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ESTIMATES FOR SECOND NON-TANGENTIAL DERIVATIVES AT THE BOUNDARY

Burcu Gök and Bülent Nafi Örnek

Abstract. In this paper, a boundary version of Schwarz lemma is investigated. We take into consideration a function $f(z)$ holomorphic in the unit disc and $f(0)=0$, $f'(0)=1$ such that $\Re f'(z)>\frac{1-\alpha}{2}$, $-1<\alpha<1$, we estimate a modulus of the second non-tangential derivative of $f(z)$ function at the boundary point z_0 with $\Re f'(z_0)=\frac{1-\alpha}{2}$, by taking into account their first nonzero two Maclaurin coefficients. Also, we shall give an estimate below $|f''(z_0)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z=0$ and $z_1\neq 0$. The sharpness of these inequalities is also proved.

1. Introduction

The classical Schwarz lemma gives information about the behavior of a holomorphic function on the unit disc $D=\{z:|z|<1\}$ at the origin, subject only to the relatively mild hypotheses that the function map the unit disc to the disc and the origin to the origin. This lemma, named after Hermann Amandus Schwarz, is a result in complex analysis about holomorphic functions defined on the unit disc. In its most basic form, the familiar Schwarz lemma says this ([6], p. 329):

Let D be the unit disc in the complex plane $\mathbb{C}$. Let $f:D\to D$ be a holomorphic function with $f(0)=0$. Under these circumstances $|f(z)|\leq |z|$ for all $z\in D$, and $|f'(0)|\leq 1$. In addition, if the equality $|f(z)|=|z|$ holds for any $z\neq 0$, or $|f'(0)|=1$, then f is a rotation, that is, $f(z)=ze^{i\theta}$, θ real. For historical background about the Schwarz lemma and its applications on the boundary of the unit disc, we refer to (see [2], [20]).

Let $f(z)=z+c_2z^2+\cdots$ be a holomorphic function in the unit disc D such that $\Re f'(z)>\frac{1-\alpha}{2}$, $-1<\alpha<1$.

Consider the function
$$\Theta(z)=\frac{f'(z)-1}{f'(z)+\alpha}.$$

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$\Theta(z)$ is a holomorphic function in the unit disc D, $\Theta(0) = 0$ and since $\Re f'(z) > \frac{1-\alpha}{2}$, it also follows that $|\Theta(z)| < 1$ for $|z| < 1$. Thus, from the Schwarz lemma, we obtain for every z_1, $|z_1| < 1$,

$$|f'(z_1)| \leq \frac{1 + \alpha |z_1|}{1 - |z_1|} \tag{1.1}$$

and

$$|f''(0)| \leq 1 + \alpha. \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z_1 \in D$) if and only if $f(z)$ is the function of the form $f(z) = -\alpha z - \frac{1+\alpha}{e^{i\theta}} \ln \left(1 - ze^{i\theta}\right)$, where $\theta = -\arg z_1$, but the equality in (1.2) holds if and only if

$$f(z) = -\alpha z - \frac{1+\alpha}{e^{i\theta}} \ln \left(1 - ze^{i\theta}\right),$$

where θ is a real number.

Robert Osserman [16] has given the inequalities which are called the boundary Schwarz lemma. He has first showed that

$$|f'(z_0)| \geq \frac{2}{1 + |f'(0)|} \tag{1.3}$$

and

$$|f'(z_0)| \geq 1 \tag{1.4}$$

under the assumption $f(0) = 0$ where f is a holomorphic function mapping the unit disc into itself and z_0 is a boundary point to which f extends continuously and $|f(z_0)| = 1$. In addition, the equality in (1.4) holds if and only if $f(z) = ze^{i\theta}$, θ real. Also, $z_0 = 1$ in the inequality (1.3) equality occurs for the function $f(z) = z\frac{z+a}{1+az}$, $0 \leq a \leq 1$.

Inequality (1.4) and its generalizations have important applications in geometric theory of functions (see, e.g., [6], [19]). Therefore, the interest to such type results is not vanished recently (see, e.g., [1], [2], [4], [5], [10], [11], [16], [17], [18], [20] and references therein).

Furthermore, if $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, $p \in \mathbb{N} = \{1, 2, \ldots\}$, then

$$|f'(z_0)| \geq p + \frac{1 - |c_p|}{1 + |c_p|} \tag{1.5}$$

and

$$|f'(z_0)| \geq p. \tag{1.6}$$

Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If the function f has an angular limit $f(z_0)$ at $z_0 \in \partial D$, $|f(z_0)| = 1$, then by the Julia-Wolff lemma the angular derivative $f'(z_0)$ exists and $1 \leq |f'(z_0)| \leq \infty$. Also, the holomorphic function f has a finite angular derivative $f'(z_0)$ if and only if f' has the finite angular limit $f'(z_0)$ at $z_0 \in \partial D$ (see [19]).

Vladimir N. Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = c_p z^p + c_{p+1} z^{p+1} + \cdots$, with a zero set $\{z_k\}$ (see [4]).

S. G. Krantz and D. M. Burns [9] and D. Chelst [3] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([12], [13], [14] and [15]).

Also, M. Jeong [7] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [8] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

2. Main results

We consider holomorphic functions f of the unit disk D fixing the origin, such that the derivative f' maps D into a right half plane ($\Re f'(z) > \frac{1-\alpha}{2}$ for a real parameter α such that $-1 < \alpha < 1$), with the normalization $f'(0) = 1$. It is assumed at some boundary point $z_0 \in \partial D$, f' has a non-tangential limit $f'(z_0)$ with $\Re f'(z_0) = \frac{1-\alpha}{2}$. The conclusion is that f has a second derivative in the non-tangential sense at z_0, and that there is an explicit lower bound for $|f''(z_0)|$ as in (1.7) in the paper, with equality attained for certain specific functions f. Further results are obtained under the assumption that $f' \neq 1$ except at the origin.

Theorem 2.1. *Let $f(z) = z + c_2 z^2 + \cdots$ be a holomorphic function in the unit disc D such that $\Re f'(z) > \frac{1-\alpha}{2}$, $-1 < \alpha < 1$. Suppose that, for some $z_0 \in \partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0 and $\Re f'(z_0) = \frac{1-\alpha}{2}$. Then f has the second non-tangential derivative at z_0 and*

$$|f''(z_0)| \geq \frac{1+\alpha}{4}. \tag{1.7}$$

Moreover, the equality in (1.7) *occurs for the function*

$$f(z) = -\alpha z - (1+\alpha)\ln(1-z).$$

Proof. Let

$$\Theta(z) = \frac{f'(z) - 1}{f'(z) + \alpha}.$$

$\Theta(z)$ is a holomorphic function in the unit disc D, $|\Theta(z)| < 1$ for $|z| < 1$, $\Theta(0) = 0$ and $|\Theta(z_0)| = 1$ for $z_0 \in \partial D$. It can be easily shown a non-tangential derivative of Θ at $z_0 \in \partial D$ (see [19]). Therefore, the second non-tangential derivative of f at z_0 is obtained.

From (1.4), we obtain

$$1 \leq |\Theta'(z_0)| = (1+\alpha)\frac{|f''(z_0)|}{|f'(z_0) + \alpha|^2}.$$

Since

$$|f'(z_0)+\alpha|^2 \geq [\Re(f'(z_0)+\alpha)]^2 = (\Re f'(z_0)+\alpha)^2 = \left(\frac{1+\alpha}{2}\right)^2,$$

we take

$$1 \leq |\Theta'(z_0)| \leq \frac{(1+\alpha)}{\left(\frac{1+\alpha}{2}\right)^2}|f''(z_0)| = \frac{4}{1+\alpha}|f''(z_0)|. \tag{1.8}$$

So, we obtain the inequality (1.7).

Now, we shall show that the inequality (1.7) is sharp. Let

$$f(z) = -\alpha z - (1+\alpha)\ln(1-z).$$

Then

$$f'(z) = -\alpha + \frac{1+\alpha}{1-z} = \frac{1+\alpha z}{1-z},$$

$$f''(z) = \frac{1+\alpha}{(1-z)^2}$$

and

$$|f''(-1)| = \frac{1+\alpha}{|(1-(-1))|^2} = \frac{1+\alpha}{4}.$$

□

Theorem 2.2. *Under the same assumptions as in Theorem 2.1, we have*

$$|f''(z_0)| \geq \frac{(1+\alpha)^2}{2(1+\alpha+|f''(0)|)}. \tag{1.9}$$

The inequality (1.9) *is sharp with equality for the function*

$$f(z) = -\alpha z + \sqrt{1-b^2}\arctan\left(\frac{b+z}{\sqrt{1-b^2}}\right) + \frac{b}{2}\ln\left(1+2bz+z^2\right) + c,$$

where $c = -\sqrt{1-b^2}\arctan\left(\frac{b}{\sqrt{1-b^2}}\right)$ *is a constant and* $b = \frac{|f''(0)|}{1+\alpha}$ *is an arbitrary number from* $[0,1]$ (*see,* (1.2)).

Proof. Let $\Theta(z)$ be as in the proof of Theorem 2.1. Using the inequality (1.3) for the function $\Theta(z)$, we obtain

$$\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(z_0)| \leq \frac{4}{1+\alpha}|f''(z_0)|.$$

Since

$$\Theta'(z) = (1+\alpha)\frac{f''(z)}{(f'(z)+\alpha)^2}$$

and

$$|\Theta'(0)| = \frac{|f''(0)|}{1+\alpha},$$

we take

$$\frac{2}{1+\frac{|f''(0)|}{1+\alpha}} \leq \frac{4}{1+\alpha}|f''(z_0)|.$$

So, we obtain the inequality (1.9).

Now, we shall show that the inequality (1.9) is sharp. Choose arbitrary $b \in [0,1]$. Let

$$f(z) = -\alpha z + \sqrt{1-b^2}\arctan\left(\frac{b+z}{\sqrt{1-b^2}}\right) + \frac{b}{2}\ln\left(1+2bz+z^2\right) + c.$$

Then

$$f'(z) = \frac{1+bz-\alpha bz-\alpha z^2}{1+2bz+z^2},$$

$$f''(z) = \frac{(b-\alpha b-2\alpha z)\left(1+2bz+z^2\right) - (2b+2z)\left(1+bz-\alpha bz-\alpha z^2\right)}{\left(1+2bz+z^2\right)^2}$$

and

$$|f''(1)| = \frac{1+\alpha}{2(1+b)}.$$

Since $b = \frac{|f''(0)|}{1+\alpha}$, (1.9) is satisfied with equality. □

If $f(z) = z + c_{p+1}z^{p+1} + \cdots$, $p \geq 1$, is a holomorphic function in D such that $\Re f'(z) > \frac{1-\alpha}{2}$, $-1 < \alpha < 1$, then

$$|f'(z)| \leq \frac{1+\alpha|z|^p}{1-|z|^p}$$

and

$$|c_{p+1}| \leq \frac{1+\alpha}{1+p}. \tag{1.10}$$

Theorem 2.3. *Let $f(z) = z + c_{p+1}z^{p+1} + \cdots$, $c_{p+1} \neq 0$, $p \geq 1$ be a holomorphic function in the unit disc D such that $\Re f'(z) > \frac{1-\alpha}{2}$, $-1 < \alpha < 1$. Suppose that, for some $z_0 \in \partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0 and $\Re f'(z_0) = \frac{1-\alpha}{2}$. Then f has the second non-tangential derivative at z_0 and*

$$|f''(z_0)| \geq \frac{1+\alpha}{4}\left(p + \frac{1+\alpha-(1+p)|c_{p+1}|}{1+\alpha+(1+p)|c_{p+1}|}\right). \tag{1.11}$$

The inequality (1.11) *is sharp with equality for the function*

$$f(z) = \int_0^z \frac{1+et-\alpha et^p-\alpha t^{p+1}}{1+et+et^p+t^{p+1}}dt,$$

where $e = \frac{1+p}{1+\alpha}|c_{p+1}|$ is an arbitrary number on $[0,1]$ (*see*, (1.10)).

Proof. Using the inequality (1.5) for the function $\Theta(z)$, we obtain

$$p + \frac{1-|a_p|}{1+|a_p|} \leq |\Theta'(z_0)| = (1+\alpha)\frac{|f''(z_0)|}{|f'(z_0)+\alpha|^2},$$

where $|a_p| = \frac{|\Theta^{(p)}(0)|}{p!} = \frac{1+p}{1+\alpha}|c_{p+1}|$.

Therefore, we take

$$p+\frac{1-\frac{1+p}{1+\alpha}\left|c_{p+1}\right|}{1+\frac{1+p}{1+\alpha}\left|c_{p+1}\right|}\leq\frac{4}{1+\alpha}\left|f''(z_0)\right|$$

and

$$\left|f''(z_0)\right|\geq\frac{1+\alpha}{4}\left(p+\frac{1+\alpha-(1+p)\left|c_{p+1}\right|}{1+\alpha+(1+p)\left|c_{p+1}\right|}\right).$$

Now, we will prove that the inequality (1.11) is sharp. Choose arbitrary $e\in[0,1]$. Let

$$f(z)=\int_0^z\frac{1+et-\alpha et^p-\alpha t^{p+1}}{1+et+et^p+t^{p+1}}dt.$$

Then

$$f'(z)=\frac{1+ez-\alpha ez^p-\alpha z^{p+1}}{1+ez+ez^p+z^{p+1}},$$

$$f''(z)=\frac{\left(e-\alpha epz^{p-1}-\alpha(p+1)z^p\right)\left(1+ez+ez^p+z^{p+1}\right)}{\left(1+ez+ez^p+z^{p+1}\right)^2}-\frac{\left(e+epz^{p-1}+(p+1)z^p\right)\left(1+ez-\alpha ez^p-\alpha z^{p+1}\right)}{\left(1+ez+ez^p+z^{p+1}\right)^2}$$

and

$$\begin{aligned}f''(1)&=\frac{(e-\alpha ep-\alpha(p+1))(1+e+e+1)}{(1+e+e+1)^2}\\&\quad-\frac{(e+ep+(p+1))(1+e-\alpha e-\alpha)}{(1+e+e+1)^2}\\&=-\frac{1+\alpha}{4}\left(p+\frac{1-e}{1+e}\right).\end{aligned}$$

Thus, we take

$$\left|f''(1)\right|=\frac{1+\alpha}{4}\left(p+\frac{1-e}{1+e}\right).$$

Since $e=\frac{1+p}{1+\alpha}\left|c_{p+1}\right|$, (1.11) is satisfied with equality. □

Theorem 2.4. *Let $f(z)=z+c_{p+1}z^{p+1}+\cdots$, $c_{p+1}\neq 0$, $p\geq 1$ be a holomorphic function in the unit disc D such that $\Re f'(z)>\frac{1-\alpha}{2}$, $-1<\alpha<1$. Suppose that, for some $z_0\in\partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0 and $\Re f'(z_0)=\frac{1-\alpha}{2}$. Let $b_1,b_2,\ldots,b_n$ be critical points of the function $f(z)-z$ in D that are different from zero. Then f has the second non-tangential derivative at z_0 and we have the inequality*

(1.12)

$$\left|f''(z_0)\right|\geq\frac{1+\alpha}{4}\left(p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-b_k|^2}+\frac{(1+\alpha)\prod_{k=1}^{n}|b_k|-(1+p)\left|c_{p+1}\right|}{(1+\alpha)\prod_{k=1}^{n}|b_k|+(1+p)\left|c_{p+1}\right|}\right).$$

In addition, the equality in (1.12) *occurs for the function*

$$f(z)=\int_0^z \frac{1-\alpha t^p \prod\limits_{k=1}^{n}\frac{t-b_k}{1-\overline{b_k}t}}{1+t^p \prod\limits_{k=1}^{n}\frac{t-b_k}{1-\overline{b_k}t}}dt,$$

where $b_1, b_2, \ldots, b_n$ *are positive real numbers.*

Proof. Let $\Theta(z)$ be as in the proof of Theorem 2.1 and $b_1, b_2, \ldots, b_n$ be critical points of the function $f(z)-z$ in D that are different from zero.

$$B(z)=\prod_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}$$

is a holomorphic function in D and $|B(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in D$, we have

$$|\Theta(z)|\leq |B(z)|\,.$$

The auxiliary function

$$\Upsilon(z)=\frac{\Theta(z)}{B(z)}=\frac{f'(z)-1}{f'(z)+\alpha}\frac{1}{\prod\limits_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}$$

is holomorphic in D, and $|\Upsilon(z)|<1$ for $|z|<1$, $\Upsilon(0)=0$ and $|\Upsilon(z_0)|=1$ for $z_0\in\partial D$. It can be easily shown a non-tangential derivative of Υ at $z_0\in\partial D$ (see [19]). So, the second non-tangential derivative of f at z_0 is obtained.

Moreover, it can be seen that

$$\frac{z_0\Theta'(z_0)}{\Theta(z_0)}=|\Theta'(z_0)|\geq |B'(z_0)|=\frac{z_0B'(z_0)}{B(z_0)}.$$

Besides, by applying some simple calculations, we take

$$|B'(z_0)|=\frac{z_0B'(z_0)}{B(z_0)}=\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-b_k|^2}.$$

Using the inequality (1.5) for the function $\Upsilon(z)$, we obtain

$$p+\frac{1-|s_p|}{1+|s_p|}\leq |\Upsilon'(z_0)|=\left|\frac{z_0\Theta'(z_0)}{\Theta(z_0)}-\frac{z_0B'(z_0)}{B(z_0)}\right|=\left\{|\Theta'(z_0)|-|B'(z_0)|\right\},$$

where $|s_p|=\frac{\left|\Upsilon^{(p)}(0)\right|}{p!}$.

Since $|s_p|=\frac{\left|\Upsilon^{(p)}(0)\right|}{p!}=\frac{(1+p)|c_{p+1}|}{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|}$, we may write

$$p+\frac{1-\frac{(1+p)|c_{p+1}|}{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|}}{1+\frac{(1+p)|c_{p+1}|}{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|}}\leq (1+\alpha)\frac{|f''(z_0)|}{|f'(z_0)+\alpha|^2}-\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-b_k|^2},$$

$$p+\frac{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|-(1+p)|c_{p+1}|}{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|+(1+p)|c_{p+1}|}\leq(1+\alpha)\frac{|f''(z_0)|}{|f'(z_0)+\alpha|^2}-\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-b_k|^2}$$

and

$$p+\sum_{k=1}^{n}\frac{1-|b_k|^2}{|z_0-b_k|^2}+\frac{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|-(1+p)|c_{p+1}|}{(1+\alpha)\prod\limits_{k=1}^{n}|b_k|+(1+p)|c_{p+1}|}\leq\frac{4}{1+\alpha}|f''(z_0)|.$$

Therefore, we take the inequality (1.12).

Now, we shall show that the inequality (1.12) is sharp. Let

$$f(z)=\int_0^z\frac{1-\alpha t^p\prod\limits_{k=1}^{n}\frac{t-b_k}{1-\overline{b_k}t}}{1+t^p\prod\limits_{k=1}^{n}\frac{t-b_k}{1-\overline{b_k}t}}dt.$$

Then

$$f'(z)=\frac{1-\alpha z^p\prod\limits_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}{1+z^p\prod\limits_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}}=-\alpha+\frac{1+\alpha}{1+z^p\prod\limits_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}},$$

$$f''(z)=-(1+\alpha)\frac{\left(pz^{p-1}\prod\limits_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}+\sum\limits_{k=1}^{n}\frac{1-|b_k|^2}{\left(1-\overline{b_k}z\right)^2}\prod\limits_{\substack{s=1\\k\neq s}}^{n}\frac{z-b_s}{1-\overline{b_s}z}z^p\right)}{\left(1+z^p\prod\limits_{k=1}^{n}\frac{z-b_k}{1-\overline{b_k}z}\right)^2}$$

and

$$f''(1)=-(1+\alpha)\frac{\left(p\prod\limits_{k=1}^{n}\frac{1-b_k}{1-\overline{b_k}}+\sum\limits_{k=1}^{n}\frac{1-|b_k|^2}{\left(1-\overline{b_k}\right)^2}\prod\limits_{\substack{s=1\\k\neq s}}^{n}\frac{1-b_s}{1-\overline{b_s}}\right)}{\left(1+\prod\limits_{k=1}^{n}\frac{1-b_k}{1-\overline{b_k}}\right)^2}.$$

Since $b_1,b_2,\ldots,b_n$ are positive real numbers, we take

$$|f''(1)|=\frac{1+\alpha}{4}\left(p+\sum_{k=1}^{n}\frac{1+b_k}{1-b_k}\right).$$

Moreover, since $|c_{p+1}|=\frac{1+\alpha}{p+1}\prod\limits_{k=1}^{n}|b_k|$, (1.12) is satisfied with equality. □

In the following theorems, if we know the second and the third coefficient in the expansion of the function $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots$, then we obtain more general results on the second non-tangential derivatives of certain classes of a holomorphic function in the unit disc at the boundary by taking into account c_{p+1}, c_{p+2} and critical points of $f(z)-z$ function. The sharpness of these inequalities is also proved.

Theorem 2.5. *Let $f(z)=z+c_{p+1}z^{p+1}+c_{p+2}z^{p+2}+\cdots$, $c_{p+1}\neq 0$, $p\geq 2$ be a holomorphic function in the unit disc D such that $\Re f'(z)>\frac{1-\alpha}{2}$, $-1<\alpha<1$. Suppose that, for some $z_0\in\partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0 and $\Re f'(z_0)=\frac{1-\alpha}{2}$. Then f has the second non-tangential derivative at z_0 and*

(1.13)
$$\left|f''(z_0)\right|\geq\frac{1+\alpha}{4}\left[p+\frac{2\left(1+\alpha-(p+1)\left|c_{p+1}\right|\right)^2}{(1+\alpha)^2-(p+1)^2\left|c_{p+1}\right|^2+(1+\alpha)(p+2)\left|c_{p+2}\right|}\right].$$

Moreover, the equality in (1.13) *occurs for the function*

$$f(z)=\int_0^z\frac{1-\alpha t^p}{1+t^p}dt.$$

Proof. Let $\Theta(z)$ be as in the proof of Theorem 2.1. $\vartheta(z)=z^p$ is a holomorphic function in the unit disc D, $|\vartheta(z)|<1$ for $|z|<1$. By the maximum principle for each $z\in D$, we have $|\Theta(z)|\leq|\vartheta(z)|$.

Therefore,

$$\varphi(z)=\frac{\Theta(z)}{\vartheta(z)}$$

is a holomorphic function in D and $|\varphi(z)|<1$ for $|z|<1$.

In particular, we have

$$|\varphi(0)|=\frac{1+p}{1+\alpha}\left|c_{p+1}\right|\leq 1 \tag{1.14}$$

and

$$|\varphi'(0)|=\frac{p+2}{\alpha+1}\left|c_{p+2}\right|.$$

In addition, it can be seen that

$$\frac{z_0\Theta'(z_0)}{\Theta(z_0)}=|\Theta'(z_0)|\geq|\vartheta'(z_0)|=\frac{z_0\vartheta'(z_0)}{\vartheta(z_0)}.$$

The function

$$\psi(z)=\frac{\varphi(z)-\varphi(0)}{1-\overline{\varphi(0)}\varphi(z)}$$

is holomorphic in the unit disc D, $|\psi(z)|<1$ for $|z|<1$, $\psi(0)=0$ and $|\psi(z_0)|=1$ for $z_0\in\partial D$. It can be easily shown that the function ψ has a non-tangential derivative at $z_0\in\partial D$ (see [19]). Therefore, the second non-tangential derivative of f at z_0 is obtained.

From (1.3), we obtain

$$\frac{2}{1+|\psi'(0)|}\leq|\psi'(z_0)|=\frac{1-|\varphi(0)|^2}{\left|1-\overline{\varphi(0)}\varphi(z_0)\right|^2}|\varphi'(z_0)|\leq\frac{1+|\varphi(0)|}{1-|\varphi(0)|}|\varphi'(z_0)|$$

$$= \frac{1+|\varphi(0)|}{1-|\varphi(0)|}\left\{|\Theta'(z_0)| - |\vartheta'(z_0)|\right\}.$$

Since

$$\psi'(z) = \frac{1-|\varphi(0)|^2}{\left(1-\overline{\varphi(0)}\varphi(z)\right)^2}\varphi'(z),$$

$$|\psi'(0)| = \frac{1-|\varphi(0)|^2}{\left(1-|\varphi(0)|^2\right)^2}|\varphi'(0)| = \frac{(1+\alpha)(p+2)|c_{p+2}|}{(1+\alpha)^2-(1+p)^2|c_{p+1}|^2},$$

we get

$$\frac{2}{1+\frac{(1+\alpha)(p+2)|c_{p+2}|}{(1+\alpha)^2-(1+p)^2|c_{p+1}|^2}} \leq \frac{1+\frac{1+p}{1+\alpha}|c_{p+1}|}{1-\frac{1+p}{1+\alpha}|c_{p+1}|}\left\{\frac{(1+\alpha)|f''(z_0)|}{|f'(z_0)+\alpha|^2} - p\right\}$$

$$= \frac{1+\alpha+(1+p)|c_{p+1}|}{1+\alpha-(1+p)|c_{p+1}|}\left\{\frac{(1+\alpha)|f''(z_0)|}{|f'(z_0)+\alpha|^2} - p\right\}.$$

Since

$$|f'(z_0)+\alpha|^2 \geq [\Re(f'(z_0)+\alpha)]^2 = \left(\frac{1+\alpha}{2}\right)^2,$$

we take

$$p + \frac{2}{1+\frac{(1+\alpha)(p+2)|c_{p+2}|}{(1+\alpha)^2-(1+p)^2|c_{p+1}|^2}}\frac{1+\alpha-(1+p)|c_{p+1}|}{1+\alpha+(1+p)|c_{p+1}|} \leq \frac{4}{1+\alpha}|f''(z_0)|.$$

So, we obtain the inequality (1.13).

To show that the inequality (1.13) is sharp, take the holomorphic function

$$f(z) = \int_0^z \frac{1-\alpha t^p}{1+t^p}dt.$$

Then

$$f'(z) = \frac{d}{dz}f(z) = \frac{1-\alpha z^p}{1+z^p},$$

$$f''(z) = \frac{-\alpha p z^{p-1}(1+z^p) - p z^{p-1}(1-\alpha z^p)}{(1+z^p)^2},$$

$$f''(1) = \frac{-2\alpha p - p(1-\alpha)}{4}$$

and

$$|f''(1)| = p\frac{1+\alpha}{4}.$$

Since $|c_{p+1}| = \frac{1+\alpha}{1+p}$, (1.13) is satisfied with equality. □

If $f(z)-z$ has no critical points different from $z=0$ in Theorem 2.5, the inequality (1.15) can be further strengthened. This is given by the following theorem.

Theorem 2.6. *Let $f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots$, $c_{p+1} > 0$, $p \geq 2$ be a holomorphic function in the unit disc D such that $\Re f'(z) > \frac{1-\alpha}{2}$, $-1 < \alpha < 1$ and let $f(z) - z$ has no critical points in D except $z = 0$. Suppose that, for some $z_0 \in \partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0, $\Re f'(z_0) = \frac{1-\alpha}{2}$. Then f has the second non-tangential derivative at z_0 and we have the inequality*

$$(1.15)\quad |f''(z_0)| \geq \frac{1+\alpha}{4}\left[p - \frac{2(p+1)\,|c_{p+1}|\left(\ln\left(\frac{(p+1)|c_{p+1}|}{1+\alpha}\right)\right)^2}{2(p+1)\,|c_{p+1}|\ln\left(\frac{(p+1)|c_{p+1}|}{1+\alpha}\right) - (p+2)\,|c_{p+2}|}\right]$$

and

$$(1.16)\qquad |c_{p+2}| \leq \frac{2}{p+2}\left|(p+1)c_{p+1}\ln\left(\frac{(p+1)\,|c_{p+1}|}{1+\alpha}\right)\right|.$$

In addition, the equality in (1.15) occurs for the function $f(z) = \int_0^z \frac{1-\alpha t^p}{1+t^p}dt$ and the equality in (1.16) occurs for the function

$$f(z) = \int_0^z \frac{1+\alpha t^p e^Q}{1-t^p e^Q}dt,$$

where $0 < c_{p+1} < 1$, $\ln\left(\frac{(p+1)c_{p+1}}{1+\alpha}\right) < 0$ and $Q = \frac{1+t}{1-t}\ln\left(\frac{(p+1)c_{p+1}}{1+\alpha}\right)$.

Proof. Let $c_{p+1} > 0$ be in the expression of the function $f(z)$. Let $\Theta(z)$, $\varphi(z)$ and $\vartheta(z)$ be as in the proof of Theorem 2.5. Having in mind the inequality (1.14) and the function $f(z) - z$ has no critical points in D except $D - \{0\}$, we denote by $\ln\varphi(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln\varphi(0) = \ln\left(\frac{1+p}{1+\alpha}c_{p+1}\right) < 0.$$

The composite function

$$\gamma(z) = \frac{\ln\varphi(z) - \ln\varphi(0)}{\ln\varphi(z) + \ln\varphi(0)}$$

is a holomorphic in the unit disc D, $|\gamma(z)| < 1$, $\gamma(0) = 0$ and $|\gamma(z_0)| = 1$ for $z_0 \in \partial D$. It can be easily shown a non-tangential derivative of γ at $z_0 \in \partial D$ (see [19]). Therefore, the second non-tangential derivative of f at z_0 is obtained.

From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+|\gamma'(0)|} \leq |\gamma'(z_0)| &= \frac{|2\ln\varphi(0)|}{|\ln\varphi(z_0) + \ln\varphi(0)|^2}\left|\frac{\varphi'(z_0)}{\varphi(z_0)}\right| \\ &= \frac{-2\ln\varphi(0)}{\ln^2\varphi(0) + \arg^2\varphi(z_0)}\left\{|\Theta'(z_0)| - |\vartheta'(z_0)|\right\}.\end{aligned}$$

Replacing $\arg^2\varphi(z_0)$ by zero, then

$$\frac{1}{1 - \frac{(p+2)|c_{p+2}|}{2(p+1)|c_{p+1}|\ln\left(\frac{1+p}{1+\alpha}|c_{p+1}|\right)}} \leq \frac{-1}{\ln\varphi(0)}\left\{\frac{(1+\alpha)\,|f''(z_0)|}{|f'(z_0)+\alpha|^2} - p\right\}$$

$$\leq \frac{-1}{\ln\left(\frac{1+p}{1+\alpha}\left|c_{p+1}\right|\right)}\left\{\frac{4}{1+\alpha}\left|f''(z_0)\right|-p\right\}.$$

Thus, we obtain the inequality (1.15) with an obvious equality case.

Likewise, $\gamma(z)$ function satisfies the assumptions of the Schwarz lemma, we obtain

$$\begin{aligned}1 \geq |\gamma'(0)| &= \frac{|2\ln\varphi(0)|}{|\ln\varphi(0)+\ln\varphi(0)|^2}\left|\frac{\varphi'(0)}{\varphi(0)}\right| \\ &= \frac{-1}{2\ln\left(\frac{1+p}{1+\alpha}\left|c_{p+1}\right|\right)}\frac{\frac{p+2}{c+1}\left|a_{p+2}\right|}{\frac{1+p}{1+\alpha}\left|c_{p+1}\right|}\end{aligned}$$

and

$$1 \geq \frac{-1}{2\ln\left(\frac{1+p}{1+\alpha}\left|c_{p+1}\right|\right)}\frac{(p+2)\left|c_{p+2}\right|}{(p+1)\left|c_{p+1}\right|}.$$

Therefore, we have

$$\left|c_{p+2}\right| \leq \frac{2}{p+2}\left|(p+1)c_{p+1}\ln\left(\frac{(p+1)\left|c_{p+1}\right|}{1+\alpha}\right)\right|.$$

We shall show that the inequality (1.16) is sharp. Let

$$f(z)=\int_0^z \frac{1+\alpha t^p e^Q}{1-t^p e^Q}dt.$$

Thus, we get

$$f'(z)=\frac{1+\alpha z^p e^Q}{1-z^p e^Q}$$

and

$$f'(z)=1+z^p g(z),$$

where

$$g(z)=(1+\alpha)\frac{e^{\frac{1+z}{1-z}\ln\left(\frac{p+1}{1+\alpha}c_{p+1}\right)}}{1-z^p e^{\frac{1+z}{1-z}\ln\left(\frac{p+1}{1+\alpha}c_{p+1}\right)}}.$$

Then

$$g'(0)=(p+2)\,c_{p+2}.$$

Under the simple calculations, we obtain

$$(p+2)\,c_{p+2}=2\ln\left(\frac{p+1}{1+\alpha}c_{p+1}\right)(p+1)\,c_{p+1}$$

and

$$\left|c_{p+2}\right|=\frac{2}{p+2}\left|(p+1)\,c_{p+1}\ln\left(\frac{p+1}{1+\alpha}\left|c_{p+1}\right|\right)\right|.$$

□

Relation (1.16) shows that inequality (1.15) is stronger than inequality (1.13). We can note that the inequality (1.3) has been used in the proofs of Theorem 2.5 and Theorem 2.6. Thus, there are both c_{p+1} and c_{p+2} in the right side of the inequalities. But, if we use (1.4) instead of (1.3), we obtain weaker but more simpler inequality (without c_{p+2}).

Theorem 2.7. *Let $f(z) = z + c_{p+1}z^{p+1} + c_{p+2}z^{p+2} + \cdots$, $c_{p+1} > 0$, $p \geq 2$ be a holomorphic function in the unit disc D such that $\Re f'(z) > \frac{1-\alpha}{2}$, $-1 < \alpha < 1$ and let $f(z) - z$ has no critical points in D except $z = 0$. Suppose that, for some $z_0 \in \partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0, $\Re f'(z_0) = \frac{1-\alpha}{2}$. Then f has the second non-tangential derivative at z_0 and*

$$|f''(z_0)| \geq \frac{1+\alpha}{4}\left(p - \frac{1}{2}\ln\left(\frac{(p+1)\,|c_{p+1}|}{1+\alpha}\right)\right). \tag{1.17}$$

The inequality (1.17) is sharp and the equality is achieved if and only if $f(z)$ is the function of the form

$$f(z) = \int_0^z \frac{1+\alpha t^p e^Q}{1 - t^p e^Q}dt,$$

where $0 < c_{p+1} < 1$, $\ln\left(\frac{(p+1)c_{p+1}}{1+\alpha}\right) < 0$, $Q = \frac{1+te^{i\theta}}{1-te^{i\theta}}\ln\left(\frac{(p+1)c_{p+1}}{1+\alpha}\right)$ and θ is a real number.

Proof. Let $c_{p+1} > 0$. Using the inequality (1.4) for the function $\gamma(z)$, we obtain

$$\begin{aligned} 1 \leq |\gamma'(z_0)| &= \frac{|2\ln\varphi(0)|}{|\ln\varphi(z_0) + \ln\varphi(0)|^2}\left|\frac{\varphi'(z_0)}{\varphi(z_0)}\right| \\ &= \frac{-2\ln\varphi(0)}{\ln^2\varphi(0) + \arg^2\varphi(z_0)}\left\{|\Theta'(z_0)| - |\vartheta'(z_0)|\right\}. \end{aligned}$$

Replacing $\arg^2\varphi(z_0)$ by zero, then

$$\begin{aligned} 1 \leq |\gamma'(z_0)| &\leq \frac{-2}{\ln\varphi(0)}\left\{\frac{(1+\alpha)\,|f''(z_0)|}{|f'(z_0)+\alpha|^2} - p\right\} \\ &\leq \frac{-2}{\ln\left(\frac{1+p}{1+\alpha}\,|c_{p+1}|\right)}\left\{\frac{4}{1+\alpha}\,|f''(z_0)| - p\right\}. \end{aligned} \tag{1.18}$$

Therefore, we obtain the inequality (1.17).

If $|f''(z_0)| = \frac{1+\alpha}{4}\left(p - \frac{1}{2}\ln\left(\frac{(p+1)|c_{p+1}|}{1+\alpha}\right)\right)$ from (1.18) and $|\gamma'(z_0)| = 1$, we obtain

$$f(z) = \int_0^z \frac{1+\alpha t^p e^{\frac{1+te^{i\theta}}{1-te^{i\theta}}\ln\left(\frac{(p+1)c_{p+1}}{1+\alpha}\right)}}{1 - t^p e^{\frac{1+te^{i\theta}}{1-te^{i\theta}}\ln\left(\frac{(p+1)c_{p+1}}{1+\alpha}\right)}}dt. \qquad \square$$

In the following theorem, we shall give an estimate below $|f''(z_0)|$ according to the first nonzero Taylor coefficient of about two zeros, namely $z = 0$ and $z_1 \neq 0$.

Theorem 2.8. *Let $f(z) = z + c_2 z^2 + \cdots$ be a holomorphic function in the unit disc D such that $\Re f'(z) > \frac{1-\alpha}{2}$, $-1 < \alpha < 1$ and $f'(z_1) = 0$, for $0 < |z_1| < 1$. Assume that, for some $z_0 \in \partial D$, f' has a non-tangential limit $f'(z_0)$ at z_0 and $\Re f'(z_0) = \frac{1-\alpha}{2}$. Then f has the second non-tangential derivative at z_0 and*

$$|f''(z_0)| \geq \frac{1+\alpha}{4}\left(1 + \frac{1-|z_1|^2}{|1-z_1|^2} + \frac{(1+\alpha)|z_1| - |f''(0)|}{(1+\alpha)|z_1| + |f''(0)|}\right.$$

$$\left. \times \left[1 + \frac{(1+\alpha)^2|z_1|^2 + |f''(0)|\left(1-|z_1|^2\right)|f''(z_1)| - (1+\alpha)\left(1-|z_1|^2\right)|f''(z_1)| - (1+\alpha)|f''(0)|}{(1+\alpha)^2|z_1|^2 + |f''(0)|\left(1-|z_1|^2\right)|f''(z_1)| + (1+\alpha)\left(1-|z_1|^2\right)|f''(z_1)| + (1+\alpha)|f''(0)|}\frac{1-|z_1|^2}{|1-z_1|^2}\right]\right). \tag{1.19}$$

The inequality (1.19) *is sharp, with equality for each possible values $|f''(0)| = (1+\alpha)\,c$ and $|f''(z_1)| = (1+\alpha)\,d$.*

Proof. Let

$$s(z) = \frac{z - z_1}{1 - \overline{z_1} z}$$

and $l : D \to D$ be a holomorphic function and a point $z_1 \in D$ in order to satisfy

$$|l(z)| \leq \frac{|l(z_1)| + |s(z)|}{1 + |l(z_1)|\,|s(z)|}. \tag{1.20}$$

If $k : D \to D$ is a holomorphic function and $0 < |z_1| < 1$, letting

$$l(z) = \frac{k(z) - k(0)}{z\left(1 - \overline{k(0)}k(z)\right)}$$

in (1.20), we obtain

$$\left|\frac{k(z) - k(0)}{1 - \overline{k(0)}k(z)}\right| \leq |z| \frac{\left|\frac{k(z_1) - k(0)}{z_1\left(1-\overline{k(0)}k(z_1)\right)}\right| + |s(z)|}{1 + \left|\frac{k(z_1)-k(0)}{z_1\left(1-\overline{k(0)}k(z_1)\right)}\right|\,|s(z)|}$$

and

$$|k(z)| \leq \frac{|k(0)| + |z|\,\frac{|H| + |s(z)|}{1+|H|\,|s(z)|}}{1 + |k(0)|\,|z|\,\frac{|H|+|s(z)|}{1+|H|\,|s(z)|}}, \tag{1.21}$$

where

$$H = \frac{k(z_1) - k(0)}{z_1\left(1 - \overline{k(0)}k(z_1)\right)}.$$

Without loss of generality, we will assume that $z_0 = 1$. If we take

$$k(z) = \frac{\Theta(z)}{z\frac{z-z_1}{1-\overline{z_1}z}},$$

then

$$k(0) = \frac{\Theta'(0)}{-z_1},\ k(z_1) = \frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}$$

and

$$H=\frac{\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}+\frac{\Theta'(0)}{z_1}}{z_1\left(1+\overline{\frac{\Theta'(0)}{z_1}}\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right)},$$

where $|H|\leq 1$. Let $|k(0)|=\kappa$ and

$$M=\frac{\left|\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{\Theta'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\Theta'(0)}{z_1}\right|\left|\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\right)}.$$

From (1.21), we take

$$|\Theta(z)|\leq |z|\,|s(z)|\,\frac{\kappa+|z|\,\frac{M+|s(z)|}{1+M|s(z)|}}{1+\kappa\,|z|\,\frac{M+|s(z)|}{1+M|s(z)|}}$$

and

(1.22)

$$\frac{1-|\Theta(z)|}{1-|z|}\geq\frac{1+\kappa\,|z|\,\frac{M+|s(z)|}{1+M|s(z)|}-\kappa\,|z|\,|s(z)|-|s(z)|\,|z|^2\,\frac{M+|s(z)|}{1+M|s(z)|}}{(1-|z|)\left(1+\kappa\,|z|\,\frac{M+|s(z)|}{1+M|s(z)|}\right)}=\Sigma(z).$$

Let $q(z)=1+\kappa\,|z|\,\frac{M+|s(z)|}{1+M|s(z)|}$ and $v(z)=1+M\,|s(z)|$. Then

$$\Sigma(z)=\frac{1-|z|^2\,|s(z)|^2}{(1-|z|)\,q(z)v(z)}+M\,|s(z)|\,\frac{1-|z|^2}{(1-|z|)\,q(z)v(z)}+M\kappa\frac{1-|s(z)|^2}{(1-|z|)\,q(z)v(z)}.$$

Since

$$\lim_{z\to 1}q(z)=1+\kappa,\ \lim_{z\to 1}v(z)=1+M$$

and

$$1-|s(z)|^2=1-\left|\frac{z-z_1}{1-\overline{z_1}z}\right|^2=\frac{\left(1-|z_1|^2\right)\left(1-|z|^2\right)}{|1-\overline{z_1}z|^2},$$

passing to the non-tangential limit in (1.22) gives

$$\begin{aligned}|\Theta'(1)|&\geq\frac{2}{(1+\kappa)\,(1+M)}\left(1+\frac{1-|z_1|^2}{|1-z_1|^2}+M+\kappa M\frac{1-|z_1|^2}{|1-z_1|^2}\right)\\&=1+\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{1-\kappa}{1+\kappa}\left(1+\frac{1-M}{1+M}\frac{1-|z_1|^2}{|1-z_1|^2}\right).\end{aligned}$$

In addition, since

$$\frac{1-\kappa}{1+\kappa}=\frac{1-|k(0)|}{1+|k(0)|}=\frac{1-\left|\frac{\Theta'(0)}{z_1}\right|}{1+\left|\frac{\Theta'(0)}{z_1}\right|}=\frac{(1+\alpha)\,|z_1|-|f''(0)|}{(1+\alpha)\,|z_1|+|f''(0)|},$$

$$\frac{1-M}{1+M}=\frac{1-\dfrac{\left|\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{\Theta'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\Theta'(0)}{z_1}\right|\left|\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\right)}}{1+\dfrac{\left|\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|+\left|\frac{\Theta'(0)}{z_1}\right|}{|z_1|\left(1+\left|\frac{\Theta'(0)}{z_1}\right|\left|\frac{\Theta'(z_1)\left(1-|z_1|^2\right)}{z_1}\right|\right)}}$$

and

$$\frac{1-M}{1+M}=\frac{(1+\alpha)^2|z_1|^2+\left|f''(0)\right|\left(1-|z_1|^2\right)\left|f''(z_1)\right|-(1+\alpha)\left(1-|z_1|^2\right)\left|f''(z_1)\right|-(1+\alpha)\left|f''(0)\right|}{(1+\alpha)^2|z_1|^2+|f''(0)|\left(1-|z_1|^2\right)|f''(z_1)|+(1+\alpha)\left(1-|z_1|^2\right)|f''(z_1)|+(1+\alpha)|f''(0)|},$$

we obtain

$$\begin{aligned}|\Theta'(1)|\geq{}&1+\frac{1-|z_1|^2}{|1-z_1|^2}+\frac{(1+\alpha)|z_1|-\left|f''(0)\right|}{(1+\alpha)|z_1|+|f''(0)|}\\&\times\left[1+\frac{(1+\alpha)^2|z_1|^2+\left|f''(0)\right|\left(1-|z_1|^2\right)\left|f''(z_1)\right|-(1+\alpha)\left(1-|z_1|^2\right)\left|f''(z_1)\right|-(1+\alpha)\left|f''(0)\right|}{(1+\alpha)^2|z_1|^2+|f''(0)|\left(1-|z_1|^2\right)|f''(z_1)|+(1+\alpha)\left(1-|z_1|^2\right)|f''(z_1)|+(1+\alpha)|f''(0)|}\frac{1-|z_1|^2}{|1-z_1|^2}\right].\end{aligned}$$

From definition of $\Theta(z)$, we have

$$\Theta'(z)=\frac{(1+\alpha)\,f''(z)}{\left(f'(z)+\alpha\right)^2}$$

and

$$|\Theta'(1)|=\frac{(1+\alpha)\,|f''(1)|}{|f'(1)+\alpha|^2}\leq\frac{4}{1+\alpha}\,|f''(1)|\,.$$

Therefore, we obtain the inequality (1.19).

Now, we shall show that the inequality (1.19) is sharp.

Since

$$k(z)=\frac{\Theta(z)}{z\frac{z-z_1}{1-\overline{z_1}z}}$$

is a holomorphic function in the unit disc and $|k(z)|\leq 1$ for $|z|<1$, we obtain

$$|\Theta'(0)|\leq|z_1|$$

and

$$|\Theta'(z_1)|\leq\frac{|z_1|}{1-|z_1|^2}.$$

We take $z_1\in(-1,0)$ and arbitrary two numbers c and d, such that $0\leq c\leq(1+\alpha)\,|z_1|$, $0\leq d\leq(1+\alpha)\,\frac{|z_1|}{1-|z_1|^2}$.

Let

$$W=\frac{\frac{\left(1-|z_1|^2\right)d}{z_1}+\frac{c}{z_1}}{z_1\left(1+cd\frac{1-|z_1|^2}{z_1}\right)}=\frac{1}{z_1^2}\frac{d\left(1-|z_1|^2\right)+c}{1+cd\frac{1-|z_1|^2}{z_1}}.$$

The auxiliary function

$$\tau(z) = z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-c}{z_1}+z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{c}{z_1}z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}$$

is holomorphic in D and $|\tau(z)|<1$ for $|z|<1$. Let

$$\frac{f'(z)-1}{f'(z)+\alpha} = z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-c}{z_1}+z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{c}{z_1}z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}$$

and

$$f'(z) = \frac{1+\alpha z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-c}{z_1}+z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{c}{z_1}z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}}{1-z\frac{z-z_1}{1-\overline{z_1}z}\frac{\frac{-c}{z_1}+z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}{1-\frac{c}{z_1}z\frac{W+\frac{z-z_1}{1-\overline{z_1}z}}{1+W\frac{z-z_1}{1-\overline{z_1}z}}}}. \tag{1.23}$$

Thus, we take $|f''(0)|=(1+\alpha)\,c$,

$$\frac{|f''(z_1)|}{1+\alpha} = \frac{z_1}{1-z_1^2}\frac{\frac{-c}{z_1}+Wz_1}{1-\frac{c}{z_1}z_1W} = \frac{z_1}{1-z_1^2}\frac{\frac{-c}{z_1}+\frac{1}{z_1^2}\frac{d\left(1-|z_1|^2\right)+c}{1+cd\frac{1-|z_1|^2}{z_1}}z_1}{1-\frac{c}{z_1}z_1\frac{1}{z_1^2}\frac{d\left(1-|z_1|^2\right)+c}{1+cd\frac{1-|z_1|^2}{z_1}}}$$

and

$$|f''(z_1)| = (1+\alpha)\,d.$$

From (1.23), with the simple calculations, we obtain

$$\begin{aligned}
|f''(1)| &= \frac{1+\alpha}{4}\left(1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-W^2}{(1+W)^2}\right)\left(1-\frac{c}{z_1}\right)+\frac{c}{z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-W^2}{(1+W)^2}\right)\left(1-\frac{c}{z_1}\right)}{\left(1-\frac{c}{z_1}\right)^2}\right)\\
&= \frac{1+\alpha}{4}\left[1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{1+\frac{c}{z_1}}{1-\frac{c}{z_1}}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{1-W^2}{(1+W)^2}\right)\right]\\
&= \frac{1+\alpha}{4}\left[1+\frac{1-z_1^2}{(1-z_1)^2}+\frac{c+z_1}{-c+z_1}\left(1+\frac{1-z_1^2}{(1-z_1)^2}\frac{z_1^2+cd\left(1-z_1^2\right)-d\left(1-z_1^2\right)-c}{z_1^2+cd\left(1-z_1^2\right)+d\left(1-z_1^2\right)+c}\right)\right].
\end{aligned}$$

Since $z_1\in(-1,0)$, the last equality show that (1.19) is sharp. $\square$

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Burcu Gök
Department of Mathematics
Amasya University
Merkez-Amasya 05100, Turkey
E-mail address: `burcugok43@gmail.com`

Bülent Nafi Örnek
Department of Computer Engineering
Amasya University
Merkez-Amasya 05100, Turkey
E-mail address: nafi.ornek@amasya.edu.tr