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SOME COMPOSITION FORMULAS OF JACOBI TYPE ORTHOGONAL POLYNOMIALS

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Abstract. The composition of Jacobi type finite classes of the classical orthogonal polynomials with two generalized Riemann-Liouville fractional derivatives are considered. The outcomes are expressed in terms of generalized Wright function or generalized hypergeometric function. Similar composition formulas are also obtained by considering the generalized Riemann-Liouville and Erdéyi-Kober fractional integral operators.

1. introduction

It is evident from various research articles [3,4,8,13,22,23,25–27,32] that the fractional calculus have wide applications in solving various integral equations, ordinary differential equations and partial differential equations in applied sciences like as turbulence and fluid dynamics, stochastic dynamical system, nonlinear control theory, nonlinear biological systems. As per the requirement, the fractional integration and differential operators have gone through several improvement and extensions, which can be seen in [1,2,9,10,14–17,30,31].

In this article, we will study the composition formula of fractional integration given in [11,13,29,32] and the fractional derivatives given in [10] and [2]. Outcomes of the composition are expressed in term of generalized hypergeometric or Wright functions [28,33,34]. Similar type of work can be seen in [18,24] and references therein.

For this purpose, first we will give a brief introductions about the fractional integral operators and its generalization. Then two type generalized fractional derivatives and its properties are discussed. At the end of this section, we introduced the finite classes require in sequel.

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1.1. Riemann-Liouville fractional integrals and its generalization

The left and right-sided generalized integral transforms defined for $x > 0$ and complex $\alpha, \beta, \eta \in \mathbb{C}$, and $\mathrm{Re}(\alpha) > 0$ are given in [11,13,29,32], respectively as

$$\left(I_{0_+}^{\alpha,\beta,\eta} f\right)(x) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{t}{x}\right) f(t)dt \tag{1.1}$$

and

$$\left(I_{-}^{\alpha,\beta,\eta} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} t^{-\alpha-\beta} {}_2F_1\left(\alpha+\beta, -\eta; \alpha; 1-\frac{x}{t}\right) f(t)dt, \tag{1.2}$$

where $\Gamma(\alpha)$ represents the gamma function [28] and $\mathrm{Re}(\alpha)$ denotes the real part of α.
The generalized hypergeometric function ${}_pF_q(a_1 \cdots a_p; c_1 \cdots c_q; x)$ is given by the representation

$$ {}_pF_q(a_1 \cdots a_p; c_1 \cdots c_q; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k \cdots (a_p)_k}{(c_1)_k \cdots (c_q)_k (1)_k} x^k, \tag{1.3}$$

where none of the denominator parameters can be zero or a negative integer and $(a)_n$ is the well-known Pochhammer symbol given by $(\lambda)_n = \lambda(\lambda+1)_{n-1}$; $(\lambda)_0 = 1$. Results related to the generalized hypergeometric functions are abundant in the literature. For example, see [28]. In particular, for $p = 2$ and $q = 1$, we get the Gaussian hypergeometric function $F(a, b; c; x) = {}_2F_1(a, b; c; x)$ is defined by

$$ {}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n,$$

where a, b are complex numbers and $c \in \mathbb{C}\backslash\{0, -1, -2, \ldots\}$. The series in (1.3) is absolutely convergent for $|x| < 1$ and $|x| = 1$, when $\mathrm{Re}(c-a-b) > 0$.

For $\beta = -\alpha$, (1.1) and (1.2) coincide respectively, with the classical left and right-hand sided Riemann-Liouville fractional integrals [15, 32] for $x > 0$ of order $\alpha \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0$,

$$\left(I_{0_+}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{0_+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad (x > 0) \tag{1.4}$$

and

$$\left(I_{-}^{\alpha,-\alpha,\eta} f\right)(x) = \left(I_{-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty (t-x)^{\alpha-1} f(t)dt, \quad (x > 0). \tag{1.5}$$

If $\beta = 0$, (1.1) and (1.2) respectively, are the so-called Erdĺeyi-Kober fractional integrals [32] defined for complex $\alpha, \eta \in \mathbb{C}$, $\mathrm{Re}(\alpha > 0)$

$$\left(I_{0_+}^{\alpha,0,\eta} f\right)(x) = \left(I_{\alpha,\eta}^{+} f\right)(x) = \frac{x^{-\alpha-\eta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} t^\eta f(t)dt, \quad (x > 0) \tag{1.6}$$

and
(1.7)
$$\left(I_{-}^{\alpha,0,\eta}f\right)(x) = \left(K_{\eta,\alpha}^{-}f\right)(x) = \frac{x^{\eta}}{\Gamma(\alpha)}\int_{x}^{\infty}(t-x)^{\alpha-1}t^{-\alpha-\eta}f(t)dt \quad (x>0).$$

1.2. Riemann-Liouville fractional derivatives and its generalization

The classical Riemann-Liouville fractional derivative of f of order μ is defined by

$$\mathtt{D}_x^{\alpha}f(x) := \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_0^x \frac{f(t)}{(x-t)^{\alpha+1}}dt & \text{if} \quad \mathrm{Re}(\alpha)<0, \\ \frac{d^m}{dx^m}\mathtt{D}_{0+}^{\alpha-m}f(x) & \text{if} \quad m-1<\mathrm{Re}(\alpha)<m. \end{cases} \tag{1.8}$$

In recent years, many authors have developed various extended fractional derivative formulas of Riemann-Liouville type. Here, we consider two generalized fractional derivative.

First, we consider the extension defined in [2] by

$$D_x^{\mu,\sigma}(f(x)) := \frac{1}{\Gamma(-\mu)}\int_0^x f(t)(z-t)^{-\mu-1}\exp\left(-\frac{\sigma x^2}{t(x-t)}\right)dt, \tag{1.9}$$

with $\mathrm{Re}\,(\mu)<0$ and $\mathrm{Re}\,(\sigma)>0$. For $f(x)=x^{\nu}$, it follows that

$$D_x^{\mu,\sigma}(x^{\nu}) = \frac{\mathtt{B}_{\sigma}(\nu+1,-\mu)}{\Gamma(-\mu)}x^{\nu-\mu} \tag{1.10}$$

for $\mathrm{Re}\,(\nu)>-1$ and $\mathrm{Re}\,(\mu)<0$. Further, if f is analytic in the disc $|x|<\rho$ and have the power series $f(x)=\sum_{n=0}^{\infty}a_nx^n$, then

$$D_x^{\mu,\sigma}\left(x^{\lambda-1}f(x)\right) = \frac{x^{\lambda-\mu}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}a_n\mathtt{B}_{\sigma}(\lambda+n,-\mu)x^n, \tag{1.11}$$

provided $\mathrm{Re}\,(\lambda)>-1$ and $\mathrm{Re}\,(\mu)<0$. Here the function $\mathtt{B}_{\sigma}$ is the extended beta function, which was introduced by Chaudhry et al. [5] and defined by

$$\mathtt{B}_{\sigma}(a,b) := \int_0^1 t^{a-1}(1-t)^{b-1}\exp\left(-\tfrac{\sigma}{t(1-t)}\right)dt, \tag{1.12}$$

when $\mathrm{Re}(a)>0$, $\mathrm{Re}(b)>0$, $\mathrm{Re}(\sigma)>0$. This extended beta functions have close association with the Macdonald, error and Whittaker functions, and using this, an extension of the hypergeometric functions can be found in [6].

Recently, Kataugampola [10] introduced a generalized fractional derivatives operator ${}^{\rho}\mathtt{D}_{0+}^{\alpha}$ defined by

$${}^{\rho}\mathtt{D}_{0+}^{\alpha}f(x) := \begin{cases} \frac{\rho^{\alpha+1}}{\Gamma(-\alpha)}\int_0^x \frac{f(t)t^{\rho-1}}{(x^{\rho}-t^{\rho})^{\alpha+1}}dt & \text{if} \quad \mathrm{Re}(\alpha)<0, \\ (x^{-\rho-1}\frac{d}{dx})^m\,{}^{\rho}\mathtt{D}_{0+}^{\alpha-m}f(x) & \text{if} \quad m-1<\mathrm{Re}(\alpha)<m, \end{cases} \tag{1.13}$$

where $m\in\mathbb{N}$ and $\rho>0$.

For the function $f(x) = x^\nu$, $\nu \in \mathbb{R}$, the generalized fractional derivative operator with $\mathrm{Re}(\alpha) < 0$ defined in (1.13) yields the formula

$$
{}^{\rho}\mathtt{D}_{0+}^{\alpha} x^{\nu} = \frac{\rho^{\alpha-1}\Gamma\left(\frac{\nu}{\rho}+1\right)}{\Gamma\left(\frac{\nu}{\rho}+1-\alpha\right)} x^{\nu-\alpha\rho}. \tag{1.14}
$$

Replacing $\rho = 1$ in (1.14) gives the classical Riemann-Liouville fractional integral for the function x^ν, which is,

$$
\mathtt{D}_{x}^{\alpha} x^{\nu} = \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-\alpha)} x^{\nu-\alpha}. \tag{1.15}
$$

1.3. Finite class of classical orthogonal polynomials

The solution of the differential equation

$$
x(x+1)y_n''(x) + ((2-p)x + (1+q))y_n'(x) - n(n-1+p)y_n(x) = 0,
$$

is the polynomial

$$
\mathtt{M}_n^{(p,q)}(x) = (-1)^n n! \sum_{k=0}^{n} \binom{p-n-1}{k} \binom{q+n}{n-k} (-x)^k. \tag{1.16}
$$

With respect to the weight function $\mathtt{w}_{p,q}(x) = x^q(1+x)^{-(p+q)}$, the polynomials defined in (1.16) are orthogonal on $[0,\infty)$ if and only if $p > 2n+1$ and $q > -1$. It is also known that the polynomials $\mathtt{M}_n^{(p,q)}$ are related with hypergeometric functions as

$$
\mathtt{M}_n^{(p,q)}(x) = (-1)^n n! \binom{q+n}{n} {}_2F_1(-n, n+1-p; q+1; -x). \tag{1.17}
$$

The Jacobi polynomials $\mathtt{P}_n^{(\alpha,\beta)}$ and $\mathtt{M}_n^{(p,q)}$ can be related by

$$
\mathtt{M}_n^{(p,q)}(x) = (-1)^n n! \mathtt{P}_n^{(q,-p-q)}(2x+1) \Leftrightarrow \mathtt{P}_n^{(p,q)}(x) = \frac{(-1)^n}{n!} \mathtt{M}_n^{(-p-q,p)}\left(\frac{x-1}{2}\right).
$$

Details related to this finite class of classical orthogonal polynomials can be found [19–21].

1.4. Outline of this work

The aim of this work is to investigate compositions of integral transforms (1.1) and (1.2) for Jacobi type finite class of classical orthogonal polynomials (1.16). In the sequel, the compositions are expressed in terms of the generalized Wright hypergeometric function ${}_p\psi_q(x)$ complex $a_i, b_j \in \mathbb{C}$, and real $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, 2, \ldots, p; j = 1, 2, \ldots, q)$ and defined by the series

$$
{}_p\psi_q(x) = {}_p\psi_q \left[\begin{array}{c} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{array} \middle| x \right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma(b_j + \beta_j k)} \frac{x^k}{k!}. \tag{1.18}
$$

Asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ was studied in [7, 33, 34] and under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1, \tag{1.19}$$

in [33, 34].

2. Composition with generalized fractional integrations

The following lemmas proved in [12], which are required to prove the result in this section.

Lemma 2.1 ([12]). *Let $\alpha, \beta, \eta \in \mathbb{C}$ be such that*

$$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\sigma) > \max[0, \mathrm{Re}(\beta - \eta)].$$

Then there exists the relation

$$\left(I_{0_+}^{\alpha,\beta,\eta} t^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)} x^{\sigma-\beta-1}.$$

In particular,

$$\left(I_{0_+}^{\alpha} t^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma)}{\Gamma(\sigma+\alpha)} x^{\sigma+\alpha-1}, \quad \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\sigma) > 0, \tag{2.1}$$

$$\left(I_{\eta,\alpha}^{+} t^{\sigma-1}\right)(x) = \frac{\Gamma(\sigma+\eta)}{\Gamma(\sigma+\alpha+\eta)} x^{\sigma+\alpha-1}, \quad \mathrm{Re}(\alpha) > 0, \mathrm{Re}(\sigma) > -\mathrm{Re}(\eta). \tag{2.2}$$

Lemma 2.2 ([12]). *Let $\alpha, \beta, \eta \in \mathbb{C}$ be such that*

$$\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\sigma) < 1 + \min[\mathrm{Re}(\beta), \mathrm{Re}(\eta)].$$

Then

$$\left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1}\right)(x) = \frac{\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1)}{\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)} x^{\sigma-\beta-1}.$$

In particular,

$$\left(I_{-}^{\alpha} t^{\sigma-1}\right)(x) = \frac{\Gamma(1-\alpha-\sigma)}{\Gamma(1-\sigma)} x^{\sigma+\alpha-1}, \quad 0 < \mathrm{Re}(\alpha) < 1 - \mathrm{Re}(\sigma), \tag{2.3}$$

$$\left(K_{\eta,\alpha}^{-} t^{\sigma-1}\right)(x) = \frac{\Gamma(\eta-\sigma+1)}{\Gamma(\alpha+\eta-\sigma+1)} x^{\sigma-1}, \quad \mathrm{Re}(\sigma) < 1 + \mathrm{Re}(\eta). \tag{2.4}$$

Theorem 2.1. *Let $\alpha, \beta, \eta, \sigma \in \mathbb{C}$ be such that*

$$\mathrm{Re}(\alpha) > 0, \quad \text{and} \quad \mathrm{Re}\,(\sigma) > \max[0, \mathrm{Re}\,(\beta - \eta)].$$

Then,

$$\left(I_{0_+}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}(t)\right)(x) = \frac{(-1)^n \Gamma(\sigma)\Gamma(\sigma+\eta-\beta)\Gamma(1+q+n) x^{\sigma-\beta-1}}{\Gamma(1+q)\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}$$

$$\times\, {}_4F_3\left[\begin{array}{c} 1+n-p,-n,\sigma,\sigma+\eta-\beta, \\ 1+q,\sigma-\beta,\sigma+\eta+\alpha, \end{array}\middle| -x\right]. \tag{2.5}$$

Proof. First of all note that ${}_4F_3$ given in (2.5) exists because this series absolutely convergent and moreover, it is an entire function if $z \in \mathbb{C}$. Now using (1.1) and changing the order of the integration, the summation yields

$$\begin{aligned}&\left(I_{0_+}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}(t)\right)(x)\\ &= \sum_{k=0}^{\infty}(-1)^k \begin{pmatrix} p-(n+1) \\ k \end{pmatrix}\begin{pmatrix} q+n \\ n-k \end{pmatrix}\left(I_{0_+}^{\alpha,\beta,\eta} t^{\sigma+k-1}\right)(x)\end{aligned}$$

for any $k = 0, 1, 2, \ldots, n$. Since

$$\mathrm{Re}(\sigma+k) > \mathrm{Re}(\sigma) > \max[0, \mathrm{Re}(\beta-\eta)],$$

applying Lemma 2.1 with σ replaced by $\sigma + k$ gives

$$\begin{aligned}&\left(I_{0_+}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}(t)\right)(x)\\ &= \frac{(-1)^n\Gamma(1+q+n)x^{\sigma-\beta-1}}{\Gamma(1+n-p)\Gamma(-n)}\\ &\quad\times\sum_{k=0}^{\infty}\frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\sigma+k)\Gamma(\sigma+\eta-\beta+k)}{\Gamma(1+q+k)\Gamma(\sigma-\beta+2k)\Gamma(\sigma+\alpha+\eta+k)\Gamma(1+k)}(-x)^k.\end{aligned} \tag{2.6}$$

Finally using the well-known identity $\Gamma(x+k) = (x)_k\Gamma(x)$, the right hand side of (2.6) can be rewrite as

$$\begin{aligned}\left(I_{0_+}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}(t)\right)(x) &= \frac{(-1)^n\Gamma(1+q+n)\Gamma(\sigma)\Gamma(\sigma+\eta-\beta)}{\Gamma(1+q)\Gamma(\sigma-\beta)\Gamma(\sigma+\alpha+\eta)}x^{\sigma-\beta-1}\\ &\quad\times\sum_{k=0}^{\infty}\frac{(-n)_k(1+n-p)_k(\sigma)_k(\sigma+\eta-\beta)_k}{(1+q)_k(\sigma-\beta)_k(\sigma+\alpha+\eta)_k k!}(-x)^k.\end{aligned}$$

This establish (2.5). ∎

Substituting $\beta = -\alpha$ in Theorem 2.1, and using (2.1) gives the following result.

Corollary 2.1. *Let* $\alpha, \sigma, \in \mathbb{C}$ *be such that* $\mathrm{Re}(\alpha) > 0$*,* $\mathrm{Re}(\sigma) > 0$*. Then*

$$\begin{aligned}&\left(I_{0_+}^{\alpha} t^{\sigma-1} M_n^{(p,q)}(t)\right)(x)\\ &= \frac{(-1)^n\Gamma(1+q+n)\Gamma(\sigma)x^{\sigma+\alpha-1}}{\Gamma(1+q)\Gamma(\sigma+\alpha)}\,{}_3F_2\left[\begin{array}{c} 1+n-p,-n,\sigma \\ 1+q,\sigma+\alpha \end{array}\middle| -x\right].\end{aligned}$$

Substituting $\beta = 0$ in Theorem 2.1, and using (2.2) implies the following result.

Corollary 2.2. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ be such that* $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}\,(\sigma) > -\mathrm{Re}\,(\eta)$. *Then*

$$\begin{aligned}&\left(I^{+}_{\eta,\alpha} t^{\sigma-1} M_n^{(p,q)}(t)\right)(x)\\ &= \frac{(-1)^n \Gamma(1+q+n)\Gamma(\sigma+\eta) x^{\sigma-1}}{\Gamma(1+q)\Gamma(\sigma+\alpha+\eta)} {}_3F_2\left[\begin{array}{c|c} 1+n-p, -n, \sigma+\eta & \\ 1+q, \sigma+\alpha+\eta & -x\end{array}\right].\end{aligned}$$

Next the generalized right-hand sided fractional integration (1.2) of the Jacobi type finite class of classical orthogonal polynomials related to Jacobi polynomials and the relation with generalized Wright functions are considered to get the following result.

Theorem 2.2. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ be such that*

$$\mathrm{Re}(\alpha) > 0 \quad \textit{and} \quad \mathrm{Re}(\sigma) < 1 + \min[\mathrm{Re}(\beta), \mathrm{Re}(\eta)].$$

Then

$$\begin{aligned}&\left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}\left(\frac{1}{t}\right)\right)(x)\\ &= \frac{\Gamma(1+q+n)\Gamma(\beta-\sigma+1)\Gamma(\eta-\sigma+1) x^{\sigma-\beta-1}}{\Gamma(1+q)\Gamma(1-\sigma)\Gamma(\alpha+\beta+\eta-\sigma+1)}\\ &\quad \times {}_4F_3\left[\begin{array}{c|c} 1+n-p, -n, \beta-\sigma+1, \eta-\sigma+1 & \\ 1+q, 1-\sigma, \alpha+\beta+\eta-\sigma+1, & -\frac{1}{x}\end{array}\right].\end{aligned}$$

Proof. Now using (1.2) and changing the order of the integration, the summation yields

$$\begin{aligned}\left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}\left(\frac{1}{t}\right)\right)(x) &= (-1)^n \Gamma(n+1) \sum_{k=0}^{\infty} \binom{p-(n-1)}{k} \binom{q+n}{n-k}\\ &\quad \times \left((-1)^k I_{-}^{\alpha,\beta,\eta} t^{\sigma-k-1}\right)(x)\end{aligned}$$

for any $k = 0, 1, 2, \ldots$.

Since

$$\mathrm{Re}(\sigma-k-1) \leq 1 + \mathrm{Re}(\sigma-1) < 1 + \min[\mathrm{Re}(\beta), \mathrm{Re}(\eta)],$$

applying Lemma 2.2 with σ replaced by $\sigma - k$, it is easy to see that

$$\begin{aligned}&\left(I_{-}^{\alpha,\beta,\eta} t^{\sigma-1} M_n^{(p,q)}\left(\frac{1}{t}\right)\right)(x)\\ &= \frac{(-1)^n \Gamma(1+q+n) x^{\sigma-\beta-1}}{\Gamma(1+n-p)\Gamma(-n)}\\ &\quad \times \sum_{k=0}^{\infty} \frac{\Gamma(1+n-p+k)\Gamma(-n+k)\Gamma(\beta-\sigma+1+k)\Gamma(\eta-\sigma+1+k)}{\Gamma(1+q+k)\Gamma(1-\sigma+k)\Gamma(\alpha+\beta+\eta-\sigma+1+k)\Gamma(1+k)} (-x)^{-k}.\end{aligned}$$

Again the use of identity $\Gamma(x+k) = (x)_k \Gamma(x)$ in the above equation will gives the required result. $\square$

Corollary 2.3. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ be such that*

$$\mathrm{Re}(\alpha) > 0 \quad \textit{and} \quad \mathrm{Re}\,(\sigma) < 1 + \min[\mathrm{Re}\,(-\alpha), \mathrm{Re}\,(\eta)]$$

and let $\sigma + \alpha \neq 1, 2, \ldots$. Then

$$\left(I_{-}^{\alpha} t^{\sigma-1} M_n^{(p,q)}\left(\frac{1}{t}\right)\right)(x)$$
$$= \frac{\Gamma(1+q+n)\Gamma(-\alpha-\sigma+1)x^{\sigma+\alpha-1}}{\Gamma(1+q)\Gamma(1-\sigma)}\, {}_3F_2\left[\begin{matrix} 1+n-p, -n, -\alpha-\sigma+1 \\ 1+q, 1-\sigma \end{matrix} \,\middle|\, -\frac{1}{x}\right].$$

Corollary 2.4. *Let $\alpha, \eta, \sigma \in \mathbb{C}$ be such that*

$$\mathrm{Re}(\alpha) > 0 \quad \textit{and} \quad \mathrm{Re}\,(\sigma) < 1 + \min[0, \mathrm{Re}\,(\eta)]$$

and let $\sigma - \eta \neq 1, 2, \ldots$. Then

$$\left(K_{\eta,\alpha}^{-} t^{\sigma-1} M_n^{(p,q)}\left(\frac{1}{t}\right)\right)(x) = \frac{(-1)^n\Gamma(1+q+n)\Gamma(\eta-\sigma+1)x^{\sigma-1}}{\Gamma(1+q)\Gamma(\alpha+\eta-\sigma+1)}$$
$$\times\, {}_3F_2\left[\begin{matrix} 1+n-p, -n, \eta-\sigma+1 \\ 1+q, \alpha+\eta-\sigma+1 \end{matrix} \,\middle|\, -\frac{1}{x}\right].$$

3. Composition with the generalized fractional derivatives

In this section, first we consider the composition of the finite class of classical orthogonal polynomials with the generalized fractional derivatives defined in (1.13) and the outcomes are represented in the closed form of the generalized Wright functions defined in (1.18).

Theorem 3.1. *Let $\rho > 0$ and $\mathrm{Re}(\alpha) < 0$. Then for $p \geq 2n+1$ and $q > -1$,*

(3.1)

$$ {}^{\rho}\mathtt{D}_{0+}^{\alpha}\left(\mathtt{M}_n^{(p,q)}(x)\right)$$
$$= \frac{(-1)^n\rho^{\alpha-1}\Gamma(q+1+n)}{x^{\rho\alpha}\Gamma(-n)\Gamma(n+1-p)}\, {}_3\Psi_2\left(\begin{matrix} (-n,1) & (n+1-p,1) & \left(1,\frac{1}{\rho}\right) \\ (q+1,1) & \left(1-\alpha,\frac{1}{\rho}\right) & \end{matrix} \,\middle|\, -x\right).$$

In particular,

$$\mathtt{D}_x^{\alpha}\left(\mathtt{M}_n^{(p,q)}(x)\right) = \frac{(-1)^n(q+1)_n}{\Gamma(1-\alpha)x^{\alpha}}\, {}_3F_2\left(-n, n+1-p, 1; q+1, 1-\alpha; -x\right). \tag{3.2}$$

Proof. From (1.17) it follows that

$$\mathtt{M}_n^{(p,q)}(x) = (-1)^n(q+1)_n \sum_{k=0}^{\infty} \frac{(-n)_k(n+1-p)_k}{(q+1)_k k!}(-x)^k.$$

Thus, an application of the operator ${}^{\rho}\mathtt{D}^{\alpha}_{0+}$ on both side of the above equation along with the relation (1.14) gives

$${}^{\rho}\mathtt{D}^{\alpha}_{0+}\left(\mathtt{M}^{(p,q)}_n(x)\right) = (-1)^n(q+1)_n\sum_{k=0}^{\infty}\frac{(-1)^k(-n)_k(n+1-p)_k}{(q+1)_k k!}\ {}^{\rho}\mathtt{D}^{\alpha}_{0+}(x^k)$$

$$= (-1)^n(q+1)_n\rho^{\alpha-1}\sum_{k=0}^{\infty}\frac{(-1)^k(-n)_k(n+1-p)_k\Gamma\left(1+\frac{k}{\rho}\right)}{(q+1)_k\Gamma\left(1-\alpha+\frac{k}{\rho}\right)k!}x^{k-\alpha\rho}. \tag{3.3}$$

Next the identity $\Gamma(x+k)=(x)_k\Gamma(x)$ yields

$${}^{\rho}\mathtt{D}^{\alpha}_{0+}\left(\mathtt{M}^{(p,q)}_n(x)\right)$$

$$= \frac{(-1)^n\Gamma(n+1+q)\rho^{\alpha-1}x^{-\rho\alpha}}{\Gamma(-n)\Gamma(n+1-p)}\sum_{k=0}^{\infty}\frac{\Gamma(-n+k)\Gamma(n+1-p+k)\Gamma\left(1+\frac{k}{\rho}\right)}{\Gamma(q+1+k)\Gamma\left(1-\alpha+\frac{k}{\rho}\right)k!}(-x)^k$$

$$= \frac{(-1)^n\Gamma(n+1+q)\rho^{\alpha-1}x^{-\alpha\rho}}{\Gamma(-n)\Gamma(n+1-p)}{}_3\Psi_2\left(\begin{matrix}(-n,1) & (n+1-p,1) & \left(1,\frac{1}{\rho}\right)\\ (q+1,1) & \left(1-\alpha,\frac{1}{\rho}\right) & \end{matrix}\ \middle|\ -x\right).$$

Since ${}^{1}\mathtt{D}^{\alpha}_{0+}=\mathtt{D}^{\alpha}_{x}$, the relation (3.3) reduces

$$\mathtt{D}^{\alpha}_{x}\left(\mathtt{M}^{(p,q)}_n(x)\right) = (-1)^n(q+1)_n x^{-\alpha}\sum_{k=0}^{\infty}\frac{(-n)_k(n+1-p)_k\Gamma\left(1+k\right)}{(q+1)_k\Gamma(1-\alpha+k)k!}(-x)^k.$$

Finally, the result follows by the use of identity $\Gamma(x+k)=(x)_k\Gamma(x)$. □

Corollary 3.1. *For $\alpha=1$, equations* (3.2) *becomes*

$$\mathtt{D}\left(\mathtt{M}^{(p,q)}_n(x)\right) = \frac{(-1)^n n(1+n-p)(q+1)_n}{(1+q)}{}_2F_1\left[-(n-1),n+1-p;q+2;-x\right], \tag{3.4}$$

which is verified by [19].

In the next result, we will give an inequality for the composition of the finite class $\mathtt{M}^{(p,q)}_n$ with the generalized fractional derivatives defined in (1.9).

Theorem 3.2. *Let $\mu,\nu,\sigma,x\in\mathbb{R}$ such that $\nu,\sigma,x>0$ and $\mu<0$. Then*

$$\left|D^{\mu,\sigma}_x\left(x^{\nu-1}\mathtt{M}^{(p,q)}_n(x)\right)\right| \leq \frac{e^{-4\sigma}\Gamma(\nu)x^{\nu-\mu-1}}{\Gamma(\nu-\mu)}{}_3F_2(-n,1+n-p,\nu;q+1,\nu-\mu;x). \tag{3.5}$$

Proof. From (1.11) and (1.17), it follows that

$$D^{\mu,\sigma}_x\left(x^{\nu-1}\mathtt{M}^{(p,q)}_n(x)\right) \tag{3.6}$$

$$= \frac{(-1)^n(q+1)_n x^{\nu+\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}\frac{(-n)_k(1+n-p)_k}{(q+1)_k k!}\mathtt{B}_{\sigma}(\nu+k,-\mu)x^k.$$

It is proved in [5] that $|\mathtt{B}_\sigma(x,y)| \leq e^{-4\sigma}\mathtt{B}(x,y)$ for $x,y,\sigma \geq 0$ and $\mathtt{B}(x,y)$ is the classical beta functions. Using this to (3.6), it follows that

$$\begin{aligned} &\left|D_x^{\mu,\sigma}\left(x^{\nu-1}\mathtt{M}_n^{(p,q)}(x)\right)\right| \\ &\leq \frac{e^{-4\sigma}(q+1)_n x^{\nu+\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}\left|\frac{(-n)_k(1+n-p)_k}{(q+1)_k k!}\right|\mathtt{B}(\nu+k,-\mu)x^k. \end{aligned} \tag{3.7}$$

Now $p \geq 2n+1$ implies $1+n-p \leq -n$ and thus, $(-n)_k(1+n-p)_k$ is non-negative for all k. Thus, we can rewrite (3.7) as

$$\begin{aligned} &\left|D_x^{\mu,\sigma}\left(x^{\nu-1}\mathtt{M}_n^{(p,q)}(x)\right)\right| \\ &\leq \frac{e^{-4\sigma}(q+1)_n x^{\nu+\mu-1}}{\Gamma(-\mu)}\sum_{n=0}^{\infty}\frac{(-n)_k(1+n-p)_k}{(q+1)_k k!}\mathtt{B}(\nu+k,-\mu)x^k, \end{aligned}$$

which is equivalent to (3.5) after using the identity

$$\mathtt{B}(\nu+k,-\mu) = \frac{\Gamma(\nu+k)\Gamma(-\mu)}{\Gamma(\nu-\mu+k)} = \frac{\Gamma(\nu)(\nu)_k}{\Gamma(\nu-\mu)(\nu-\mu)_k}.$$

This prove the inequality. □

Remark 3.1. For $\sigma = 0$ and $\nu = 1$, we have from Theorem 3.2 that

$$\left|D_x^{\mu}\left(\mathtt{M}_n^{(p,q)}(x)\right)\right| \leq \frac{x^{-\mu}}{\Gamma(1-\mu)}\,{}_3F_2(-n,1+n-p,1;q+1,1-\mu;x).$$

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