem 0.6]).

Figure 3 visualizes Theorem 12 for the Fuchsian group generated by $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. Figure 3(A) is a precise fundamental domain of Γ, which contains three geodesics rays in D corresponding to 0, 1, and $i\infty$. The Siegel sets and the compact set C in (2.8) are illustrated in Figure 3(B).



(A) A fundamental set D (B) A fundamental domain Ω

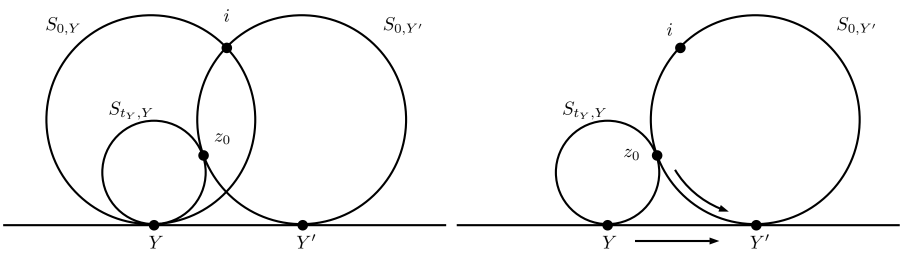
Figure 3. Reduction theory of Garland and Raghunathan

The next lemma is also from [13], and will be used in our proof.

Lemma 13 ([13, Lemma 4.3]). *Let* $\mathbf{k}_Y : G \to K$, $\mathbf{a}_Y : G \to A_P$, $\mathbf{n}_Y : G \to N_P$ *be the natural projections of the Iwasawa decomposition* $G = K \times A_P \times N_P$. *Let* Y, Y' *be distinct vectors in* $\mathfrak{p}^1$, *and* $f_Y : \mathbb{R} \to \mathbb{R}$ *be a smooth function satisfying* $\mathbf{a}_{Y'}(\exp tY) = \exp f_Y(t) Y'$ *where* $t_Y = \sup\{t \mid f_Y(t) = 0\}$. *For any* $M > 0$, *there exists* $\beta > 0$ *such that for all* $Y \in \mathfrak{p}^1$ *satisfying* $\|Y' - Y\| < \beta$ *and* $Y' \neq Y$,

$$d(\mathbf{n}_{Y'}(\exp t_Y Y) \cdot x_0, x_0) \geq M. \tag{2.9}$$

Let us describe the geometric idea of Lemma 13 with the upper half-plane $\mathbf{H}$. Each vector $Y \in \mathfrak{p}^1$ is identified with a unique point at infinity in $\mathbf{H}(\infty)$. Let $P_Y \subset SL_2(\mathbb{R})$ be the corresponding parabolic subgroup. For $t \in \mathbb{R}$, a *horocycle of level t at Y* is the set $S_{t,Y} = \{g \cdot i \in \mathbf{H} \mid \mathbf{a}_Y(g)^\alpha = t,\ \alpha \in \Phi^+(P_Y, A_{P_Y})\}$ (cf. Figure 4(A)). Note that the base point i always lies on the horocycle of level 0. For another vector $Y' \in \mathfrak{p}^1$, let t_Y be the level of the horocycle tangent to $S_{0,Y'}$ at z_0. The distance (2.9) is the distance between i and z_0. Lemma 13 states that as Y approaches to Y' the distance between i and z_0 diverges (cf. Figure 4(B)).



(A) The constant t_Y (B) The behavior of t_Y

FIGURE 4. A geometric idea of Lemma 13

3. Proof of the main theorem

In this section, we give proof of our main theorem.

(1) In §3.1, we construct a Γ-space $\overline{X}_\Gamma$, and describe its topology.
(2) In §3.2, we define the action of Γ on $\overline{X}_\Gamma$. In Proposition 17, we show that the Γ-action on the symmetric space X extends continuously on $\overline{X}_\Gamma$. In Proposition 20, we show that the Γ-action is proper.
(3) In §3.3, we prove that the space $\overline{X}_\Gamma$ is a model for the classifying space $\underline{E}\Gamma$. In Proposition 24, we show that for any finite subgroup H of Γ the fixed point set $\left(\overline{X}_\Gamma\right)^H$ is contractible.

3.1. Construction of the space $\overline{X}_\Gamma$

Let $\mathbf{G}$ be a semisimple algebraic Lie group of $\mathbb{R}$-rank 1. Let $G = \mathbf{G}(\mathbb{R})$ and K be a maximal compact subgroup of G and $X = G/K$ be the corresponding symmetric space. Since the $\mathbb{R}$-rank is 1, the boundary symmetric space X_P is a one-point space. Thus for any real parabolic subgroup $P \subset G$, the horospherical decomposition is $X \cong N_P \times A_P$.

Definition 14. Let Γ be a non-uniform lattice in G. A parabolic subgroup $P \subset G$ is called Γ-*rational* if the intersection $\Gamma \cap N_P$ is a cocompact lattice in N_P.

Let Δ_Γ be the collection of all Γ-rational parabolic subgroups in G. For each parabolic subgroup P, define $e(P) = N_P$. Let $\overline{X}_\Gamma$ be the union

$$\overline{X}_\Gamma = X \cup \coprod_{P\in\Delta_\Gamma} e(P),$$

with the following convergence class of sequences $\mathcal{C}$:

(S1) $y_j \xrightarrow{\mathcal{C}} y_\infty$ if $y_j \to y_\infty$ in X;

(S2) $n_j \xrightarrow{\mathcal{C}} n_\infty$ if $y_j \to n_\infty$ in N_P;

(S3) $y_j \xrightarrow{\mathcal{C}} n_\infty$ if there exists a Γ-rational parabolic subgroup $P \subset G$ such that (a) $y_j = (n_j, a_j) \in X = N_P \times A_P$, (b) $n_j \to n_\infty$ in N_P, and (c) $a_j^\alpha \to \infty$ for $\alpha \in \Phi^+(P, A_P)$.

Lemma 15. *The collection of following subset of* $\overline{X}_\Gamma$ *form a closed basis*:

(C1) *A closure* (*in* $\overline{X}_\Gamma$) *of an open subset in* X;

(C2) *A closed subset in* $\coprod_{P\in\Delta_\Gamma} e(P)$.

Proof. Let A be a closed subset. If $A \cap X = \emptyset$, then A is of the type (C2). Suppose that $A \cap X \neq \emptyset$ and $y_j \in A$ be a convergent sequence in X. Then y_j is essentially of the type (S1) or (S3). If y_j is of the type (S1), then y_j lies on a closed subset of X, which is of the type (C1). If y_j is of the type (S2), then y_j lies on the closure of a Siegel set in X, which is of the type (C2). □

Proposition 16. *The collection of following subsets of* $\overline{X}_\Gamma$ *form an open basis*:

(O1) *Open subsets in* X;

(O2) *the unions of Siegel sets.*

Proof. First, we show that subsets of types (O1) and (O2) are open (in $\overline{X}_\Gamma$). Let V_1 be a subset of the type (O1). Then

$$\overline{X}_\Gamma - V_1 = (\underbrace{X - V}_{\text{type (C1)}}) \cup \underbrace{\coprod_{P\in\Delta_\Gamma} e(P)}_{\text{type (C2)}}.$$

Now let V_2 be a subset of the type (O2). Then

$$\overline{X}_\Gamma - V_2 = (e(P) - U) \cup (X - \mathfrak{S}_{P,U,t}) \cup \bigcup_{Q\in\Delta_\Gamma,\, Q\neq P} e(Q).$$

We claim that $\overline{X}_\Gamma - V_2$ is the closure (in $\overline{X}_\Gamma$) of $X - \mathfrak{S}_{P,U,t}$. Note that

$$X - \mathfrak{S}_{P,U,t} = ((N_P - U) \times A_{P,t}) \cup (N_P \times (A_P - A_{P,t}))\,.$$

Thus $e(P) - U$ is contained in the closure of $X - \mathfrak{S}_{P,U,t}$. Let $Q \neq P$ be a Γ-rational parabolic subgroup. We can choose sufficiently large $T \gg 1$ such that $\mathfrak{S}_{Q,U',T} \cap \mathfrak{S}_{P,U,t} = \emptyset$. Thus $e(Q)$ is contained in the closure of $X - \mathfrak{S}_{P,U,t}$.

To claim that the collection of subsets of the types (O1) and (O2) is an open basis of $\overline{X}_\Gamma$, it suffices to show that the complement of each types of closed set can be generated by open sets of types (O1) and (O2). Let C_1 be a closed set of type (C1). If C_1 is bounded, then for each $P \in \Delta_\Gamma$, then there exists a sufficiently large constant $T_P \gg 1$ such that $\mathfrak{S}_{P,N_P,t_P} \cap C_1 = \emptyset$. Thus

$$\overline{X}_\Gamma - C_1 = (\underbrace{X - C_1}_{\text{type O1}}) \cup \bigcup_{P \in \Delta_\Gamma} \underbrace{\mathfrak{S}_{P,N_P,t_P} \cup e(P)}_{\text{type O2}}.$$

Let C_2 be of the type (C2). Without loss of generality, assume $C_2 = \coprod_{P \in \Delta} C_P$ where $\Delta \subset \Delta_\Gamma$ and C_P is a closed subset in $e(P)$. For each $P \in \Delta$, let us denote $U_P = N_P - C_P$. Then for any $t > 0$,

$$\overline{X}_\Gamma - C_2 = \bigcup_{P \in \Delta_\Gamma - \Delta} (\underbrace{\mathfrak{S}_{P,N_P,t} \cup e(P)}_{\text{type O2}}) \cup \bigcup_{P \in \Delta} (\underbrace{\mathfrak{S}_{P,U_P,t} \cup U_P}_{\text{type O2}}).$$

□

3.2. The proper action of Γ on $\overline{X}_\Gamma$

We define the canonical action of Γ on X as in (2.2), (2.3). On a boundary point $n' \in e(P) = N_P$, we define the action of $\gamma = k \cdot p \in \Gamma$ as follows:

$$\gamma \cdot z = {}^k(n({}^{am}n')) \in e({}^kP). \tag{3.1}$$

Proposition 17. *The* Γ*-action on* $\overline{X}_\Gamma$ *is continuous.*

Proof. Since Γ is discrete, it is enough to show that for a fixed $\gamma \in \Gamma$ and a convergent sequence $y_j \to y_\infty$ in $\overline{X}_\Gamma$, the sequence $\gamma \cdot y_j$ converges to $\gamma \cdot y_\infty$. The explicit actions (2.2), (2.3), (3.1) show that the statement holds for convergent sequences of the type (S1) and (S2). Suppose that $y_j = (n_j, a_j, m_j)$ is a convergent sequence of the type (S3). If $\gamma = k \cdot nam$, then $\gamma \cdot y_j = {}^k(n^{(am)}n_j, {}^k(aa_j), {}^k(mm_j))$. Let $\alpha \in \Phi^+(P, A_P)$ be a (unique) positive restricted root. Since $a_j^\alpha \to \infty$ as $j \to \infty$, $({}^k(aa_j))^{\alpha^k} = (aa_j)^\alpha = a^\alpha a_j^\alpha \to \infty$. □

Lemma 18. *Let* $y_j \in X$ *be a convergent sequence of the type* (S3). *There is no infinite sequence* γ_j *of* Γ *where* $y'_j = \gamma_j \cdot y_j \in X$ *is a convergent sequence of the type* (S1).

Proof. We suppose that there is a sequence γ_j such that $y'_j \to y'_\infty \in X$, and show that this leads to a contradiction. Since Γ is discrete, there exists $\epsilon > 0$ such that $d(\gamma \cdot y'_\infty, y'_\infty) > \epsilon$ for all $\gamma \in \Gamma$. Thus

$$\begin{aligned} 2d(y'_\infty, y'_j) + d(\gamma \cdot y'_j, y'_j) &= d(\gamma \cdot y'_\infty, \gamma \cdot y'_j) + d(\gamma \cdot y'_j, y'_j) + d(y'_j, y'_\infty) \\ &\geq d(\gamma \cdot y'_\infty, y'_\infty) > \epsilon > 0. \end{aligned} \tag{3.2}$$

For a positive $\epsilon' > 0$ such that $\epsilon' < \epsilon$, we can choose a sufficiently large $N \gg 1$ such that

$$2d(y'_\infty, y'_j) < \epsilon' < \epsilon \text{ for all } j > N. \tag{3.3}$$

From (3.2) and (3.3), for every nontrivial $\gamma \in \Gamma$ and $j > N$,

$$d(\gamma \cdot y'_j, y'_j) > \epsilon - \epsilon' > 0. \tag{3.4}$$

Note that both ϵ and ϵ' do not depend on the choice of γ. For a non-trivial element $\gamma \in \Gamma \cap N_P$, let $\gamma'_j = \gamma_j \gamma \gamma_j^{-1}$. Since $d(\gamma \cdot y_j, y_j) \to 0$, we get $d(\gamma \cdot y_j, y_j) = d(\gamma_j \gamma \cdot y_j, y'_j) = d(\gamma'_j \cdot y'_j, y'_j) \to 0$. This contradicts (3.4). □

Proposition 19. *Let $P_1, \ldots, P_n$ be the parabolic subgroups in* (2.8)*. A parabolic subgroup $P \subset G$ is Γ-rational if and only if it is Γ-conjugate to P_i for some $i = 1, \ldots, n$.*

Proof. From Theorem 12 and Definition 14, all such P_i are Γ-rational. Conversely, let P be a Γ-rational parabolic subgroup of G such that $P \neq P_i$ for all $i = 1, \ldots, n$. We will show that there exists $\gamma \in \Gamma$ such that ${}^{\gamma}P = P_i$ for some i.

Let $Y \in \mathfrak{p}^1$ be the vector corresponding to P. For unbounded sequence $t_j > 0$ and a fixed point $x_0 \in X$, let $y_j = \exp t_j Y \cdot x_0$ be a unbounded sequence. By Theorem 12, $y_j \notin D(x_0, \Gamma)$ for all but finitely many j. For each j, choose an element $\gamma_j \in \Gamma$ such that $\gamma_j \cdot y_j \in D(x_0, \Gamma)$. Let us denote $y'_j = \gamma_j \cdot y_j$. By Lemma 18, the sequence y'_j is unbounded. Again by Theorem 12, a subsequence of y'_j lies on one of Siegel sets in (2.8), say $\mathfrak{S}_{P',U',t'}$.

Let $Y_j \in \mathfrak{p}^1$ be the vector corresponding to ${}^{\gamma_j}P$. Since $\mathfrak{p}^1$ is compact, Y_j converges to $Y_\infty \in \mathfrak{p}^1$. Let $Y' \in \mathfrak{p}^1$ be the vector corresponding to P'. We claim that $Y' = Y_\infty$. If not, for any given Siegel set $\mathfrak{S}_\infty$ relative to Y_∞, there exist a sufficiently large $t' \gg 1$ such that the Siegel set $\mathfrak{S}_{P',U',t'}$ is disjoint from $\mathfrak{S}_\infty$. This is a contradiction.

Suppose that $Y_j \neq Y'$ for all j. By Lemma 13, as $Y_j \to Y'$ the sequence $x'_j = \mathbf{n}_{Y'}(\exp t_{Y_j} Y) \cdot x_0$ diverges in $N_{P'}$. This implies that the sequence $x_j = {}^{\gamma_j}x'_j \in N_P$ also diverges in N_P. Recall that $x_j = \mathbf{n}_Y(y_j)$. Since y_j lies in a Siegel set $\mathfrak{S}_{P,U,t}$, the sequence x_j lies in U. Since P is Γ-rational, the set U is relatively compact. Thus the sequence x_j must be bounded, which is a contradiction. Therefore, $Y_j = Y'$ for some j, and thus ${}^{\gamma_j}P = P_i$. □

Proposition 20. *The Γ-action on $\overline{X}_\Gamma$ is properly discontinuous.*

Proof. Since Γ is discrete, it is enough to show that for every compact subset C in $\overline{X}_\Gamma$, the set $\Gamma' = \{\gamma \in \Gamma \mid \gamma \cdot C \cap C \neq \emptyset\}$ is finite. Let C be a compact subset of the type (C1) in $\overline{X}_\Gamma$. If $C \subset X$, then Γ' is finite, since the Γ-action on X is properly discontinuous. Otherwise, without loss of generality, we can assume C is of the form $C = \overline{\mathfrak{S}}_{P,U,t}$ where $P \in \Delta_\Gamma$, $t > 0$, and U is a relatively compact open subset of N_P. By Proposition 19, such C is Γ-conjugate to the closure $\overline{\mathfrak{S}}_{P_i,U_i,t_i}$ for some P_i in (2.8). By Theorem 12, the set Γ' is finite. □

3.3. The space $\overline{X}_\Gamma$ as a model for $\underline{E}\Gamma$

Lemma 21. *The space $\overline{X}_\Gamma$ is an analytic manifold with boundary.*

Proof. For each Γ-rational parabolic subgroup $P \subset G$, let $\psi_P : N_P \xrightarrow{\cong} \mathbb{R}^n$, $\alpha : A_P \xrightarrow{\cong} \mathbb{R}_{>0}$ be the canonical diffeomorphisms. From Proposition 19 and

Lemma 16, the collection $\mathcal{A} = \{X \cup e(P) \mid P \in \Delta_\Gamma\}$ is the open covering of $\overline{X}_\Gamma$. The map define on $X \cup e(P)$ by

$$\phi_P(z) = \begin{cases} (\psi_P(n), 1/a^\alpha) & \text{if } z = (n,a) \in N_P \times A_P \cong X, \\ (\psi_P(z), 0) & \text{if } z \in N_P = e(P). \end{cases}$$

is the local chart on $\overline{X}_\Gamma$.

For two Γ-rational parabolic subgroups $P \neq Q$, choose $k \in K$ such that $P = {}^kQ$. Then we have the following diagram commutes:

$$\begin{array}{ccc} N_Q \times A_Q & \xrightarrow{k\text{-conj.}} & N_P \times A_P \\ {\scriptstyle \phi_Q^{-1}}\big\uparrow & & \big\downarrow{\scriptstyle \phi_P} \\ \mathbb{R}^n \times \mathbb{R}_{>0} & \xrightarrow{\phi_P \circ \phi_Q^{-1}} & \mathbb{R}^n \times \mathbb{R}_{>0} \end{array}$$

Since the G-action is analytic and the maps ϕ_P and ϕ_Q are diffeomorphisms, the transition map $\phi_P \circ \phi_Q^{-1}$ is analytic. □

Proposition 22. *The space $\overline{X}_\Gamma$ has a Γ-CW-complex structure.*

Proof. The space $\overline{X}_\Gamma$ is a subanalytic manifold of its double cover (cf. [19, Definition 3.2]). By [19, Theorem B], the space $\overline{X}_\Gamma$ admits a Γ-CW-complex structure. □

Proposition 23. *Let Ω be the fundamental set of Γ in X defined in* (2.7). *Then the closure $\overline{\Omega}$ in $\overline{X}_\Gamma$ is a compact fundamental set for Γ.*

Proof. The compactness follows from the fact that every $U_i \subset N_{P_i}$ is relatively compact. By Proposition 19 and Theorem 12, the closure $\overline{\Omega}$ is a locally finite subset of $\overline{X}_\Gamma$ satisfying $\Gamma \cdot \overline{\Omega} = \overline{X}_\Gamma$. □

Proposition 24. *For any finite subgroup $H \subset \Gamma$, the fixed point set $\left(\overline{X}_\Gamma\right)^H$ is contractible.*

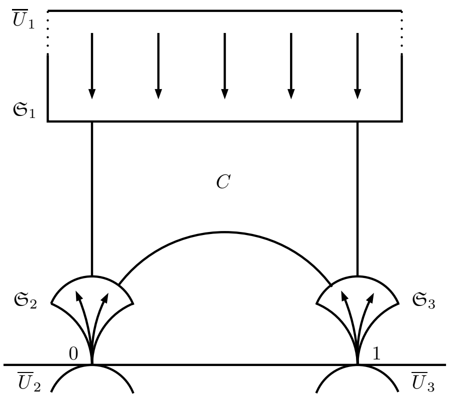
Proof. From Proposition 22 and Proposition 23, the space $\overline{X}_\Gamma$ is a cofinite Γ-CW-complex. By Proposition20, every isotropy group is finite. Let H be a non-trivial finite subgroup of Γ. If H fixes a point on the boundary component $e(P)$, then it fixes a geodesic ray converging to that point. Thus the fixed point set $\left(\overline{X}\right)^H$ retracts into X^H. Since the fixed point set X^H is a geodesic submanifold of X, the set $\left(\overline{X}\right)^H$ is contractible.

It remains to show that the space $\overline{X}_\Gamma$ itself is contractible. We will first construct a retraction h_t of $\overline{\Omega}$, and then extends it to the retraction H_t of $\overline{X}_\Gamma$. For each $i = 1, \dots, n$, let $\mathfrak{S}_{P_i,U_i,t_i}$ be the mutually disjoint Siegel sets from (2.8) and α_i the unique positive simple root in $\Phi^+(P_i, A_{P_i})$. Since the $\mathbb{R}$-rank of G is 1, the map $\alpha_i : A_{P_i} \to \mathbb{R}_>$ is a diffeomorphism. Thus, for any $0 < t \leq 1$, there exists a unique element $a_i(t) \in A_{P_i}$ such that $\alpha_i(a_i(t)) = t_i/t$. Let us define a map $h_t : \overline{\Omega} \times [0,1] \to \overline{\Omega}$ as follows (cf. Figure 5(A)):

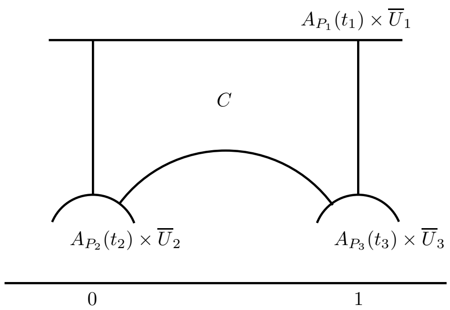
$$h_t(z) = \begin{cases} z & \text{if } z \in \overline{\Omega} - \bigcup_{i=1}^{n} \overline{\mathfrak{S}}_{P_i,U_i,t_i/t}, \\ (n, a_i(t)) & \text{if } z \in \mathfrak{S}_{P_i,U_i,t_i/t} \text{ and } z = (n,a) \in N_{P_i} \times A_{P_i}, \\ (z, a_i(t)) & \text{if } z \in \overline{U}_i, \end{cases}$$
$$h_0(z) = z.$$

We claim that h_t is continuous. Let z be a point in the interior Ω. For sufficiently small $\epsilon > 0$, the point z does not belong to any Siegel sets $\overline{\mathfrak{S}}_{P_i,U_i,t_i/t}$ for all $t < \epsilon$ and $i = 1, \ldots, n$. Let z be a point on the boundary $e(P_i)$ for some $i = 1, \ldots, n$. As $t \to 1$, the path $s(t) = h_{1-t}(z)$ approaches to z. Let $A_{P_i}(t_i) = \{a \in A_{P_i} \mid \alpha_i(a) = t_i\}$, $S_{P_i,\overline{U}_i,t_i} = \overline{U}_i \times A_{P_i}(t_i)$. Then the homotopy retract $h_1(\overline{\Omega})$ is $C \cup \bigcup_{i=1}^{n} S_{P_i,\overline{U}_i,t_i}$ (cf. Figure 5(B)).

Now we extend h_t to H_t. For each $z \in \overline{X}_\Gamma$, choose an element $\gamma \in \Gamma$ such that $\gamma \cdot z \in \overline{\Omega}$. Define $H_t : \overline{X}_\Gamma \times [0,1] \to \overline{X}_\Gamma$ as $H_t(z) = \gamma^{-1} \cdot h_t(\gamma \cdot z)$. We claim that $H_t : X \times [0,1] \to X$ is well-defined. For a point $z \in \overline{X}_\Gamma$, suppose that there are two distinct elements $\gamma_1, \gamma_2 \in \Gamma$ such that $\gamma_1 \cdot z \neq \gamma_2 \cdot z \in \mathfrak{S}_{P_i,U_i,t_i}$ for some $i = 1, \ldots, n$. For each $j = 1, 2$, the path $s_j(t) = h_{1-t}(\gamma_j \cdot z)$ eminates from $s_j(0) = \gamma_j \cdot z$. Let us extends $s_j(t)$ a semi-infinite geodesic $\tilde{s}_j(t)$. Then $\tilde{s}_1(\infty) = \tilde{s}_2(\infty)$ is fixed by the element $\gamma_2\gamma_1^{-1} \in P$. Since $\Gamma \cap P = \Gamma \cap N_P$, we get $\gamma_2\gamma_1^{-1} h_t(\gamma_1 z) = h_t(\gamma_2 z)$. The homotopy retract of $\overline{X}_\Gamma$ is then the union of all Γ-translates of $h_1(\overline{\Omega})$. Since $S_{P_i,\overline{U}_i,t_i} \subset (\Gamma \cap N_{P_i}) \cdot C$, we have $H_1(\overline{X}_\Gamma) = \Gamma \cdot C$. So H_t is well-defined.



(A) The homotopy retraction h_t



(B) The retract $h_t(\overline{\Omega})$

FIGURE 5. Contractibility of the space $\overline{X}_\Gamma$

In conclusion, the space $\overline{X}_\Gamma$ contracts into C which is contractible. Therefore, $\overline{X}_\Gamma$ is contractible. □

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