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ON INDEFINITE LOCALLY CONFORMAL COSYMPLECTIC MANIFOLDS

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Abstract. We prove that there exist foliations whose leaves are the maximal integral null manifolds immersed as submanifolds of indefinite locally conformal cosymplectic manifolds. Necessary and sufficient conditions for such leaves to be screen conformal, as well as possessing integrable distributions are given. Using Newton transformations, we show that any compact ascreen null leaf with a symmetric Ricci tensor admits a totally geodesic screen distribution. Supporting examples are also obtained.

1. Introduction

Almost contact structures provide a counterpart of almost complex structures in odd dimension and include several classes of special importance as contact, Sasakian and cosymplectic ones [3]. The notion of almost cosymplectic manifolds was introduced by Goldberg and Yano in [10]. In fact, they extended earlier results on almost Kähler manifolds which says that if the curvature transformation of the almost Kähler metric commutes with the almost complex metric, then the latter is integrable. The simplest examples of such manifolds are those locally formed by the products of almost Kählerian manifolds and the real line $\mathbb{R}$ (or the circle S^1).

In this paper, we are specially interested in indefinite locally conformal deformations of almost cosymplectic manifolds, by paying attention to the geometry of one of its canonical foliations. In the Riemannian case and under some special conformal deformation, Olszak in [16] proved that such manifolds are almost α-Kemnotsu. In the same case, the first two authors in [13] proved that the class of these deformations contain the one of bundle-like metric structures.

Null geometry of submanifolds of semi-Riemannian manifolds is remarkably different from the geometry of submanifolds immersed in a Riemannian manifold by the fact that the normal vector bundle of a null submanifold intersects with its tangent bundle. This aspect makes null geometry difficult to study

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despite having numerous applications in other fields like mathematical physics. The study of null submanifolds of a semi Riemannian manifold was initiated by Duggal-Bejancu [8] and Kupeli [12] and later by many authors [11], [14], [6] and many other references therein. Geometry of submanifolds in locally conformal cosymplectic manifolds as well as cosymplectic manifolds has also been studied by many authors, for instance [5], [10], [13], [15] and [16].

We consider a locally conformal cosymplectic manifold endowed with an indefinite metric, and we study the leaves (as submanifolds) of the foliations which are coming from the distributions generated by the Pfaffian equation $\omega = 0$, ω being the characteristic 1-form of the ambient manifold under consideration, $P : x \in M(c) \mapsto \mathbb{R}V_x \oplus \mathbb{R}B_x$, where $c = g(B_x, B_x)$ and $\mathbb{R}V_x$ and $\mathbb{R}B_x$ denotes line bundles locally spanned by V_x and B_x, respectively. In this case the Lee form ω is not required to be parallel as it is the case with locally conformal Kähler. But according to different positions of the Lee vector field B with respect to the structure vector ξ and due to the causal character of the Lee vector field, we obtain some rich informations about the geometry of leaves in M. We give the necessary and sufficient conditions for leaves to be screen conformal as well as some distributions on them to be integrable. Also, we give the necessary condition for the induced connection on the leaves to be a metric connection. By considering a suitable conformal vector field on M, we show that any ascreen null leaf, with a symmetric Ricci tensor admits a totally geodesic screen distribution, using the concept of Newton transformations [1], [2], and [6].

The paper is arranged as follows. In Section 2 we list the basic notions on locally conformal cosymplectic manifolds as well null geometry. In Section 3 we discuss the geometry of leaves admitting a non-tangential structure vector field. Finally, in Section 4 we consider a special conformal vector on M and use it to study the geometry of screen distributions of leaves with a symmetric induced Ricci tensor.

2. Preliminaries

Let M be a $(2m+1)$-dimensional almost contact manifold endowed with an almost contact metric structure (ϕ, ξ, η), where ϕ is tensor field of type $(1, 1)$ on M, a vector field ξ and a 1-form η satisfying the following relations

$$\phi^2 = -\mathbb{I} + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \phi\xi = 0, \quad \eta\phi = 0. \tag{2.1}$$

Then the structure (ϕ, ξ, η, g) is called an indefinite almost contact metric structure on M if (ϕ, ξ, η) is an almost contact structure on M and g is a semi-Riemannian metric on M such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X and Y on M. It follows that $\eta(X) = g(X, \xi)$. The fundamental 2-form of M is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y on M.

M is said to be *almost cosymplectic* if the forms η and Φ are closed, that is, $d\eta = 0$ and $d\Phi = 0$, d being the operator of the exterior differentiation (see [10]). If M is almost cosymplectic and its almost contact structure (ϕ, ξ, η) is normal, then M is called *cosymplectic*. The normality condition says that the torsion tensor field

$$[\phi, \phi] + 2d\eta \otimes \xi = 0, \tag{2.3}$$

where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y].$$

It is well-known that a necessary and sufficient condition for the almost contact metric manifold M to be cosymplectic is $\nabla\phi = 0$, where ∇ is the Levi-Civita connection of M.

Throughout the paper, all manifolds are supposed to be paracompact and smooth. We denote by $\Gamma(\Xi)$ the set of smooth sections of the vector bundle Ξ.

Now, let (M, ϕ, ξ, η, g) be an almost contact metric manifold. Such a manifold is said to be *l.c. almost cosymplectic* [16] if M has an open covering $\{U_t\}_{t\in I}$ endowed with smooth functions $\sigma_t : U_t \longrightarrow \mathbb{R}$ such that over each U_t the almost contact metric structure $(\phi_t, \xi_t, \eta_t, g_t)$ defined by

$$\phi_t = \phi, \;\; \xi_t = \exp(\sigma(t))\xi, \;\; \eta_t = \exp(-\sigma(t))\eta, \;\; g_t = \exp(-2\sigma(t))g, \tag{2.4}$$

is *almost cosymplectic*. If the structures $(\phi_t, \xi_t, \eta_t, g_t)$ defined in (2.4) are cosymplectic, then M is called *l.c. cosymplectic*.

L.c. almost cosymplectic manifolds were characterized by Vaisman in [17]. This is stated as follows: *An almost contact metric manifold M is an l.c. almost cosymplectic manifold if and only if there exists a 1-form ω on M such that*

$$d\Phi = 2\omega \wedge \Phi, \;\; d\eta = \omega \wedge \eta \;\; \textit{and} \;\; d\omega = 0. \tag{2.5}$$

Moreover, an l.c. almost cosymplectic (respectively, an l.c. cosymplectic) manifold M is *almost cosymplectic* (respectively, *cosymplectic*) if and only if $\omega = 0$. If ω has no singular points, M was termed, by Capursi and Dragomir in [4], *strongly non-cosymplectic*.

Assume that (M, ϕ, ξ, η, g) is an l.c. almost cosymplectic manifold. Then the relations in (2.5) are satisfied for a certain 1-form ω. For any t, over open set U_t, the structure $(\phi_t, \xi_t, \eta_t, g_t)$ given by (2.4) is almost cosymplectic and $d\sigma_t = \omega$.

Now, we give a proof to the formula (3.3) in [16]. Let ∇ and ∇^t be the Levi-Civita connections associated with the metrics g and g_t, respectively. Then, for any vector fields X and Y on M,

$$\nabla^t_X Y = \nabla_X Y - \omega(X)Y - \omega(Y)X + g(X, Y)B, \tag{2.6}$$

where B is the vector field defined by $g(B, X) = \omega(X)$.

Note that the vector field B defined in (2.6) is explicitly given by $B = \operatorname{grad}\sigma_t$, over any U_t.

As proved in [13], an almost contact metric manifold M is *l.c. almost cosymplectic* if and only if there exists a 1-form ω on M such that $d\omega = 0$ and

$$\begin{aligned} 2g((\nabla_X\phi)Y, Z) &= g(N_1(Y,Z), \phi X) + 2\omega(\phi Y)g(X,Z) - 2\omega(\phi Z)g(X,Y) \\ &\quad - 2\omega(Y)g(\phi X, Z) - 2\omega(Z)g(X, \phi Y) \end{aligned} \tag{2.7}$$

for any vector fields X, Y and Z on M, where $N_1(X,Y) = [\phi,\phi](X,Y) + 2d\eta(X,Y)\xi$.

For the covariant derivative $\nabla\phi$ and using (2.7), we have

$$(\nabla_\xi\phi)\xi = \phi B \ \text{ and } \ (\nabla_\xi\phi)X = \omega(\phi X)\xi + \eta(X)\phi B. \tag{2.8}$$

Let us consider a $(1,1)$-tensor field h on M by [16]

$$hX = \nabla_X\xi - \omega(\xi)X + \eta(X)B \tag{2.9}$$

for any $X \in \Gamma(TM)$. This leads to

$$\nabla_\xi\xi = -B + \omega(\xi)\xi. \tag{2.10}$$

Using (2.4) and (2.1), we obtain on each U_t, $\exp(-\sigma_t)\nabla^t_X\xi_t = hX$. Note that the linear operator h is symmetric and satisfies (see [16] for details)

$$h\phi + \phi h = 0, \ \ h\xi = 0 \ \text{ and } \ \mathrm{trace}(h) = 0. \tag{2.11}$$

As an example of l.c. almost cosymplectic manifold, we have the following.

Example 2.1. Consider a 9-dimensional semi-Riemannian manifold $M^9 = \{p \in \mathbb{R}^9 \,|\, x_1 > 1, y_1 > 1\}$, where $p = (x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4, z)$ are the standard coordinates in $\mathbb{R}^9$. The vectors fields $X_1 = e^{-z-x_1y_1}\{\partial x_1 + \partial y_1\}$, $X_2 = e^{-z-x_1y_1}\{\partial x_2 + \partial y_2\}$, $X_3 = e^{-z-x_1y_1}\partial x_3$, $X_4 = e^{-z-x_1y_1}\partial x_4$, $Y_1 = e^{-z-x_1y_1}\{\partial x_1 - \partial y_1\}$, $Y_2 = e^{-z-x_1y_1}\{\partial x_2 - \partial y_2\}$, $Y_3 = -e^{-z-x_1y_1}\partial y_3$, $Y_4 = -e^{-z-x_1y_1}\partial y_4$, $Z = e^{-z-x_1y_1}\partial z$ are linearly independent at each point of M. Let g be the indefinite metric on M defined by $g(X_i, X_i) = g(Y_i, Y_i) = -1$ for $i = 1, 2$ and $g(X_i, X_i) = g(Y_i, Y_i) = 1$ for $i = 3, 4$, $g(X_i, Y_j) = 0$ and $g(\xi, \xi) = 1$. Let η be the 1-form on M defined by $\eta = e^{z+x_1y_1}dz$, then the structure vector field is $\xi = e^{-z-x_1y_1}\partial z$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi X_1 = -Y_1$, $\phi Y_1 = X_1$, $\phi X_2 = -Y_2$, $\phi Y_2 = X_2$, $\phi X_3 = Y_3$, $\phi Y_3 = -X_3$, $\phi X_4 = Y_4$, $\phi Y_4 = -X_4$, $\phi\xi = 0$. By linearity of ϕ and g, the relations (2.2) are satisfied thus, (ϕ, ξ, η, g) defines an almost contact metric structure on M^9. We have also $d\eta = e^{z+x_1y_1}\{y_1dx_1 \wedge dz + x_1dy_1 \wedge dz\}$. By straightforward calculations we obtain $\Phi = e^{2(z+x_1y_1)}\{\frac{1}{2}dx_1 \wedge dy_1 + \frac{1}{2}dx_2 \wedge dy_2 + dx_3 \wedge dy_3 + dx_4 \wedge dy_4\}$. By letting $\omega = y_1dx_1 + x_1dy_1 + dz$, we have $d\eta = \omega \wedge \eta$ and $d\Phi = 2\omega \wedge \Phi$ and $d\omega = 0$, which show that $(M^9, \phi, \xi, \eta, g)$ is an l.c. almost cosymplectic manifold with the dual vector field B of ω given by $B = e^{-2(z+x_1y_1)}\{\frac{y_1}{2}\partial x_1 + x_1\partial y_1 + \partial z\}$.

Let M be a $(2n+1)$-dimensional indefinite l.c. almost cosymplectic manifold of index q, $0 < q < 2n+1$. Let us set $c = g(B,B) \in \mathcal{C}^\infty(M)$ and $\mathrm{Sign}(B) = \{x \in M : B_x = 0\}$. Note that c and $\mathrm{Sing}(B)$ determine the causal character of B, so it may be $c = 0$ and $\mathrm{Sing}(B) = \emptyset$ when B is null.

From now on, the characteristic 1-form ω given in (2.5) does not vanish, *unless otherwise started*.

Since M is an l.c. almost cosymplectic, it admits a canonical foliation $\mathcal{F}$ of codimension r whose leaves are the maximal connected integral manifolds of the Pfaffian equation $\omega = 0$ (see [4] for details and references therein).

Let $(T\mathcal{F})^c$ be the complementary distribution to $T\mathcal{F}$ in TM. Then, its dimension is r.

First, assume that $c = g(B, B) \neq 0$. Then, if it is easy to see that the index of each leaf L of $\mathcal{F}$ is given by

$$\operatorname{ind}(L) = q - s,$$

where $s = \operatorname{ind}((T\mathcal{F})^c)$ with $0 \leq s \leq r$.

Now we assume that $c = 0$. Then $B \in T\mathcal{F}$. Set

$$\operatorname{Rad}(T\mathcal{F})_x = (T\mathcal{F})_x \cap (T\mathcal{F}^\perp)_x, \quad x \in M.$$

It is easy to see that $B \in \operatorname{Rad}(T\mathcal{F})$. Let $S(T\mathcal{F})$ be a distribution on M such that

$$T\mathcal{F} = S(T\mathcal{F}) \perp \operatorname{Rad}(T\mathcal{F}). \tag{2.12}$$

The screen distribution $S(T\mathcal{F})$ is seen as the complementary bundle of $\operatorname{Rad}(T\mathcal{F})$ in $T\mathcal{F}$. It is then a rank $(n - p - \dim_{\mathbb{R}} \operatorname{Rad}(T\mathcal{F}))$ non-degenerate distribution over $\mathcal{F}$. In fact, there are infinitely many possibilities of choices for such a distribution provided the foliation $\mathcal{F}$ is paracompact, but each of them is canonically isomorphic to the factor vector bundle $T\mathcal{F}/\operatorname{Rad}(T\mathcal{F})$.

Case 1: If $\omega(\xi) = 0$, i.e., $\xi \in T\mathcal{F}$ and using (2.2), one has $g(\phi B, \phi B) = g(B, B) - \omega(\xi)^2 = 0$, and since $g(\phi B, B) = 0$, the vector field ϕB belongs to $T\mathcal{F}$ and is also null and it may be in the radical distribution or not.

As the structure vector field ξ belongs to $T\mathcal{F}$, we assume that $\xi \in S(T\mathcal{F})$.

If $r = 1$, then by Proposition 2.2 in [8] $\dim_{\mathbb{R}}(\operatorname{Rad}(T\mathcal{F}))_x = 1$ for any $x \in M$. Let $\mathbb{R}B$ be the line bundle spanned by the vector field B. Since $\mathtt{Sign}(B) = \emptyset$, we have $\operatorname{Rad}(T\mathcal{F}) = \mathbb{R}B$. Also $\phi B \notin \operatorname{Rad}(T\mathcal{F})$ which means that $\phi B \in S(T\mathcal{F})$. Therefore L is a null hypersurface immersed in (M, g). Let $S(T\mathcal{F})^\perp$ be an orthogonal complementary vector bundle to $S(T\mathcal{F})$ in $TM|_{\mathcal{F}}$. Consider a complementary vector bundle F of $\mathbb{R}B$ in $S(T\mathcal{F})^\perp$ and take $V \in \Gamma(F|_{\mathcal{U}})$ a locally non-zero section defined on the open subset $\mathcal{U} \subset M$. Then $\omega(V) \neq 0$, otherwise $S(T\mathcal{F})^\perp$ would be degenerate at a point of $\mathcal{U}$ (see [8, p. 79] for more details). We define on $\mathcal{U}$ a vector field

$$N_V = \frac{1}{\omega(V)} \left\{ V - \frac{g(V, V)}{2\omega(V)} B \right\}. \tag{2.13}$$

It is easy to see that

$$\omega(N_V) = 1 \ \text{ and } \ g(N_V, N_V) = g(N_V, W) = 0 \tag{2.14}$$

for any $W \in \Gamma(S(T\mathcal{F})|_{\mathcal{U}})$. If we consider another coordinate neighborhood $\mathcal{U}^* \subseteq M$ such that $\mathcal{U} \cap \mathcal{U}^* \neq \emptyset$. As both $\mathbb{R}B$ and F are vector bundles over

$\mathcal{F}$ of rank 1, we have $B^* = \beta B$ and $V^* = \gamma V$, where β and γ are non-zero smooth functions on $\mathcal{U} \cap \mathcal{U}^*$. It follows that $N^*_{V^*}$ is related with N_V on $\mathcal{U} \cap \mathcal{U}^*$ by $N^*_{V^*} = (1/\beta)N_V$. Therefore, the vector bundle F induces a vector bundle $\mathrm{tr}(T\mathcal{F})$ of rank 1 over $\mathcal{F}$ such that, locally, the equations in (2.14) are satisfied. Finally, we consider another complementary vector bundle E to $\mathbb{R}B$ in $S(T\mathcal{F})^\perp$ and by using (2.13), for both F and E, we obtain the same $\mathrm{tr}(T\mathcal{F})$. As $g(\phi N_V, N_V) = 0$, we have $\phi N_V \in S(T\mathcal{F})$. From (2.2), we have $g(\phi N_V, \phi B) = 1$. Therefore, $\{\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V\}$ (direct sum but not orthogonal) is a non-degenerate vector subbundle of $S(T\mathcal{F})$ of rank 2. Since $\xi \in S(T\mathcal{F})$ and $g(\phi N_V, \xi) = g(\phi B, \xi) = 0$, there exists a non-degenerate invariant distribution D_0 of rank $2n-4$ such that

$$S(T\mathcal{F}) = \{\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V\} \perp D_0 \perp \mathbb{R}\xi, \tag{2.15}$$

and the tangent space of $\mathcal{F}$ is decomposed as follows:

$$T\mathcal{F} = \{\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V\} \perp D_0 \perp \mathbb{R}\xi \perp \mathbb{R}B. \tag{2.16}$$

If $r > 1$, then the radical $\mathrm{Rad}(T\mathcal{F})$ is of rank p with $1 \leq p < \min\{2n+1-r, r\}$ and L is a p-null submanifold.

Case 2: If $\omega(\xi) \neq 0$, i.e., $\xi \notin T\mathcal{F}$. Therefore, L is a null submanifold immersed in M. This holds even when $\phi B \notin \mathrm{Rad}(T\mathcal{F})$. In this case, ξ takes the form

$$\xi = \xi_{T\mathcal{F}} + \xi_{\mathrm{tr}(T\mathcal{F})},$$

where $\xi_{T\mathcal{F}}$ and $\xi_{\mathrm{tr}(T\mathcal{F})}$ are the tangential and transversal components of ξ in M, respectively. But if $\phi B \in \mathrm{Rad}(T\mathcal{F})$, then $r \geq 2$ and there exists a distribution D_2 of k with $0 \leq k < \min\{2n+1-r, r\}$ in $T\mathcal{F}$ such that

$$\mathrm{Rad}(T\mathcal{F}) = D_1 \oplus D_2, \tag{2.17}$$

where $D_1 = \{B, \phi B\}$. This means D_1 is invariant under ϕ. By Lemma 1.2 given in [8, p. 142], we have the following. Choose a screen transversal bundle $S(T\mathcal{F}^\perp)$, which is semi-Riemannian and complementary to $\mathrm{Rad}(\mathcal{F})$ in $T\mathcal{F}^\perp$. Since, for any local basis $\{E_0 = B, E_1 = \phi B, E_k\}$ of $\mathrm{Rad}(T\mathcal{F})$, there exists a local null frame $\{N_0, N_1 = \phi N_0, N_k\}$ of sections with values in the orthogonal complement of $S(TM^\perp)$ in $S(TM)^\perp$ such that $g(E_i, N_j) = \delta_{ij}$, it follows that there exists a null transversal vector bundle $l\mathrm{tr}(T\mathcal{F})$ locally spanned by $\{N_0, N_1 = \phi N_0, N_k\}$ [8]. Then,

$$\mathrm{tr}(T\mathcal{F}) = l\mathrm{tr}(T\mathcal{F}) \perp S(T\mathcal{F}^\perp), \tag{2.18}$$

$$TM = S(T\mathcal{F}) \perp S(T\mathcal{F}^\perp) \perp \{\mathrm{Rad}(T\mathcal{F}) \oplus l\mathrm{tr}(T\mathcal{F})\}. \tag{2.19}$$

It is easy to check that $\phi D_2 \subseteq S(T\mathcal{F})$. The latter means there exists a subbundle L_2 of rank k in $l\mathrm{tr}(T\mathcal{F})$ such that $\phi L_2 \subseteq S(T\mathcal{F})$. Also there exists a subbundle $\mathcal{S}$ in $S(T\mathcal{F}^\perp)$ such that $\phi\mathcal{S} \subseteq S(T\mathcal{F})$. The bundle $\{\phi D_2 \oplus \phi L_2 \oplus \phi\mathcal{S}\}$ is a subbundle of $S(T\mathcal{F})$ of rank at least 2. Therefore there exists a non-degenerate invariant distribution $\mathcal{D}_0$ of even rank such that

$$S(T\mathcal{F}) = \{\phi D_2 \oplus \phi L_2 \oplus \phi\mathcal{S}\} \perp D_0. \tag{2.20}$$

Thus, in this case, L is a quasi generalized CR-null submanifold immersed in M (see [14] for more details of quasi generalized CR concept). Therefore, we have the following theorem.

Theorem 2.2. *Let M be a $(2n+1)$-dimensional indefinite l.c. almost cosymplectic manifold of index q, where $0 < q < 2n+1$ with $\mathtt{Sign}(B) = \emptyset$. Then*

(i) *If $c \neq 0$, then the index of each leaf L of $\mathcal{F}$ is given by $\mathrm{ind}(L) = q - s$, where $s = \mathrm{ind}((T\mathcal{F})^c)$ with $0 \leq s \leq r$. Moreover, L is totally geodesic r codimensional semi-Riemannian submanifold of (M, g) if and only if the Lee form ω is parallel.*

(ii) *If $c = 0$, then each leaf of $\mathcal{F}$ is either a null hypersurface or a quasi generalized CR-null submanifold of (M, g).*

Example 2.1 shows that $c = g(B, B) = \frac{1}{2}e^{-2(z+x_1y_1)}\{-2x_1^2 - y_1^2 + 2\}$, which is always different from zero, since $-2x_1^2 - y_1^2 + 2 \neq 0$ for $x_1 > 0$ and $y_1 > 0$. The item (ii) in Theorem 2.2 is supported by the following example.

Example 2.3. Consider M a 7-dimensional semi-Riemannian manifold $M^7 = \{p \in \mathbb{R}^7 \,|\, x_1 > 0, y_3 > 0\}$, where $p = (x_1, x_2, x_3, y_1, y_2, y_3, z)$ are the standard coordinates in $\mathbb{R}^7$.

The vectors fields $X_1 = \frac{1}{x_1+y_3}\partial x_1$, $Y_1 = \frac{1}{x_1+y_3}\partial y_1$, $X_2 = \frac{1}{x_1+y_3}\partial x_2$, $Y_2 = \frac{1}{x_1+y_3}\partial y_2$, $X_3 = \frac{1}{x_1+y_3}\partial x_3$, $Y_3 = -\frac{1}{x_1+y_3}\partial y_3$, $Z = \frac{1}{x_1+y_3}\partial z$ are linearly independent at each point of M. Let $\overline{g}$ be the indefinite metric on $\overline{M}$ defined by $g(X_i, X_j) = g(Y_i, Y_j) = -\delta_{i,j}$ for any $i, j = 1, 2$, $g(X_3, X_3) = g(Y_3, Y_3) = 1$, $g(\xi, \xi) = 1$, $g(X_l, X_k) = g(Y_l, Y_k) = 0$ for all $l \neq k, l, k = 1, 2, \ldots, 7$. Let η be the 1-form on M defined by $\eta = (x_1 + y_3)dz$ and the structure vector field given by $\xi = \frac{1}{x_1+y_3}\partial z$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi X_1 = -Y_1$, $\phi Y_1 = X_1$, $\phi X_2 = -Y_2$, $\phi Y_2 = X_2, \phi X_3 = Y_3$, $\phi Y_3 = -X_3$, $\phi X_4 = Y_4$, $\phi Y_4 = -X_4$, $\phi\xi = 0$. By linearity of ϕ and g the quadruplet (ϕ, ξ, η, g) defines an almost contact metric structure on M^7. Take $\sigma = \ln(x_1 + y_3)$. It follows that $\omega = \frac{1}{x_1+y_3}(dx_1 + dy_3)$, then clearly we have $d\eta = \omega \wedge \eta$. The 2-form fundamental is given by $\Phi = (x_1+y_3)^2\{-dx_1\wedge dy_1 - dx_2\wedge dy_2 + dx_3\wedge dy_3\}$, which satisfies $d\Phi = 2\omega \wedge \Phi$. The Lee vector field (i.e., the dual vector field of ω) is given by $B = \frac{1}{(x_1+y_3)^2}(X_1 + Y_3)$. It follows that $c = g(B, B) = 0$ and thus B is a null vector field. It is easy to see that $\omega(\xi) = 0$ and for $p \in M^7$, the distribution $D_p = \{X \in T_pM^7 : \omega(X) = 0\}$ is spanned by $\{X_2, X_3, Y_1, Y_2, B, \xi\}$. The non-vanishing components of the Lie brackets are $[X_{2,3}, B] = \frac{2}{(x_1+y_3)^4}X_{2,3}$ and $[Y_{1,2}, B] = \frac{2}{(x_1+y_3)^4}Y_{1,2}$, which prove that the distribution D is integrable and therefore admits a foliation $\mathcal{F}$ whose leaves are null hypersurfaces immersed in M^7. In this case the anti-Lee vector field $V = -\phi B = \frac{1}{(x_1+y_3)^2}\{X_3 - Y_1\} \in T\mathcal{F}$. The transversal vector field is given by $N = \frac{1}{2}(x_1 + y_3)^2\{-X_1 + Y_3\}$.

Note that if the ambient space M is an indefinite l.c. cosymplectic manifold, then $h = 0$ and $B = \omega(\xi)\xi$ (see [13] and [16]). In this case the condition $c = 0$ implies $\omega(\xi) = 0$. Therefore, we have the following.

Lemma 2.4. *There exist no null hypersurfaces immersed in an indefinite l.c. cosymplectic manifold with* $\mathrm{Sign}(B) = \emptyset$ *such that* $c = 0$ *and* $\omega(\xi) \neq 0$.

From now on, we consider the *leaf* L *of the foliation* $\mathcal{F}$ *to be a null hypersurface immersed in* M *with* $\mathrm{Sign}(B) = \emptyset$ *such that* $c = 0$ *and* $\omega(\xi) \neq 0$ (*Theorem 2.2*).

According to the terminology in [8, p. 79], the portion of $\mathrm{tr}(T\mathcal{F})$ over a leaf L of $\mathcal{F}$ is the null transversal vector bundle of L with respect to the screen distribution $S(T\mathcal{F})|_L$ (see [7] for more details). By definition of null hypersurface, (2.18) and (2.19), we obtain the decomposition

$$TM = S(T\mathcal{F}) \perp \{T\mathcal{F}^{\perp} \oplus \mathrm{tr}(T\mathcal{F})\} = T\mathcal{F} \oplus \mathrm{tr}(T\mathcal{F}). \tag{2.21}$$

Let $\tan : TM \longrightarrow T\mathcal{F}$ and $\mathrm{tra} : TM \longrightarrow \mathrm{tr}(T\mathcal{F})$ be the projections associated with (2.21). We set

$$\begin{aligned} \nabla^{\mathcal{F}}_X Y &= \tan(\nabla_X Y), \quad \mathcal{H}(X, Y) = \mathrm{tra}(\nabla_X Y) \\ A_V X &= -\tan(\nabla_X V), \quad \nabla^{\mathrm{tr}}_X V = \mathrm{tra}(\nabla_X V) \end{aligned}$$

for any X, $Y \in T\mathcal{F}$ and any $V \in \mathrm{tr}(T\mathcal{F})$. Then $\nabla^{\mathcal{F}}$ is a connection in $T\mathcal{F} \longrightarrow M$, $\mathcal{H}$ is a symmetric $\mathrm{tr}(T\mathcal{F})$-valued bilinear form on $T\mathcal{F}$, A_V is an endomorphism of $T\mathcal{F}$, and ∇^{tra} is a connection in $\mathrm{tr}(T\mathcal{F}) \longrightarrow M$. Then, the Gauss and Weingarten formulae of $\mathcal{F}$ in (M, g) are giving by

$$\nabla_X Y = \nabla^{\mathcal{F}}_X Y + \mathcal{H}(X, Y), \quad \nabla_X V = -A_V X + \nabla^{\mathrm{tr}}_X V. \tag{2.22}$$

Similarly, if P denotes the projection morphism of $T\mathcal{F}$ onto $S(T\mathcal{F})$ with respect to the decomposition (2.12), we obtain

$$\nabla^{\mathcal{F}}_X PY = \nabla^{*\mathcal{F}}_X PY + \mathcal{H}^*(X, PY), \quad \nabla^{\mathcal{F}}_X U = -A^*_U X - \nabla^{*\mathrm{tr}}_X U. \tag{2.23}$$

The details given in [8, p. 83 and 85] show clearly that the pointwise restrictions of $\nabla^{\mathcal{F}}$, ∇^{tr}, $\mathcal{H}$ and A_V to a leaf L of the foliation $\mathcal{F}$ are respectively the induced connections, the second fundamental form and the shape operator of L in (M, g). The pointwise restrictions of $\nabla^{*\mathcal{F}}$, $\mathcal{H}^*$ and A^*_U to L are respectively the linear connection, the second fundamental form and the shape operator on the vector bundle $S(TL) \longrightarrow L$, while the pointwise restriction of $\nabla^{*\mathrm{tr}}$ to L is linear of connection on the vector bundle $TL^{\perp} \longrightarrow L$.

Keeping the same notations of geometric objects above for the pointwise restrictions to a leaf L of $\mathcal{F}$, and locally supposing $\{B, N\}$ is a pair of sections on a coordinate neighborhood $\mathcal{U} \cap L \subset L$ (see [8, Theorem 1.1, p. 79], then the local Gauss-Weingarten equations of $\mathcal{F}$ are given by

$$\nabla_X Y = \nabla^{\mathcal{F}}_X Y + \mathcal{B}(X, Y)N, \quad \nabla_X N = -A_N X + \tau(X)N, \tag{2.24}$$

$$\nabla^{\mathcal{F}}_X PY = \nabla^{*\mathcal{F}}_X PY + \mathcal{C}(X, PY)B, \quad \nabla^{\mathcal{F}}_X B = -A^*_B X - \tau(X)B \tag{2.25}$$

for all $B \in \Gamma(TL^{\perp})$, $N \in \Gamma(\mathrm{tr}(TL))$, where $\mathcal{B}$ and $\mathcal{C}$ are the local second fundamental forms of L and $S(TL)$, respectively, and τ is a differential 1-form on L. Notice that $\nabla^{*\mathcal{F}}$ is a metric connection on $S(TL)$ while $\nabla^{\mathcal{F}}$ is generally not a metric connection and satisfies the following relation

$$(\nabla^{\mathcal{F}}_X g)(Y,Z) = \mathcal{B}(X,Y)\lambda(Z) + \mathcal{B}(X,Z)\lambda(Y) \tag{2.26}$$

for all $X, Y, Z \in \Gamma(T\mathcal{F})$, where λ is a 1-form on L given $\lambda(\cdot) = \overline{g}(\cdot, N)$. It is well-known from [8] that $\mathcal{B}$ is independent of the choice of $S(TL)$ and it satisfy

$$\mathcal{B}(X,B) = 0, \quad X \in \Gamma(TL). \tag{2.27}$$

The local second fundamental forms $\mathcal{B}$ and $\mathcal{C}$ are related to their shape operators by the following equations $g(A^*_B X, Y) = \mathcal{B}(X,Y)$, $\overline{g}(A^*_B X, N) = 0$, $g(A_N X, PY) = \mathcal{C}(X, PY)$ and $\overline{g}(A_N X, N) = 0$ for all $X, Y \in \Gamma(TL)$. Note that A^*_B is $S(T\mathcal{F})$-valued, self-adjoint and satisfies $A^*_B B = 0$.

In this case, ξ is decomposed as follows.

$$\xi = \xi_S + aB + bN, \tag{2.28}$$

where ξ_S denotes the component of ξ on $S(TL)$ while a and b are non-zero smooth functions on M. If $\xi_S = 0$, then L is called an *ascreen null hypersurface* [11].

Theorem 2.5. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M such that $c = 0$ and $\omega(\xi) \neq 0$. Then L is an ascreen null hypersurface of $\mathcal{F}$ if and only if $\phi\mathrm{Rad}(T\mathcal{F}) = \phi l\mathrm{tr}(T\mathcal{F})$.*

Proof. The proof follows from a straightforward calculation. □

Example 2.6. Consider a 7-dimensional semi-Riemannian manifold $M^7 = \{p \in \mathbb{R}^7 \mid x_1 > 0,\, y_1 > 0,\, z > 0\}$ with a metric of signature $(-,+,+,-,+,+,+)$ with respect to the canonical basis $\{\partial x_i, \partial y_i, \partial z\}$ for $i = 1,2,3$. The vectors fields $X_1 = e^{-\sigma}\partial x_1$, $Y_1 = e^{-\sigma}\partial y_1$, $X_2 = e^{-\sigma}\partial x_2$, $Y_2 = e^{-\sigma}\partial y_2$, $X_3 = e^{-\sigma}\partial x_3$, $Y_3 = -e^{-\sigma}\partial y_3$, $Z = e^{-\sigma}\partial z$, where $\sigma = x_1 + y_1 + \sqrt{2}z$, are linearly independent at each point of M. Let g be the indefinite metric on M defined by $g(X_1, X_1) = g(Y_1, Y_1) = -1$, $g(X_i, X_j) = g(Y_i, Y_j) = \delta_{ij}$ for $i, j = 2, 3$, $g(X_l, X_k) = g(Y_l, Y_k) = 0$ for any $l \neq k, l, k = 1, 2, 3$ and $g(\xi, \xi) = 1$. Let η be the 1-form on M defined by $\eta = e^{\sigma}dz$ and the structure vector field given by $\xi = e^{-\sigma}\partial z$. Let ϕ be the $(1,1)$-tensor field defined by, $\phi X_1 = -Y_1$, $\phi Y_1 = X_1$, $\phi X_2 = -Y_2$, $\phi Y_2 = X_2, \phi X_3 = Y_3$, $\phi Y_3 = -X_3$, $\phi X_4 = Y_4$, $\phi Y_4 = -X_4$, $\phi\xi = 0$. By linearity of ϕ and g the quadruplet (ϕ, ξ, η, g) defines an almost contact metric structure on M. The smooth 1-form ω is locally given by $\omega = d\sigma = dx_1 + dy_1 + \sqrt{2}dz$ and satisfies $d\eta = \omega \wedge \eta$. The 2-form fundamental Φ is given by $\Phi = e^{-2\sigma}\{-dx_1 \wedge dy_1 + dx_2 \wedge dy_2 + dx_3 \wedge dy_3\}$, and verifies $d\Phi = 2\omega \wedge \Phi$. The Lee vector field is given by $B = \partial x_1 + \partial y_1 + \sqrt{2}\partial z$ and satisfies $c = g(B, B) = 0$. Thus B is a null vector field. It is easy to see $\omega(\xi) = e^{-\sigma}\sqrt{2} \neq 0$. The distribution $D_p = \ker \omega_p$ with $p \in M^7$ is spanned by $\{X_2, X_3, Y_2, Y_3, B\}$. The non-vanishing components of the Lie brackets are $[X_{2,3}, B] = 4X_{2,3}$ and $[Y_{2,3}, B] = 4Y_{2,3}$, which prove that the distribution D is integrable and therefore admits a foliation $\mathcal{F}$ of codimension 1 and its leaves are null hypersurfaces immersed in M^7. The transversal vector field is given by $N = -\frac{1}{4}\{\partial x_1 + \partial y_1 - \sqrt{2}\partial z\}$. We can easily see that $\xi = \frac{e^{-\sigma}}{2\sqrt{2}}(B + 4N)$ and also $\phi B = -4\phi N$. Hence, the leaves of $\mathcal{F}$ are ascreen null hypersurfaces of M^7.

From Theorem 2.5, we notice that if L is an ascreen null hypersurface of $\mathcal{F}$ then $\dim(\phi\mathbb{R}B \oplus \phi\mathbb{R}N_V) = 1$ and hence TL decomposes as follows

$$TL = \mathbb{R}B \perp \phi\mathbb{R}B \perp D_0, \tag{2.29}$$

where D_0 is a non-degenerate ϕ invariant distribution, i.e., $\phi D_0 = D_0$.

As the geometry of null hypersurfaces depends on the vector bundles $S(TL)$ and $\mathrm{tr}(TL)$, it is important to investigate the relationship between geometric objects induced by two screen distributions. The components of the structural vector field ξ in (2.28) depends on both the screen distribution $S(TL)$ and the transversal bundle $\mathrm{tr}(TL)$ and this is proven as follows. Suppose a screen $S(TL)$ changes to another screen $S(TL)'$. The following are some of the local transformation equations due to this change (see [8] for details):

$$K_i' = \sum_{j=1}^{2n-1} K_i^j \left(K_j - \epsilon_j c_j B\right), \tag{2.30}$$

$$N'(X) = N - \frac{1}{2}g(K, K)B + K, \tag{2.31}$$

$$\nabla_X^{\mathcal{F}'} Y = \nabla_X^{\mathcal{F}} Y + \mathcal{B}(X, Y)\{\frac{1}{2}g(K, K)B - K\} \tag{2.32}$$

for any $X,\ Y \in \Gamma(TL|_{\mathcal{U}\cap L})$, where $K = \sum_{i=1}^{2n-1} c_i K_i$, $\{K_i\}$ and $\{K_i'\}$ are the local orthonormal bases of $S(T\mathcal{F})$ and $S(T\mathcal{F})'$ with respective transversal sections N and N' for the same null section B. Here c_i and K_i^j are smooth functions on $\mathcal{U}$ and $\{\epsilon_1, \ldots, \epsilon_{2n-1}\}$ is the signature of the basis $\{K_1, \ldots, K_{2n-1}\}$. Denote by κ is the dual 1-form of K, characteristic vector field of the screen change, with respect to the induced metric $g = g|_L$ of $L \hookrightarrow M$ [8], that is,

$$\kappa(X) = g(X, K), \quad \forall\, X \in \Gamma(TL). \tag{2.33}$$

Suppose that the structure vector field ξ in (2.28) is written for a given screen distribution $S(TL)$. Let $\xi = \xi_{S'} + a'B + b'N'$ be another form of the structure vector field ξ in the distribution $S(TL)'$. Then we have the following.

Lemma 2.7. *If the screen distribution $S(TL)$ changes to another screen $S(TL)'$, then $b' = b$ and $\xi_{S'} = \xi_S + \{a - a' + \frac{1}{2}g(K, K)b\}B - bK$. Moreover, the combination in* (2.28) *is independent of $S(TL)$ if and only if 1-form κ vanishes identically on L.*

3. Geometry of non-tangential leaves of $\mathcal{F}$

This section deals with the geometry of the leaves of the foliations $\mathcal{F}$. First of all, we define the following.

A leaf L of $\mathcal{F}$ is called *non-tangential* if ξ satisfies relation (2.28). From (2.9) we can set

$$\nabla_X \xi = hX + \mathcal{A}X, \quad \forall\, X \in \Gamma(T\mathcal{F}), \tag{3.1}$$

where A is a $(1,1)$-tensor field defined by $\mathcal{A}X := \omega(\xi)X - \eta(X)B$. It is easy to see that $\mathcal{A}$ is symmetric with respect to g, i.e., $g(\mathcal{A}X, Y) = g(X, \mathcal{A}Y)$ for any $X, Y \in \Gamma(T\mathcal{F})$, $\mathcal{A}\xi = \omega(\xi)\xi - B$, $\mathcal{A}B = 0$ and $\mathcal{A}\phi X - \phi\mathcal{A}X = \eta(X)\phi B$.

A null hypersurface L of $\mathcal{F}$ with $c = 0$ is said to be *screen conformal* [8] if there exists a non-vanishing smooth function φ such that $A_N = \varphi A^*_B$, and screen homothetic if φ is a constant function.

Theorem 3.1. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is a non-tangential null hypersurface. Then L is screen conformal if h satisfies*

$$h = \nabla^{*\mathcal{F}}\xi_S + \{2\eta - b\lambda - \lambda \circ h\} \otimes B - (\omega \circ h) \otimes N - b\mathbb{I},$$

where $\mathbb{I}$ denotes the identity on $\mathcal{F}$.

Proof. By straightforward calculations using (3.1), (2.28) and Gauss-Weingetein formulas for L one gets, for any $X \in \Gamma(T\mathcal{F})$,

$$\begin{aligned} aA^*_B X + bA_N X = {}& \nabla^{*\mathcal{F}}_X \xi_S + \{X(a) - a\tau(X) + \mathcal{C}(X, \xi_S)\}B \\ & + \{X(b) + b\tau(X) + \mathcal{B}(X, \xi_S)\}N - \mathcal{A}X - hX. \end{aligned} \tag{3.2}$$

Then taking the g-product of (3.2) with B and N in turn, we get

$$X(b) + b\tau(X) + \mathcal{B}(X, \xi_S) = -g(\mathcal{A}X, B) - g(hX, B) \text{ and} \tag{3.3}$$

$$X(a) - a\tau(X) + \mathcal{C}(X, \xi_S) = -\overline{g}(\mathcal{A}X, N) - g(hX, N) \tag{3.4}$$

for any $X \in \Gamma(T\mathcal{F})$. Applying the definition of A to (3.3) and (3.4), we get $g(\mathcal{A}X, B) = 0$ and $g(\mathcal{A}X, N) = b\lambda(X) - \eta(X)$. Hence, (3.2) reduces to

$$\begin{aligned} aA^*_B X + bA_N X = {}& \nabla^{*\mathcal{F}}_X \xi_S + \{\eta(X) - b\lambda(X) - g(hX, N)\}B \\ & - g(hX, B)N - AX - hX, \end{aligned} \tag{3.5}$$

from which our assertion follows and $\varphi = -\frac{a}{b}$, which completes the proof. □

Theorem 3.2. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is non-tangential null hypersurface in M. Then $S(TL)$ is integrable if and only if $g(\nabla^{*\mathcal{F}}_X \xi_S, Y) = g(\nabla^{*\mathcal{F}}_Y \xi_S, X)$ for all $X, Y \in (S(TL))$.*

Proof. By straightforward calculations using (3.5) and the fact that h is symmetric, we have $g([X, Y], N) = \frac{1}{b}\{g(\nabla^{*\mathcal{F}}_X \xi_S, Y) - g(\nabla^{*\mathcal{F}}_Y \xi_S, X)\}$ for any $X, Y \in \Gamma(S(TL))$, which completes the proof. □

The following corollary is obvious.

Corollary 3.3. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\operatorname{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. If L is an ascreen null hypersurface, then $S(TL)$ is integrable.*

Using the Koszul's formula, the non-vanishing components of the covariant derivatives on the basis of the TL defined in Example 2.6 are given by $\nabla_{X^i} X_i = -4N$ and $\nabla_{Y^i} Y_i = -4N$ for $i = 2, 3$, from which we deduce $\mathcal{B}(X_i, X_i) = -4$ and $\mathcal{B}(Y_i, Y_i) = -4$ and zero otherwise. Also, $g(\nabla_U B, N) = 0$ for all $U \in \Gamma(T\mathcal{F})$ which means $\nabla^{\mathcal{F}}_U B$ has no component along $\operatorname{Rad} TL$ and hence $\mathcal{C} = 0$ on $\mathcal{F}$. This means that $S(TL)$ is totally geodesic and therefore integrable.

Next, we study the geometry of distribution D_0 in (2.29). Suppose that $\xi_S = 0$, that is L is an ascreen null hypersurface immersed in M. First, we notice that if $Y \in \Gamma(D_0)$, then $\omega(Y) = \omega(\phi Y) = 0$. Let F be the projection of TL on to D_0. Then by decomposition (2.29) we have

$$X = FX + \lambda(X)B - \frac{1}{b^2} g(X, \phi B)\phi B, \ \ \forall\, X \in \Gamma(T\mathcal{F}). \tag{3.6}$$

Applying ϕ to (3.6) we get

$$\phi X = fX + \frac{1}{b^2} g(X, \phi B)B + \lambda(X)\phi B - \frac{1}{b} g(X, \phi B)\xi \tag{3.7}$$

for all $X \in \Gamma(T\mathcal{F})$, where $fX = \phi FX$.

Theorem 3.4. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\operatorname{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is an ascreen null hypersurface. Then D_0 is integrable if and only if, for any $X, Y \in \Gamma(D_0)$ and $Z \in \Gamma(S(TL))$, $2g((\nabla^{\mathcal{F}}_X f)Y - (\nabla^{\mathcal{F}}_Y f)X, Z) = g(N_1(Y, Z), fX) - g(N_1(X, Z), fY)$, and in this case f is anti-symmetric on $S(TL)$.*

Proof. Let $X, Y \in \Gamma(D_0)$, then $\nabla_X \phi Y = \nabla_X fY$. Then, using this equation together with (2.7) we derive

$$\begin{aligned}
&2g((\nabla^{\mathcal{F}}_X f)Y, Z) \\
&= -2\lambda(Z)\mathcal{B}(X, fY) - 2\lambda(\nabla^{\mathcal{F}}_X Y)\omega(\phi Z) + \frac{2}{b^2} g(\nabla^{\mathcal{F}}_X Y, \phi B)\omega(Z) \\
&\quad - \frac{2}{b} g(\nabla^{\mathcal{F}}_X Y, \phi B)\eta(Z) - 2g(X, \phi Y)\omega(Z)B - g(X, Y)\omega(\phi Z) \\
&\quad g(N_1(Y, Z), \phi X), \quad \forall\, Z \in \Gamma(T\mathcal{F}).
\end{aligned} \tag{3.8}$$

Then from (3.8) we get

$$\begin{aligned}
&2g((\nabla^{\mathcal{F}}_X f)Y - (\nabla^{\mathcal{F}}_Y f)X, Z) + 2\lambda(Z)\{\mathcal{B}(X, fY) - \mathcal{B}(Y, fX)\} \\
&= \frac{2}{b^2} g([X, Y], \phi B)\omega(Z) - 2\lambda([X, Y])\omega(\phi Z) - \frac{2}{b} g([X, Y], \phi B)\eta(Z) \\
&\quad + g(N_1(Y, Z), \phi X) - g(N_1(X, Z), \phi Y) + 4g(\phi X, Y)\omega(Z).
\end{aligned} \tag{3.9}$$

Hence, from (3.9) we can see that if D_0 is integrable, then

$$2g((\nabla^{\mathcal{F}}_X f)Y - (\nabla^{\mathcal{F}}_Y f)X, Z) = g(N_1(Y, Z), fX) - g(N_1(X, Z), fY)$$

for all $Z \in \Gamma(S(T\mathcal{F}))$. Conversely, using this relation and (3.9) we can easily see that $g([X, Y], \phi B) = 0$ and $\lambda([X, Y]) = 0$, which together shows that D_0 is integrable. $\square$

Corollary 3.5. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is an ascreen null hypersurface. Then D_0 is integrable if and only if, $\mathcal{B}(X, fY) - \mathcal{B}(Y, fX) = \frac{1}{2\lambda(Z)}\{g(N_1(Y, Z), fX) - g(N_1(X, Z), fY)\}$, $\forall X, Y \in \Gamma(D_0)$, $Z \in \Gamma((TL^{\perp}))$.*

A leaf L of $\mathcal{F}$ will be called D_0-totally geodesic if for any $X, Y \in \Gamma(D_0)$ we have $h(X, Y) = 0$, or equivalently, $\mathcal{B}(X, Y) = 0$.

Theorem 3.6. *Let L be a leaf of a foliation $\mathcal{F}$ in an l.c. almost cosymplectic manifold M with* $\mathrm{Sign}(B) = \emptyset$ *such that $c = 0$ and $\omega(\xi) \neq 0$. Suppose that L is an ascreen null hypersurface. Then L is D_0-totally geodesic if and only if $h + \mathcal{A} = -\omega(\xi)A_N$ on D_0.*

Proof. By straightforward calculations, using (2.2), (2.24) and (3.1), we have

$$\overline{g}(\mathcal{H}(X, Y), B) = \overline{g}(\phi\nabla_X Y, \phi B) - \omega(\xi)g((h + \mathcal{A})X, Y) \tag{3.10}$$

for any $X, Y \in \Gamma(D_0)$. Now, applying (2.24) to (3.10) we get

$$\overline{g}(\mathcal{H}(X, Y), B) = \frac{1}{2}\mathcal{B}(X, Y) - b^2\lambda(\nabla^{\mathcal{F}}_X Y) - \omega(\xi)g((h + \mathcal{A})X, Y),$$

from which we deduce that $\mathcal{B}(X, Y) = -2b^2\lambda(\nabla^{\mathcal{F}}_X Y) - 2\omega(\xi)g((h + \mathcal{A})X, Y)$, which completes the proof. $\square$

It is important to investigate the relationship between some geometric objects induced, studied above, with the change of the screen distributions. We know that the local second fundamental form $\mathcal{B}$ of L on $\mathcal{U} \cap L$ is independent of the vector bundles $(S(TL), S(TL^{\perp}))$ and $\mathrm{tr}(TL)$. This means that all results above depending only on $\mathcal{B}$ are stable with respect to any change of those vector bundles. Let P and P' be projections of TL on $S(TL)$ and $S(TL)'$, respectively, with respect to the orthogonal decomposition of TL. Any vector field X on $L \hookrightarrow M$ can be written as $X = PX + \lambda(X)B = P'X + \lambda'(X)B$ with $\lambda'(X) = \lambda(X) + \kappa(X)$. Then we have $P'X = PX - \kappa(X)B$ and $\mathcal{C}'(X, P'X) = \mathcal{C}'(X, PY)$. The relationship between the local second fundamental forms $\mathcal{C}$ and $\mathcal{C}'$ of the screen distributions $S(TL)$ and $S(TL)'$, respectively is given using (2.31) by $\mathcal{C}'(X, PY) = \mathcal{C}(X, PY) - \frac{1}{2}\kappa(\nabla^{\mathcal{F}}_X PY + \mathcal{B}(X, Y)K)$. All equations in this section depending only on the local second fundamental form $\mathcal{C}$ (making equations non unique), are independent of $S(TL)$ if and only if $\kappa(\nabla^{\mathcal{F}}_X PY + \mathcal{B}(X, Y)K) = 0$.

Using the changes $\tau'(X) = \tau(X) + \mathcal{B}(X, K)$ and $A^{*'}_B X = A^*_B X - \mathcal{B}(X, K)B$, the linear connections $\nabla^{*\mathcal{F}}$ and $\nabla^{*\mathcal{F}'}$ associated to the change are related by

$$\begin{aligned}\nabla^{*\mathcal{F}'}_X P'Y = \nabla^{*\mathcal{F}}_X PY - \mathcal{B}(X, PY)K - \kappa(Y)A^*_B X - X(\kappa(Y))B \\ - \{\kappa(Y)\tau(X) + \frac{1}{2}\kappa(\nabla^{\mathcal{F}}_X PY + \mathcal{B}(X, Y)K) \\ - \frac{1}{2}\mathcal{B}(X, PY)g(K, K)\}B.\end{aligned} \tag{3.11}$$

4. Higher order geodesibility of leaves of $\mathcal{F}$

Let L be a leaf of the foliation. In this section L is considered to be an ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold M with $\mathrm{Sign}(B) = \emptyset$ such that $c = 0$ and $\omega(\xi) \neq 0$. Denote the vector ξ_t given in (2.4) by $\overline{Q}$. Since L is an ascreen null hypersurface, we have

$$\overline{Q} = \xi_t = e^{\sigma(t)}\xi = e^{\sigma(t)}(aB + bN). \tag{4.1}$$

Let denote the tangential part $ae^{\sigma_t}B$ of $\overline{Q}$ by Q. Then

$$Q = \overline{Q} - be^{\sigma(t)}N. \tag{4.2}$$

Now, we study the umbilicity of L via the divergence of T_rQ, where T_r denotes the Newton transformation with respect to the operator A_N. Applying ∇_X to $\overline{Q}$ and using (3.1) we have

$$\nabla_X \overline{Q} = X(\sigma(t))\overline{Q} + e^{\sigma(t)}(h + \mathcal{A})X, \quad \forall\, X \in \Gamma(T\mathcal{F}). \tag{4.3}$$

In a similar way using (4.2) and (4.3) we have

$$\begin{aligned}\nabla^{\mathcal{F}}_X Q = e^{\sigma(t)}(h + \mathcal{A})X + be^{\sigma(t)}A_N X + X(\sigma(t))\overline{Q} \\ - \{X(b)e^{\sigma(t)} + X(\sigma(t))be^{\sigma(t)} + be^{\sigma(t)}\tau(X) + \mathcal{B}(X, Q)\}N\end{aligned} \tag{4.4}$$

for any $X \in \Gamma(T\mathcal{F})$. Then from (4.4) we deduce that

$$g(\nabla^{\mathcal{F}}_X Q, Y) = e^{\sigma(t)}g((h + \mathcal{A})X, Y) + be^{\sigma(t)}g(A_N X, Y) \text{ and,} \tag{4.5}$$

$$g(\nabla^{\mathcal{F}}_X Q, N) = X(\sigma(t))g(\overline{Q}, N) + e^{\sigma(t)}g(hX, N) \tag{4.6}$$

for any $X \in \Gamma(T\mathcal{F})$ and $Y \in \Gamma(S(T\mathcal{F}))$.

Proposition 4.1. *Let $(L, g, c = 0)$ be an ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold, with $\mathrm{Sign}(B) = \emptyset$, $\omega(\xi) \neq 0$ and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric, then there exists a pair $\{B, N\}$ on $\mathcal{U} \subset L$ such that the corresponding 1-form τ vanishes on any $\mathcal{U} \cap L$. Moreover, $g(\overline{Q}, B) \neq 0$ and $g(\overline{Q}, N) \neq 0$.*

Proof. Since L is ascreen, then $\overline{Q} = e^{\sigma(t)}\xi = e^{\sigma(t)}(aB+bN)$ and thus, $g(\overline{Q}, B) = be^{\sigma(t)} \neq 0$ and $g(\overline{Q}, N) = ae^{\sigma(t)} \neq 0$. Furthermore, since the Ricci tensor with respect $\nabla^{\mathcal{F}}$ is symmetric, then the induced 1-form τ is closed [8]. That is $d\tau = 0$; so we can set $\tau = d\alpha$. Thus, $\tau(X) = X(\alpha)$. If we take $\overline{B} = fB$ and

$\overline{N} = \frac{1}{f}N$, then the corresponding 1-form $\overline{\tau}$ is given by $\overline{\tau}(X) = \overline{g}(\nabla^t_X \overline{N}, \overline{B}) = -X(\log f) + \tau(X)$, where f is a smooth function. Then, one can choose $f = e^{\alpha}$ and hence $\overline{\tau}(X) = 0$ for any $X \in \Gamma(T\mathcal{F}|_{\mathcal{U}})$. Since $\overline{g}(\overline{Q}, B) \neq 0$ and $\overline{g}(\overline{Q}, N) \neq 0$, then $\{\overline{B}, \overline{N}\}$ are the corresponding vectors which satisfies Proposition 4.1. $\square$

We have seen that if L is ascreen null hypersurface immersed in an l.c. almost cosymplectic manifold M with $\mathrm{Sign}(B) = \emptyset$, $c = 0$ and $\omega(\xi) \neq 0$, then $S(TL)$ is integrable (see Corollary 3.3). Further still, A is screen-valued and $AB = 0$, which leads to $A_N B = 0$. The operator A_N is also symmetric on $S(TL)$ and hence diagonalizable. Let $l_0 = 0, l_1, \ldots, l_m$ its principal curvatures with respect to the quasi-orthonormal basis $\{B, Z_1, \ldots, Z_m\}$, where $\{Z_1, \ldots, Z_m\}$ is the basis $S(TL)$. Associated to the operator A_N are the m algebraic invariants $S_r = e_r(l_0, l_1, \ldots, l_m)$, where $e_r : \mathbb{R}^{m+1} \longrightarrow \mathbb{R}$ denotes the r-th symmetric polynomial in variables $l_0, l_1, \ldots, l_m$. We usually set $S_0 = 1$ and it is also easy to see that $S_1 = \mathrm{tr}(A_N)$, the mean curvature. Furthermore, S_r is called the r-th mean curvature with respect to A_N. Then the Newton transformations T_r with respect to the operator A_N are defined by $T_r : TL \longrightarrow TL$ and explicitly given by the recurrence relation

$$T_r = (-1)^r S_r \mathbb{I} + A_N \circ T_{r-1}, \quad 0 \leq r \leq m. \tag{4.7}$$

It is important to know that T_r is also symmetric and commutes with A_N. Let $H_r = \binom{m+1}{r}^{-1} S_r$ denote the normalized mean curvature with respect to A_N and let further $c_r = (m+1-r)\binom{m+1}{r}$. The following properties of T_r can be deduced from (4.7).

$$\mathrm{tr}(T_r) = (-1)^r (m+1-r) S_r = (-1)^r c_r H_r, \tag{4.8}$$

$$\mathrm{tr}(A_N \circ T_r) = (-1)^r (r+1) S_{r+1} = (-1)^r c_r H_{r+1}. \tag{4.9}$$

Details on Newton transformations can be found in [1], [6] and many more references therein.

Note that the interrelation between the second fundamental forms of the null hypersurface L and its screen distribution and their respective shape operators indicates that the null geometry depends on the choice of a screen distribution. By [8, p. 87], A_N and $A'_{N'}$ are related by

$$\begin{aligned} A'_{N'}X = {} & A_N X + \delta(X)B + \sum_{j=1}^{2n-1} \mu_j(X) K_j - \sum_{j=1}^{2n-1} c_j \nabla^{\mathcal{F}}_X K_j \\ & - \frac{1}{2} g(K, K) A^*_B X, \end{aligned} \tag{4.10}$$

where $\delta = \sum_{j=1}^{2n-1} \{\epsilon_j c_j X(c_j) - \tau(X)\epsilon_j (c_j)^2 + \frac{1}{2}\epsilon_j (c_j)^2 \mathcal{B}(X, K) - c_j \mathcal{C}(X, K_j)\}$ and $\mu_j = c_j(\tau(X) + \mathcal{B}(X, K)) - X(c_j)$.

The dependence of T_r on $S(TL)$ is as follows. Let $Z_i \in \Gamma(S(TL))$ be an eigenvector of A_N, then it is easy to show that $T_r Z_i = (-1)^r S^i_r Z_i$. Notice that

$(-1)^r S_r^i$ is an eigenvalue of T_r corresponding to eigenvector Z_i. Then by direct calculations we have

$$T_r Z_i = (-1)^r S_r \mathbb{I} + (-1)^{r-1} S_{r-1}^i A_N Z_i, \text{ and} \tag{4.11}$$

$$T'_r Z_i = (-1)^r S'_r \mathbb{I} + (-1)^{r-1} S'^{i}_{r-1} A'_{N'} Z_i. \tag{4.12}$$

Subtracting (4.11) from (4.12) we deduce that

$$\begin{aligned} T'_r &= T_r + (-1)^r (S'_r - S_r)\mathbb{I} + \mathfrak{S}_{r-1} A_N \\ &+ (-1)^{r-1} S'^i_{r-1} \left\{ \delta B + \sum_{j=1}^{2n-1} \mu_j K_j - \sum_{j=1}^{2n-1} c_j \nabla^{\mathcal{F}} K_j - \frac{1}{2} g(K,K) A^*_B \right\}, \end{aligned} \tag{4.13}$$

where $\mathfrak{S}_r = (-1)^r (S'^i_r - S^i_r)$. Hence, from (4.13) we can see that the operators T_r depends on a chosen section N and on $S(TL)$. Note that T_r is unique if and only if M is r-maximal (i.e., $S_r = 0$ for all r).

Next, the divergence of T_r on the screen distribution will be denoted by $\operatorname{div}^{\nabla^*}(T_r)$ and given by

$$\operatorname{div}^{\nabla^*}(T_r) = \sum_{i=1}^{m} (\nabla^{\mathcal{F}}_{Z_i} T_r) Z_i. \tag{4.14}$$

Since L is null, the divergence $\operatorname{div}^{\nabla^{\mathcal{F}}}(Y)$ of a vector $Y \in \Gamma(T\mathcal{F})$ with respect to the degenerate metric g on L is intrinsically defined by (see [9, p. 136], for more details and references therein)

$$\operatorname{div}^{\nabla^{\mathcal{F}}}(Y) = \operatorname{div}^{\nabla^*}(Y) + g(\nabla^{\mathcal{F}}_B Y, N). \tag{4.15}$$

Let dV_M be the volume element of M with respect to g and a given orientation. Then, we denote the volume form on $\mathcal{F}$ by

$$dV = i_N dV_M,$$

where i_N is the contraction with respect to the vector field N. We have the following.

Theorem 4.2. *Let $(L, g, c = 0)$ be a compact ascreen null hypersurface of a $\mathcal{F}$ in an l.c. almost cosymplectic manifold M of constant sectional curvature, with* $\operatorname{Sign}(B) = \emptyset$, *$\omega(\xi) \neq 0$ and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric, then*

$$\int_L (B \cdot \overline{g}(T_r Q, N) + e^{\sigma(t)} \operatorname{tr}(T_r \circ h) + (-1)^r c_r \omega(\overline{Q})\{H_r + H_{r+1}\}) dV = 0.$$

Proof. Our proof follows by computation of the divergence of the vector field $T_r Q$ from (4.15). That is;

$$\operatorname{div}^{\nabla^{\mathcal{F}}}(T_r Q) = \operatorname{div}^{\nabla^*}(T_r Q) + \overline{g}(\nabla^{\mathcal{F}}_B T_r Q, N). \tag{4.16}$$

Applying (4.14) to (4.16) we obtain

$$\operatorname{div}^{\nabla^{\mathcal{F}}}(T_rQ) = \overline{g}(\operatorname{div}^{\nabla^*}(T_r), Q) + \sum_{i=1}^{m} \epsilon_i \overline{g}(\nabla^{\mathcal{F}}_{Z_i}Q, T_rZ_i) + \overline{g}(\nabla_B T_rQ, N),$$

from which, after applying (4.5), Proposition 4.1 and the fact that M is a space form of constant sectional curvature, we get

$$\begin{aligned}\operatorname{div}^{\nabla^{\mathcal{F}}}(T_rQ) &= e^{\sigma(t)}\operatorname{tr}(T_r \circ h) + e^{\sigma(t)}\operatorname{tr}(T_r \circ \mathcal{A})\\ &\quad + be^{\sigma(t)}\operatorname{tr}(T_r \circ A_N) + B \cdot \overline{g}(T_rQ, N).\end{aligned} \tag{4.17}$$

When L is ascreen, we see from (3.5) that A is screen-valued operator and in fact $AX = \omega(\xi)X$ for any $X \in \Gamma(S(T\mathcal{F}))$. Thus, (4.16) reduces to

$$\begin{aligned}\operatorname{div}^{\nabla^{\mathcal{F}}}(T_rQ) &= e^{\sigma(t)}\operatorname{tr}(T_r \circ h) + e^{\sigma(t)}\omega(\xi)\operatorname{tr}(T_r)\\ &\quad + be^{\sigma(t)}\operatorname{tr}(T_r \circ A_N) + B \cdot \overline{g}(T_rQ, N).\end{aligned} \tag{4.18}$$

Finally, our result follows from (4.18) by considering (4.14) and the fact that L is compact. □

Next we look at some applications of Theorem 4.2 in which the functions $a = \eta(N)$, $b = \eta(B) = \omega(\xi)$ and $\sigma(t)$ are all constants. Hypersurfaces with constant higher order mean curvatures are of great importance to modern differential geometry and have been a focal point of study for the past decades. For instance, in the analysis of minimal surfaces (surfaces with zero mean curvatures) and in the study of physical interfaces between fluids, which are assumed to have constant mean curvatures (see [2] and many more references therein). We suppose that L is of constant higher order mean curvature in the rest of the paper.

Theorem 4.3. *Under the assumptions of Theorem 4.2, if the functions a, b and σ are all constant, then*

$$\int_L (aB(S_r) + (-1)^{r-1}\operatorname{tr}(T_r \circ h) + \omega(\xi)c_r\{H_r + H_{r+1}\})dV = 0. \tag{4.19}$$

Proof. By Proposition 4.1 and the fact that $B(\overline{g}(T_rQ, N)) = (-1)^r B(S_r\lambda(Q))$ $= (-1)^r ae^{\sigma(t)}B(S_r)$, we complete the proof. □

Theorem 4.4. *Let $(L, g, c = 0)$ be a compact ascreen null hypersurface of a $\mathcal{F}$ in an l.c. almost cosymplectic manifold M of constant sectional curvature, with $\operatorname{Sign}(B) = \emptyset$, $\omega(\xi) \neq 0$ and a conformal vector field $\overline{Q} = e^{\sigma(t)}\xi$. Let a, b and σ be constant such that h is tangent to $\mathcal{F}$. If the Ricci tensor of the induced connection $\nabla^{\mathcal{F}}$ is symmetric and H_1 is constant, then $S(TL)$ is totally geodesic.*

Proof. By considering $r = 0$ in (4.19) and multiplying the resultant equation by H_1, we get

$$\int_L (H_1 + H_1^2) dV = 0. \tag{4.20}$$

Then substituting $r = 1$, (4.19) and using T_1, properties of h, the fact that $B(S_1) = 0$, we get

$$\int_L (H_1 + H_2) dV = 0. \tag{4.21}$$

Then, from (4.20) and (4.21) we have $\int_L (H_1^2 - H_2) dV = 0$. But, for $l_1 = \cdots = l_m$ we have

$$H_1^2 - H_2 = \frac{1}{m(m-1)} \left(\frac{m-1}{m} \left(\sum_{i=1}^m l_i \right)^2 - 2 \sum_{i=1}^m l_i^2 \right). \tag{4.22}$$

Using Cauchy-Schwarz inequality on (4.22) we get that

$$H_1^2 - H_2 \geq \frac{m-2}{m(m-1)} \sum_{i=1}^m l_i^2 \geq 0, \tag{4.23}$$

with equality if $l_1 = \cdots = l_m = 0$. Hence, $S(T\mathcal{F})$ is totally geodesic. □

Corollary 4.5. *Under the assumptions of Theorem 4.4, if H_2 is a positive constant (or H_{r-1} and H_r for $r = 1, \ldots, m$, are both constant) and $\mathrm{tr}(T_r \circ h) = 0$, then $S(TL)$ is also totally geodesic.*

Note that all results above depending only on the local second fundamental form $\mathcal{B}$ are independent of any change of screen distributions.

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References

[1] K. Andrzejewski, W. Kozlowski, and K. Niedzialomski, *Generalized Newton transformation and its applications to extrinsic geometry*, Asian J. Math. **20** (2016), no. 2, 293–322.

[2] K. Andrzejewski and P. G. Walczak, *The Newton transformation and new integral formulae for foliated manifolds*, Ann. Global Anal. Geom. **37** (2010), no. 2, 103–111.

[3] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, vol. 203. Birkhüser, New York, 2002.

[4] M. Capursi and S. Dragomir, *On manifolds admitting metrics which are locally conformal to cosymplectic metrics: their canonical foliations, Boothby-Wang fiberings, and real homology type*, Colloq. Math. **64** (1993), no. 1, 29–40.

[5] D. Chinea and J. C. Marrero, *Conformal changes of almost cosymplectic manifolds*, Demonstratio Math. **25** (1992), no. 3, 641–656.

[6] J. Dong and X. Liu, *Totally umbilical lightlike hypersurfaces in Robertson-Walker spacetimes*, ISRN Geom. **2014** (2014), Art. ID 974695, 10 pages.

[7] S. Dragomir and K. L. Duggal, *Indefinite locally conformal Kähler manifolds*, Differential Geom. Appl. **25** (2007), no. 1, 8–22.

[8] K. L. Duggal and A. Bejancu, *Lightlike submanifolds of semi-Riemannian manifolds and applications*, Mathematics and Its Applications. Kluwer Publishers, 1996.
[9] K. L. Duggal and B. Sahin, *Differential Geometry of Lightlike Submanifolds, Frontiers in Mathematics*, Birkhäuser Verlag, Basel, 2010.
[10] S. I. Goldberg and K. Yano, *Integrability of almost cosymplectic structures*, Pacific J. Math. **31** (1969), 373–382.
[11] D. H. Jin, *Ascreen lightlike hypersurfaces of a semi-Riemannian space form with a semi-symmetric non-metric connection*, Commun. Korean Math. Soc. **29** (2014), no. 2, 311–317.
[12] D. N. Kupeli, *Singular semi-Riemannian geometry, Mathematics and Its Applications*, 366, Kluwer Academic Publishers, 1996.
[13] F. Massamba and A. Maloko Mavambou, *A class of locally conformal almost cosymplectic manifolds*, Bull. Malays. Math. Sci. Soc. DOI 10.1007/s40840-016-0309-3, 2016.
[14] F. Massamba and S. Ssekajja, *Quasi generalized CR-lightlike submanifolds of indefinite nearly Sasakian manifolds*, Arab. J. Math. **5** (2016), no. 2, 87–101.
[15] Z. Olszak, *On almost cosymplectic manifolds*, Kodai Math. J. **4** (1981), no. 2, 239–250.
[16] ———, *Locally conformal almost cosymplectic manifolds*, Colloq. Math. **57** (1989), no. 1, 73–87.
[17] I. Vaisman, *Conformal changes of almost contact metric structures*, Geometry and differential geometry (Proc. Conf., Univ. Haifa, Haifa, 1979), pp. 435–443, Lecture Notes in Math., 792, Springer, Berlin, 1980.

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