

CENTROIDS AND SOME CHARACTERIZATIONS OF CATENARIES

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ABSTRACT. For every interval $[a, b]$, we denote by (\bar{x}_A, \bar{y}_A) and (\bar{x}_L, \bar{y}_L) the geometric centroid of the area under a catenary $y = k \cosh((x-c)/k)$ defined on this interval and the centroid of the curve itself, respectively. Then, it is well-known that $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$.

In this paper, we show that one of $\bar{x}_L = \bar{x}_A$ and $\bar{y}_L = 2\bar{y}_A$ characterizes the family of catenaries among nonconstant C^2 functions. Furthermore, we show that among nonconstant and nonlinear C^2 functions, $\bar{y}_L/\bar{x}_L = 2\bar{y}_A/\bar{x}_A$ is also a characteristic property of catenaries.

1. Introduction

A well-known property of the catenary $y = k \cosh((x-c)/k)$, $k > 0$ is that the ratio of the area under the curve to the arc length of the curve is independent of the interval over which these quantities are concurrently measured. For a positive C^1 function $y(x)$ defined on an interval I and an interval $[a, b] \subset I$, we consider the area $A(a, b)$ over the interval $[a, b]$ and the arc length $L(a, b)$ of the graph of $y(x)$. Then, the catenary $y = k \cosh((x-c)/k)$, $k > 0$ satisfies for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$. This property characterizes the family of catenaries $y = k \cosh((x-c)/k)$ among nonconstant C^2 functions ([11]). Thus, we have the following.

Proposition 1.1. *For a nonconstant positive C^2 function $y(x)$ defined on an interval I , the followings are equivalent.*

- (1) *There exists a positive constant k such that for every interval $[a, b] \subset I$, $A(a, b) = kL(a, b)$.*
- (2) *The function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$, where k is a positive constant.*

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- (3) For some $k > 0$ and $c \in \mathbb{R}$,

$$y(x) = k \cosh\left(\frac{x-c}{k}\right).$$

Two higher dimensional generalizations of Proposition 1.1 were established in [1]. For a positive C^1 function $y(x)$ defined on an interval I and an interval $[a, b] \subset I$, we denote by $(\bar{x}_A, \bar{y}_A) = (\bar{x}_A(a, b), \bar{y}_A(a, b))$ and $(\bar{x}_L, \bar{y}_L) = (\bar{x}_L(a, b), \bar{y}_L(a, b))$ the geometric centroid of the area under the graph of $y(x)$ defined on this interval and the centroid of the graph itself, respectively. Then, for a catenary we have the following ([11]).

Proposition 1.2. *A catenary $y = k \cosh((x-c)/k)$ satisfies the following.*

- (1) For every interval $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.
- (2) For every interval $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.

In this paper, first of all, in Section 2 we establish the following characterization theorem for catenaries.

Theorem 1.3. *For a nonconstant positive C^2 function $y(x)$ defined on an interval I , the followings are equivalent.*

- (1) For every interval $[a, b] \subset I$, $\bar{x}_L(a, b) = \bar{x}_A(a, b)$.
- (2) For every interval $[a, b] \subset I$, $\bar{y}_L(a, b) = 2\bar{y}_A(a, b)$.
- (3) For some $k > 0$ and $c \in \mathbb{R}$,

$$y(x) = k \cosh\left(\frac{x-c}{k}\right).$$

In Section 3, we prove the following characterization theorem for catenaries.

Theorem 1.4. *For a nonconstant and nonlinear positive C^2 function $y(x)$ defined on an interval I , the followings are equivalent.*

- (1) For every interval $[a, b] \subset I$,

$$\frac{\bar{y}_L}{\bar{x}_L} = 2\frac{\bar{y}_A}{\bar{x}_A}.$$

- (2) For some $k > 0$ and $c \in \mathbb{R}$,

$$y(x) = k \cosh\left(\frac{x-c}{k}\right).$$

In order to find the centroid of polygons, see [3]. For the perimeter centroid of a polygon, we refer to [2]. In [9], mathematical definitions of centroid of planar bounded domains were given. For various centroids of higher dimensional simplexes, see [10]. The relationships between various centroids of a quadrangle were given in [5, 8].

Archimedes proved the area properties of parabolic sections and then formulated the centroid of parabolic sections ([12]). Some characterizations of parabolas using these properties were given in [4, 6, 7].

2. Proof of Theorem 1.3

In this section, we prove Theorem 1.3 stated in Section 1.

Suppose that a nonconstant positive C^2 function $y(x)$ defined on an interval I satisfies $\bar{x}_L(a, b) = \bar{x}_A(a, b)$. Then for all $a, b \in I$ with $a < b$, we have

$$(2.1) \quad \int_a^b y(x)dx \int_a^b xw(x)dx = \int_a^b w(x)dx \int_a^b xy(x)dx,$$

where $w(x)$ is a function defined by $w(x) = \sqrt{1 + y'(x)^2}$. Note that (2.1) is valid for all $a, b \in I$.

By differentiating (2.1) with respect to the variable b , the fundamental theorem of calculus gives

$$(2.2) \quad y(b) \int_a^b xw(x)dx + bw(b) \int_a^b y(x)dx = w(b) \int_a^b xy(x)dx + by(b) \int_a^b w(x)dx.$$

With respect to a , we differentiate (2.2) to have

$$(2.3) \quad y(b)aw(a) + bw(b)y(a) = w(b)ay(a) + by(b)w(a),$$

which shows that for all $a, b \in I$

$$(2.4) \quad (b - a)\{y(b)w(a) - y(a)w(b)\} = 0.$$

It follows from (2.4) that on the interval I , $y(x)/w(x)$ is a constant k . That is, the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Hence, Proposition 1.1 implies that (1) \Rightarrow (3).

Now, suppose that a nonconstant positive C^2 function $y(x)$ defined on an interval I satisfies $\bar{y}_L(a, b) = \bar{y}_A(a, b)$. Then for all $a, b \in I$ with $a < b$, we have

$$(2.5) \quad \int_a^b y(x)dx \int_a^b y(x)w(x)dx = \int_a^b w(x)dx \int_a^b y(x)^2dx,$$

where $w(x) = \sqrt{1 + y'(x)^2}$. Note that (2.5) is valid for all $a, b \in I$.

Differentiating (2.5) with respect to b gives

$$(2.6) \quad \begin{aligned} & y(b) \int_a^b y(x)w(x)dx + y(b)w(b) \int_a^b y(x)dx \\ &= w(b) \int_a^b y(x)^2dx + y(b)^2 \int_a^b w(x)dx. \end{aligned}$$

We differentiate (2.6) with respect to a . Then we have

$$(2.7) \quad y(b)y(a)w(a) + y(b)w(b)y(a) = w(b)y(a)^2 + y(b)^2w(a),$$

from which for all $a, b \in I$ we get

$$(2.8) \quad \{y(b) - y(a)\}\{y(b)w(a) - y(a)w(b)\} = 0.$$

We fix a point $a_0 \in I$. Then we have from (2.8)

$$(2.9) \quad \{y(b) - y(a_0)\}\{y(b)w(a_0) - y(a_0)w(b)\} = 0.$$

Let us denote by J the open set given by

$$J = \{b \in I \mid y(b)w(a_0) - y(a_0)w(b) \neq 0\}.$$

We divide by two cases as follows.

Case 1. $J = \phi$. In this case, from the definition of the open set J we get

$$(2.10) \quad \frac{y(b)}{w(b)} = \frac{y(a_0)}{w(a_0)} (= k).$$

Hence, the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Thus, Proposition 1.1 implies that the function $y(x)$ is a catenary.

Case 2. $J \neq \phi$. In this case, it follows from (2.9) that for all $x \in J$, $y(x) = y(a_0)$. We let $k = y(a_0)$ and fix a point $x_0 \in J$. We denote by $K = (x_1, x_2)$ the maximal open interval containing x_0 such that $y(x) = k$. If the maximal interval K satisfies $K = I$, then the function $y(x)$ is a constant function. This contradiction shows that $K \neq I$, and hence one of x_1 and x_2 belongs to I . Thus we may assume that $x_2 \in I$. For a sufficiently small $\epsilon > 0$, the interval $(x_2, x_2 + \epsilon)$ does not intersect J . Hence, it follows from (2.8) with $a = x_2$ that for all $b \in (x_2, x_2 + \epsilon)$

$$(2.11) \quad \frac{y(b)}{w(b)} = \frac{y(x_2)}{w(x_2)} (= k),$$

where we use $y(x_2) = k, y'(x_2) = 0$ and $w(x_2) = 1$. Hence, on the interval $(x_2, x_2 + \epsilon)$ the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Together with (2.11) and Proposition 1.1, the maximality of K shows that

$$(2.12) \quad y(x) = \begin{cases} k, & \text{if } x \in (x_1, x_2], \\ k \cosh\left(\frac{x-x_2}{k}\right), & \text{if } x \in (x_2, x_2 + \epsilon). \end{cases}$$

This yields that the function $y(x)$ cannot be C^2 , a contradiction.

Summarizing the above cases, we see that (2) \Rightarrow (3).

Conversely, it follows from Proposition 1.2 that (3) \Rightarrow (1) and (2). This completes the proof of Theorem 1.3.

3. Proof of Theorem 1.4

In this section, we prove Theorem 1.4 stated in Section 1.

Suppose that a nonconstant positive C^2 function $y(x)$ defined on an interval I satisfies $\bar{y}_L/\bar{x}_L = 2\bar{y}_A/\bar{x}_A$. Then for all $a, b \in I$ with $a < b$, we have

$$(3.1) \quad \int_a^b xy(x)dx \int_a^b y(x)w(x)dx = \int_a^b xw(x)dx \int_a^b y(x)^2dx,$$

where $w(x) = \sqrt{1 + y'(x)^2}$. Note that (2.1) is valid for all $a, b \in I$.

By differentiating (3.1) with respect to b , we obtain

$$(3.2) \quad \begin{aligned} & by(b) \int_a^b y(x)w(x)dx + y(b)w(b) \int_a^b xy(x)dx \\ &= bw(b) \int_a^b y(x)^2 dx + y(b)^2 \int_a^b xw(x)dx. \end{aligned}$$

We differentiate (3.2) with respect to a . Then we have

$$(3.3) \quad by(b)y(a)w(a) + y(b)w(b)ay(a) = bw(b)y(a)^2 + y(b)^2aw(a),$$

which shows that for all $a, b \in I$

$$(3.4) \quad \{by(a) - ay(b)\}\{y(b)w(a) - y(a)w(b)\} = 0.$$

We fix a nonzero $a_0 \in I$. Then we get from (3.4)

$$(3.5) \quad \{by(a_0) - a_0y(b)\}\{y(b)w(a_0) - y(a_0)w(b)\} = 0.$$

Putting $J = \{b \in I \mid y(b)w(a_0) - y(a_0)w(b) \neq 0\}$, we divide by three cases as follows.

Case 1. $J = \phi$. In this case, from the definition of the open set J we get

$$(3.6) \quad \frac{y(b)}{w(b)} = \frac{y(a_0)}{w(a_0)} (= k).$$

Hence, the function $y(x)$ satisfies $y(x) = k\sqrt{1 + y'(x)^2}$. Thus, Proposition 1.1 implies that the function $y(x)$ is a catenary.

Case 2. $J = I$. In this case, from (3.5) we get

$$(3.7) \quad y(x) = kx, \quad k = \frac{y(a_0)}{a_0},$$

which leads a contradiction.

Case 3. $J \neq \phi$ and $J \neq I$. In this case, it follows from (3.5) that for all $x \in J$, $y(x) = kx$, where $k = y(a_0)/a_0$. We fix a point $x_0 \in J$ and denote by $K = (x_1, x_2)$ the maximal open interval containing x_0 such that $y(x) = kx$. If the maximal interval K satisfies $K = I$, then the function $y(x)$ is a linear function. This contradiction yields $K \neq I$, and hence one of x_1 and x_2 belongs to I . Thus we may assume that $x_2 \in I$. For a sufficiently small $\epsilon > 0$, the interval $(x_2, x_2 + \epsilon)$ does not intersect J . Hence, it follows from (3.4) with $a = x_2$ that for all $b \in (x_2, x_2 + \epsilon)$

$$(3.8) \quad \frac{y(b)}{w(b)} = \frac{y(x_2)}{w(x_2)} (= l), \quad l = \frac{kx_2}{\sqrt{1 + k^2}},$$

where we use $y(x_2) = kx_2$, $y'(x_2) = k$ and $w(x_2) = \sqrt{1 + k^2}$. Hence, on the interval $(x_2, x_2 + \epsilon)$ the function $y(x)$ satisfies $y(x) = l\sqrt{1 + y'(x)^2}$. Therefore, Proposition 1.1 shows that

$$(3.9) \quad y(x) = \begin{cases} kx, & \text{if } x \in (x_1, x_2], \\ l \quad \text{or} \quad l \cosh\left(\frac{x-x_2}{l}\right), & \text{if } x \in (x_2, x_2 + \epsilon), \end{cases}$$

which cannot be a C^2 function.

Due to the above three cases, we see that (1) \Rightarrow (2).

Conversely, it follows from Proposition 1.2 that (2) \Rightarrow (1). This completes the proof of Theorem 1.4.

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