Commun. Korean Math. Soc. **32** (2017), No. 3, pp. 661–676
https://doi.org/10.4134/CKMS.c160191
pISSN: 1225-1763 / eISSN: 2234-3024

ON OPERATORS SATISFYING $T^{*m}(T^*|T|^{2k}T)^{1/(k+1)}T^m \geq T^{*m}|T|^2T^m$

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Abstract. Let T be a bounded linear operator acting on a complex Hilbert space $\mathscr{H}$. In this paper we introduce the class, denoted $\mathcal{Q}(A(k), m)$, of operators satisfying $T^{m*}(T^*|T|^{2k}T)^{1/(k+1)}T^m \geq T^{*m}|T|^2T^m$, where m is a positive integer and k is a positive real number and we prove basic structural properties of these operators. Using these results, we prove that if P is the Riesz idempotent for isolated point λ of the spectrum of $T \in \mathcal{Q}(A(k), m)$, then P is self-adjoint, and we give a necessary and sufficient condition for $T \otimes S$ to be in $\mathcal{Q}(A(k), m)$ when T and S are both non-zero operators. Moreover, we characterize the quasinilpotent part $H_0(T-\lambda)$ of class $A(k)$ operator.

1. Introduction

Let $\mathscr{H}$ be a complex Hilbert space and let $\mathscr{L}(\mathscr{H})$ be the algebra of all bounded linear operators acting on $\mathscr{H}$. An operator $T \in \mathscr{L}(\mathscr{H})$ has a unique polar decomposition $T = U|T|$, where $|T| = (T^*T)^{1/2}$ and U is partial isometry satisfying $\ker(U) = \ker(T) = \ker(|T|)$ and $\ker(U) = \ker(T^*)$.

An operator T is said to be positive (denoted by $T \geq 0$) if $\langle Tx, x\rangle \geq 0$ for all $x \in \mathscr{H}$ and also T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. An operator T is called p-hyponormal if $|T|^{2p} \geq |T^*|^{2p}$ for every $0 < p \leq 1$ and log-hyponormal if T is invertible and $\log(T^*T) \geq \log(TT^*)$, T is called paranormal if $\|T^2x\| \geq \|Tx\|^2$ for every unit vector $x \in \mathscr{H}$, and T is called normaloid if $\|T\| = r(T)$, the spectral radius of T. Following [9, 10], we say that $T \in \mathscr{L}(\mathscr{H})$ belongs to class A if $|T^2| \geq |T|^2$ and class $A(k)$ for $k > 0$ (abbreviation $T \in \mathcal{A}(k)$) if $(T^*|T|^{2k}T)^{1/(k+1)} \geq |T|^2$, we note that T is class A if and only if T is class $A(1)$. According to [2], an operator $T \in \mathscr{L}(\mathscr{H})$ is said to be w-hyponormal if $|\widetilde{T}| \geq |T| \geq |\widetilde{T^*}|$, where $\widetilde{T}$ is the Aluthge transformation $\widetilde{T} = |T|^{1/2}U|T|^{1/2}$. As a generalization of w-hyponormal and class $A(k)$, Ito [10] introduced class $wA(s,t)$ as follows. An operator T is called class $wA(s,t)$ for $s > 0$ and $t > 0$ if $|\widetilde{T_{s,t}}|^{2t/(s+t)} \geq |T|^{2t}$

Received September 17, 2016; Accepted December 9, 2016.
2010 *Mathematics Subject Classification.* 47A55, 47A10, 47A11.
Key words and phrases. Riesz idempotent, tensor product, class $A(k)$, m-quasi-class $A(k)$.

and $|T|^{2s} \geq |\widetilde{T^*_{s,t}}|^{2s/(s+t)}$, where $\widetilde{T_{s,t}}$ is generalized Aluthge transformation, i.e., $\widetilde{T_{s,t}} = |T|^s U |T|^t$. An operator $T \in \mathscr{L}(\mathscr{H})$ is called k-paranormal for positive integer k, if $\|T^{k+1}x\| \geq \|Tx\|^{k+1}$ for every unit vector $x \in \mathscr{H}$.

Definition 1.1. We say that an operator $T \in \mathscr{L}(\mathscr{H})$ is of m-quasi class A_k (abbreviate $\mathcal{Q}(A(k), m)$), if

$$T^{*m}(T^*|T|^{2k}T)^{1/(k+1)}T^m \geq T^{m*}|T|^2 T^m,$$

where m is a positive integers and $k > 0$. If $m = 1$, then T is called a quasi-class $A(k)$ and $k = m = 1$, then $\mathcal{Q}(A(k), m)$ coincides with quasi-class A operator.

Example 1.2. Let $\mathscr{H} = \bigoplus_{n=0}^{\infty} \mathbb{C}^2$ and define an operator T on $\mathscr{H}$ by

$$T(\cdots \oplus x_{-2} \oplus x_{-1} \oplus x_0^{(0)} \oplus x_1 \oplus \cdots) = \cdots \oplus Ax_{-2} \oplus Ax_{-1} \oplus Bx_0 \oplus Bx_1 \oplus \cdots,$$

where $A = \frac{1}{4}\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Then T is of m-quasi-class $A(k)$ for each $k \geq \frac{1}{4}$. In fact, for each $k \geq \frac{1}{4}$,

$$\begin{aligned} & \left\langle T^{*m}\left((T^*|T|^{2k}T)^{1/(k+1)} - |T|^2\right)T^m x, x\right\rangle \\ = & \left\langle A^m\left((ABA)^{1/(k+1)} - A^2\right)A^m x_{-1}, x_{-1}\right\rangle \\ = & \left(\frac{1}{16}\right)^m \left\{\left(\frac{1}{32}\right)^{1/(k+1)} - \left(\frac{1}{16}\right)\right\} \left\|\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} x_{-1}\right\|^2 \geq 0 \end{aligned}$$

for each $x \in \mathscr{H}$.

Let $0 < \alpha < 1$ and $A = \alpha\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$. Then $T \in \mathcal{Q}(A(k), m)$ with $k \geq \frac{-\log 2}{2\log\alpha}$. Since $\frac{-\log 2}{2\log\alpha} \to 0$ as $\alpha \to 0$ for any $k > 0$. Then $T \in \mathcal{Q}(A(k), m)$ for each $k > 0$ and m is a positive integer.

Since $T \geq 0$ implies $R^*TR \geq 0$, we have:

Proposition 1.3. *Let $T \in \mathscr{L}(\mathscr{H})$. If $T \in \mathcal{A}(k)$, then $T \in \mathcal{Q}(A(k), m)$.*

Throughout this paper, we shall denote the spectrum, the point spectrum and the isolated points of the spectrum of $T \in \mathscr{L}(\mathscr{H})$ by $\sigma(T), \sigma_p(T)$ and $iso\sigma(T)$, respectively. The range and the kernel of $T \in \mathscr{L}(\mathscr{H})$ will be denoted by $\Re(T)$ and $\ker(T)$, respectively. We shall denote the set of all complex numbers and the complex conjugate of a complex number λ by $\mathbb{C}$ and $\overline{\lambda}$, respectively. The closure of a set S will be denoted by $\overline{S}$ and we shall henceforth shorten $T - \lambda I$ to $T - \lambda$.

In Section 2, we prove basic properties of $\mathcal{Q}(A(k), m)$ operators and using these properties, in Section 3, we prove that if P is the Riesz idempotent for a non-zero isolated point λ of the spectrum of $T \in \mathcal{Q}(A(k), m)$, then P is self-adjoint and $\Re(P) = \ker(T - \lambda) = \ker(T - \lambda)^*$ and if $\lambda = 0$, then $\Re(P) = H_0(T) = \ker(T^{m+1})$. This is a complete extension of results proved for quasi-class A operators and quasi-class (A, m) operators in [12, 26], respectively. In Section 4, we give a necessary and sufficient condition for $T \otimes S$ to be in $\mathcal{Q}(A(k), m)$ when T and S are both non-zero operators. This gives an analogous result proved for quasi-class A operators and quasi-class (A, m) operators in [12, 26], respectively.

2. Properties of $\mathcal{Q}(A(k), m)$ operators

To prove these properties we need the following lemma.

Lemma 2.1 ([13]). *If $A, B \in \mathscr{L}(\mathscr{H})$ satisfying $A \geq 0$ and $\|B\| \leq 1$, then*

$$(B^*AB)^\alpha \geq B^*A^\alpha B \quad \textit{for all } \alpha \in (0, 1].$$

Lemma 2.2. *Let $T \in \mathcal{Q}(A(k), m)$ and T not have a dense range. Then*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathscr{H} = \overline{\Re(T^m)} \oplus \ker(T^{*m}),$$

where $T_1 = T|_{\overline{\Re(T^m)}}$ is the restriction of T to $\overline{\Re(T^m)}$, and $T_1 \in \mathcal{A}(k)$ and T_3 is nilpotent of nilpotency m. Moreover, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Consider the matrix representation of T with respect to the decomposition $\mathscr{H} = \overline{\Re(T^m)} \oplus \ker(T^{*m})$;

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$$

Let P be the orthogonal projection onto $\overline{\Re(T^m)}$. Then $\left(\begin{smallmatrix} T_1 & 0 \\ 0 & 0 \end{smallmatrix}\right) = TP = PTP$. Since $T \in \mathcal{Q}(A(k), m)$, we have

$$P\left((T^*|T|^{2k}T)^{1/(k+1)} - |T|^2\right)P \geq 0.$$

Then by Lemma 2.1

$$\begin{aligned} P(T^*|T|^{2k}T)^{1/(k+1)}P &= P(T^*|T|^{2k}T)^{1/(k+1)}P \\ &\leq \left(PT^*|T|^{2k}TP\right)^{\frac{1}{k+1}} \leq \left(PT^*(PT^*TP)^kTP\right)^{\frac{1}{k+1}} \\ &= \begin{pmatrix} (T_1^*|T_1|^{2k}T_1)^{1/(k+1)} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$P|T|^2P = PT^*TP = \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} \begin{pmatrix} (T_1^*|T_1|^{2k}T_1)^{1/(k+1)} & 0 \\ 0 & 0 \end{pmatrix} &\geq P(T^*|T|^{2k}T)^{1/(k+1)}P \geq P|T|^2P \\ &= \begin{pmatrix} |T_1|^2 & 0 \\ 0 & 0 \end{pmatrix}, \end{aligned}$$

i.e., $T_1 \in \mathcal{A}(k)$. On the other hand, if $u = \binom{u_1}{u_2} \in \mathscr{H}$,

$$\langle T_3 u_2, u_2\rangle = \langle T(I-P)u, (I-P)u\rangle = \langle (I-P)u, T^*(I-P)u\rangle = 0,$$

which implies that $T_3 = 0$. It is well known that $\sigma(T_1) \cup \sigma(T_3) = \sigma(T) \cup \mathcal{C}$, where $\mathcal{C}$ is the union of certain of the holes in $\sigma(T)$ which happen to be subset of $\sigma(T_1) \cap \sigma(T_3)$ and $\sigma(T_1) \cap \sigma(T_3)$ has no interior points. Therefore, we have

$$\sigma(T) = \sigma(T_1) \cup \sigma(T_3) = \sigma(T_1) \cup \{0\}\,. \qquad \square$$

Theorem 2.3. *Let $T \in \mathscr{L}(\mathscr{H})$ be a $\mathcal{QA}_k$ operator and $\mathscr{M}$ be its invariant subspace. Then the restriction $T|_{\mathscr{M}}$ of T to $\mathscr{M}$ is also $\mathcal{Q}(A(k), m)$ operator.*

Proof. Let Q be the orthogonal projection onto $\mathscr{M}$. Put $T_1 = T|_{\mathscr{M}}$. Then $TQ = QTQ$ and $T_1 = (QTQ)|_{\mathscr{M}}$. Since T is a $\mathcal{Q}(A(k), m)$ operator, we have

$$QT^* \left(T^*|T|^{2k}T\right)^{1/(k+1)} TQ \geq QT^*|T|^2TQ.$$

Since

$$\begin{aligned} QT^* \left(T^*|T|^{2k}T\right)^{1/(k+1)} TQ &= QT^*Q \left(T^*|T|^{2k}T\right)^{1/(k+1)} QTQ \\ &\leq QT^*Q \left(QT^*(T^*T)^kTQ\right)^{\frac{1}{1+k}} QTQ \\ &\leq QT^*Q \left(QT^*(QT^*TQ)^kTQ\right)^{\frac{1}{1+k}} QTQ \\ &= \begin{pmatrix} (T_1^*|T_1|^{2k}T_1)^{1/(k+1)} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

and

$$QT^*|T|^2TQ = QT^*QT^*TQTQ = \begin{pmatrix} T_1^*|T_1|^2T_1 & 0 \\ 0 & 0 \end{pmatrix},$$

we have

$$\begin{aligned} \begin{pmatrix} (T_1^*|T_1|^{2k}T_1)^{1/(k+1)} & 0 \\ 0 & 0 \end{pmatrix} &\geq QT^* \left(T^*|T|^{2k}T\right)^{1/(k+1)} T \\ &\geq QT^*(|T|^2)TQ = \begin{pmatrix} T_1^*|T_1|^2T_1 & 0 \\ 0 & 0 \end{pmatrix}. \end{aligned}$$

This implies that $T_1 \in \mathcal{Q}(A(k), m)$. $\square$

Theorem 2.4. *Let $T \in \mathcal{Q}(A(k), m)$. Then the following assertions holds*:

(a) *If $\mathscr{M}$ is an invariant subspace of T and $T|_{\mathscr{M}}$ is an injective normal operator, then $\mathscr{M}$ reduces T.*

(b) *If $(T-\lambda)x = 0$ and $\lambda \neq 0$, then $(T-\lambda)^* = 0$.*

Proof. (a) Decompose T into

$$T = \begin{pmatrix} S & A \\ 0 & B \end{pmatrix} \quad \text{on} \quad \mathscr{H} = \mathscr{M} \oplus \mathscr{M}^\perp$$

and let $S = T|_{\mathscr{M}}$ be an injective normal operator. Let Q be the orthogonal projection of $\mathscr{H}$ onto $\mathscr{M}$. Since $T^m = \begin{pmatrix} S^m & * \\ 0 & B^m \end{pmatrix}$ and $\ker(S) = \ker(S^*) = \{0\}$, we have

$$\mathscr{M} = \overline{\Re(S)} = \overline{\Re(S^m)} \subset \overline{\Re(T^m)}.$$

Then

$$\begin{aligned} \begin{pmatrix} |S|^2 & 0 \\ 0 & 0 \end{pmatrix} &= Q|T|^2Q \leq Q(T^*|T|^{2k}T)^{1/(k+1)}Q \\ &\leq \big(QT^*|T|^{2k}T)Q\big)^{1/(k+1)} \\ &\leq \big(QT^*(QT^*TQ)^kT)Q\big)^{1/(k+1)} = \begin{pmatrix} |S^{k+1}|^{2/(k+1)} & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

by Lemma 2.1. Therefore,

$$Q(T^*|T|^{2k}T)^{1/(k+1)}Q = \begin{pmatrix} |S|^2 & 0 \\ 0 & 0 \end{pmatrix} = Q|T|^2Q.$$

Since S is normal, we can write $(T^*|T|^{2k}T)^{1/(k+1)} = \begin{pmatrix} |S|^2 & C \\ C^* & D \end{pmatrix}$. Since

$$\begin{pmatrix} |S|^{2(k+1)} & 0 \\ 0 & 0 \end{pmatrix} = Q(T^*|T|^{2k}T)Q = Q((T^*|T|^{2k}T)^{k+1})^{1/(k+1)}Q,$$

we can easily show that $C = 0$. Therefore,

$$(T^*|T|^{2k}T)^{1/(k+1)} = \begin{pmatrix} |S|^2 & 0 \\ 0 & D \end{pmatrix}$$

and hence

$$T^*|T|^{2k}T = \begin{pmatrix} |S|^{2(k+1)} & 0 \\ 0 & D^{k+1} \end{pmatrix} = T^*(T^*T)^kT.$$

This implies that $D = (B^*|B|^{2k}B)^{1/(k+1)}$. Therefore,

$$\begin{aligned} 0 &\leq T^{*m}((T^*(T^*T)^kT)^{1/(k+1)} - |T|^2)T^m \\ &= \begin{pmatrix} 0 & Y \\ Y^* & B^{*m}((B^*|B|^{2k}B)^{1/(k+1)} - |B|^2)B^m \end{pmatrix}. \end{aligned}$$

Hence $A = 0$ and B is a $\mathcal{Q}(A(k), m)$ operator.

(b) Let $\mathscr{M} = span\,\{x\}$. Then $T|_{\mathscr{M}} = \lambda \neq 0$ and $T|_{\mathscr{M}}$ is an injective normal operator. Hence $\mathscr{M}$ reduces T and $T = \begin{pmatrix} \lambda & 0 \\ 0 & B \end{pmatrix}$ on $\mathscr{H} = \mathscr{M} \oplus \mathscr{M}^{\perp}$. Then $(T - \lambda)^* = 0$. $\square$

An operator $T \in \mathscr{L}(\mathscr{H})$ is called isoloid if every isolated point of $\sigma(T)$ is an eigenvalue of T. In [21], Rashid proved every class $wF(p, r, q)$ operators are isoloid, we extend this result to m-quasi-class $A(k)$ operators.

Lemma 2.5. *Let $T \in \mathcal{Q}(A(k), m)$. Then T is an isoloid.*

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\Re(T^m)} \oplus \ker(T^{*m})$, and assume that $\mu \in iso\sigma(T)$. Then $\mu \in iso\sigma(T_1)$ or $\mu = 0$ by Lemma 2.2. If $\mu \in iso\sigma(T_1)$, then $\mu \in \sigma_p(T_1)$ because $T_1 \in \mathcal{A}(k)$ and a class $A(k)$ is an isoloid by Theorem 2.10 of [22]. Thus we may assume that $\mu = 0$ and $\mu \notin \sigma(T_1)$, so $\dim \ker(T_3) > 0$. Therefore, if $x \in \ker(T_3)$, then $-T_1^{-1}T_2 x \oplus x \ker(T)$. Hence μ is an eigenvalue of T. $\square$

Let $Hol(\sigma(T))$ be the space of all functions that analytic in an open neighborhoods of $\sigma(T)$. Following [7] we say that $T \in \mathscr{L}(\mathscr{H})$ has the single-valued extension property (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathscr{H}$ which satisfies the equation $(T - \mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathscr{L}(\mathscr{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathscr{L}(\mathscr{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [18, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

Definition 2.6 ([4])**.** An operator T is said to have *Bishop's property* (β) at $\lambda \in \mathbb{C}$ if for every open neighborhood G of λ, the function $f_n \in Hol(G)$ with $(T - \lambda)f_n(\mu) \to 0$ uniformly on every compact subset of G implies that $f_n(\mu) \to 0$ uniformly on every compact subset of G, where $Hol(G)$ means the space of all analytic functions on G. When T has Bishop's property (β) at each $\lambda \in \mathbb{C}$, simply say that T has property (β).

Lemma 2.7 ([17])**.** *Let G be open subset of complex plane $\mathbb{C}$ and let $f_n \in Hol(G)$ be functions such that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, then $f_n(\mu) \to 0$ uniformly on every compact subset of G.*

Following [8], we say that an operator $T \in \mathscr{L}(\mathscr{H})$ belongs to class $A(s,t)$ for every $s > 0$ and $t > 0$ if

$$(|T^*|^t|T|^{2s}|T^*|^s)^{t/(t+s)} \geq |T^*|^{2t}.$$

It is easy to see that $T \in A(k)$ if and only $T \in A(k,1)$ because if T is a class $A(k)$, then

$$\begin{aligned}(T^*|T|^{2k}T)^{1/(k+1)} &= (U^*|T^*||T|^{2k}|T^*|U)^{1/(k+1)} \\ &= U^*(|T^*||T|^{2k}|T^*|)^{1/(k+1)}U \geq |T|^2 \text{ and} \\ (|T^*||T|^{2k}|T^*|)^{1/(k+1)} &\geq U|T|^2U^* = |T^*|^2.\end{aligned}$$

Hence, $T \in A(k,1)$. If $T \in A(k,1)$, then

$$\begin{aligned}|T^*|^2 &\leq (|T^*||T|^{2k}|T^*|)^{1/(k+1)} \\ &\leq (UT^*|T|^{2k}TU^*)^{1/(k+1)} = U(T^*|T|^{2k}T)^{1/(k+1)}U^* \text{ and} \\ (T^*|T|^{2k}T)^{1/(k+1)} &\geq U^*|T^*|^2U = |T|^2.\end{aligned}$$

So, $T \in A(k)$.

The relations between T and its transformation $\widetilde{T}_{s,t}$ are

$$\widetilde{T}_{s,t}|T|^s = |T|^sU|T|^t|T|^s = |T|^sT, \tag{2.1}$$

and

$$U|T|^t\widetilde{T}_{s,t} = U|T|^t|T|^sU|T|^t = TU|T|^t \tag{2.2}$$

for each $s > 0$ and $t > 0$.

Theorem 2.8. *Let T belong the class $A(k)$ for $k > 0$. Then T has the property (β).*

Proof. Since $\widetilde{T}_{k,1}$ is $\frac{\min(k,1)}{k+1}$-hyponormal ([10]) it is suffices to show that T has property (β) if and only if $\widetilde{T}_{k,1}$ has property (β).

Let G be an open neighborhood of λ and let $f_n \in Hol(\sigma(T))$ be functions such that $(\mu - \widetilde{T}_{k,1})f_n(\mu) \to 0$ uniformly on every compact subset of G. By Equations 2.2, $(\mu - T)(U|T|^kf_n(\mu)) = U|T|^k(\mu - \widetilde{T}_{k,1})f_n(\mu) \to 0$ uniformly on every compact subset of G. Hence $\widetilde{T}_{k,1}f_n(\mu) \to 0$ uniformly on every compact subset of G, and $\widetilde{T}_{k,1}$ having property β follows by Lemma 2.7.

Suppose that $\widetilde{T}_{k,1}$ has property (β). Let G be an open neighborhood of λ and let $f_n \in Hol(\sigma(T))$ be functions such that $(\mu - T)f_n(\mu) \to 0$ uniformly on every compact subset of G. Since $(\widetilde{T}_{k,1} - \mu)|T|^kf_n(\mu) = |T|^k(T - \mu)f_n(\mu) \to 0$ uniformly on every compact subset of G. Hence $Tf_n(\mu) = U|T|^k|T|f_n(\mu) \to 0$ uniformly on every compact subset of G for $\widetilde{T}_{k,1}$ has property $(\beta$, so that $\mu f_n(\mu) \to 0$ uniformly on every compact subset of G, and T has property (β) follows by Lemma 2.7. □

The quasinilpotent part of $T - \lambda$ is defined as

$$H_0(T - \lambda) = \left\{x \in \mathscr{H} : \lim_{n\to\infty} \|(T - \lambda)^n\|^{1/n} = 0\right\}.$$

In general, $\ker(T - \lambda) \subset H_0(T - \lambda)$ and $H_0(T - \lambda)$ is not closed. Let $F \subset \mathbb{C}$ be closed set. Then the global spectral subspace is defined by

$$\chi_T(F) = \{x \in \mathscr{H} \mid \exists \,\text{analytic}\, f(z) : (T - \lambda)f(z) = x \,\text{on}\, \mathbb{C} \setminus F\}.$$

Theorem 2.9. *Let $T \in \mathcal{A}(k)$. Then $H_0(T - \lambda) = \ker(T - \lambda)$ for $\lambda \in \mathbb{C}$.*

Proof. Let $F \subset \mathbb{C}$ be closed set. It is known that $H_0(T - \lambda) = \chi_T(\{\lambda\})$ by Theorem 2.20 of [1]. As T has Bishop's property by Theorem 2.8, $\chi_T(F)$ is closed and $\sigma(T|_{\chi_T(F)}) \subset F$ by Proposition 1.2.19 of [19]. Hence $H_0(T - \lambda)$ is closed and $T|_{H_0(T-\lambda)}$ is class A_k by Theorem 2.3. If $\sigma(T|_{H_0(T-\lambda)}) \subset \{\lambda\}$, $T|_{H_0(T-\lambda)}$ is normal by Theorem 2.4. If $\sigma(T|_{H_0(T-\lambda)}) = \emptyset$, then $H_0(T - \lambda) = \{0\}$ and $\ker(T - \lambda) = \{0\}$. If $\sigma(T|_{H_0(T-\lambda)}) = \{\lambda\}$, then $T|_{H_0(T-\lambda)} = \lambda$ and $H_0(T - \lambda) = \ker(T - \lambda)$. □

Rashid [20] proved that quasi-class (A, k) has Bishop's property, in the following we prove analogous result for m-quasi-class $A(k)$ operators.

Lemma 2.10. *Let $T \in \mathcal{Q}(A(k), m)$. Then T has Bishop's property (β).*

Proof. Let $f_n(z)$ be analytic on G. Let $(T-z)f_n(z) \longrightarrow 0$ uniformly on each compact subset of G. Then, using the representation of Lemma 2.2 we have

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_{n1}(z) \\ f_{n2}(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_{n1}(z) + T_2 f_{n2}(z) \\ (T_3 - z)f_{n2}(z) \end{pmatrix} \longrightarrow 0.$$

Since T_3 is nilpotent, T_3 has Bishop's property (β). Hence $f_{n2}(z) \longrightarrow 0$ uniformly on every compact subset of G. Then $(T_1 - z)f_{n1}(z) \longrightarrow 0$. Since T_1 is of class $A(k)$, T_1 has Bishop's property (β) by Theorem 2.8. Hence $f_{n1}(z) \longrightarrow 0$ uniformly on every compact subset of G. Thus T has Bishop's property (β). □

Lemma 2.11. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class $A(k)$. Let $\lambda \in \mathbb{C}$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda I$.*

Proof. We consider two cases:

Case (I) $(\lambda = 0)$: Since T is a class $A(k)$, T is normaloid. Therefore $T = 0$.

Case (II) $(\lambda \neq 0)$: Here T is invertible, and since T is a class A_k, we see that T^{-1} is also belongs class $A(k)$. Therefore T^{-1} is normaloid. On the other hand, $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\|\|T^{-1}\| = |\lambda||\frac{1}{\lambda}| = 1$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda$. □

Lemma 2.12. *Let $T \in \mathscr{L}(\mathscr{H})$ be a $\mathcal{Q}(A(k), m)$ operator and $\sigma(T) = \{\lambda\}$. Then $T = \lambda$ if $\lambda \neq 0$, and $T^{m+1} = 0$ if $\lambda = 0$.*

Proof. If the range of T is dense, then T is a class $A(k)$. Hence $T = \lambda$ by Lemma 2.11. If the range of T is not dense, then

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad \text{on} \quad \mathscr{H} = \overline{\Re(T^m)} \oplus \ker(T^{*m}),$$

where $T_1 = T|_{\overline{\Re(T^m)}}$ is the restriction of T to $\overline{\Re(T^m)}$, and $T_1 \in \mathcal{Q}(A(k), m)$, $T_3^m = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$ by Lemma 2.2. Hence $T_1 = 0$ by Lemma 2.11. Thus

$$T^{m+1} = \begin{pmatrix} 0 & T_2 \\ 0 & T_3 \end{pmatrix}^{m+1} = \begin{pmatrix} 0 & T_2 T_3^m \\ 0 & T_3^{m+1} \end{pmatrix} = 0. \qquad \square$$

3. Riesz idempotent for an isolated point of the spectrum

Let $T \in \mathscr{L}(\mathscr{H})$ and $\mu \in iso\sigma(T)$. Then there exists a positive number $r > 0$ such that $\{\lambda \in \mathbb{C} : |\lambda - \mu| \leq r\} \cap \sigma(T) = \{\mu\}$. Let γ be the boundary of $\{\lambda \in \mathbb{C} : |\lambda - \mu| \leq r\}$. Then $P := \frac{1}{2\pi i}\int_\gamma (\lambda - T) d\lambda$ is called the Riesz idempotent of T for μ. Then it is well known that

$$P^2 = P, \quad PT = TP, \quad \sigma(T|_{\Re(P)}) = \{\mu\} \quad \text{and} \quad \Re(P) \supseteq \ker(T - \mu).$$

In general, it is well known that the Riesz idempotent P is not an orthogonal projection and necessary and sufficient condition for P to be orthogonal is that P is self-adjoint [6]. For a hyponormal operator T, Stampfli [24] have shown that the Riesz idempotent for an isolated point of the spectrum of T

is self-adjoint. Uchiyama and Tanahashi [27] proved this property for class A. Recently, Jeon and Kim [12] showed that this property also holds for quasi-class A. In this section we extend these result to class $A(k)$ operators and $\mathcal{Q}(A(k), m)$ operators.

Theorem 3.1. *Let $T \in \mathscr{L}(\mathscr{H})$ be a class $A(k)$ operator and λ be a non-zero isolated point of $\sigma(T)$. Then the Riesz idempotent satisfies that*

$$\Re(P) = \ker(T - \lambda I) = \ker(T - \lambda I)^*.$$

In particular T is self-adjoint.

Proof. Since class A_k operators are isoloid by Lemma 2.5. Then λ is an isolated point of $\sigma(T)$. Let γ be the boundary of a closed disc $\mathbb{D}_\lambda = \{\mu \in \mathbb{C} : |\mu - \lambda| \leq r\}$ for which $0 \notin \mathbb{D}_\mu$ such that $\gamma \cap \sigma(T) = \{\lambda\}$. Then the range of Riesz idempotent $P = \frac{1}{2\pi i}\int_\gamma (T - \lambda I)^{-1}\, d\lambda$ is an invariant closed subspace of T and $\sigma(T|_{\Re(P)}) = \{\lambda\}$.

If $\lambda = 0$, then $\sigma(T|_{\Re(P)}) = \{0\}$. Since $T|_{\Re(P)}$ is class $A(k)$ by Theorem 2.3, $T|_{\Re(P)} = 0$ by Lemma 2.11. Therefore, 0 is an eigenvalue of T.

If $\lambda \neq 0$, then $T|_{\Re(P)}$ is an invertible class $A(k)$ operator and hence $(T|_{\Re(P)})^{-1}$ is also class $A(k)$. We see that $\left\|T|_{\Re(P)}\right\| = |\lambda|$ and $\left\|(T|_{\Re(P)})^{-1}\right\| = \frac{1}{|\lambda|}$. Let $x \in \Re(P)$ be arbitrary vector. Then

$$\|x\| \leq \left\|(T|_{\Re(P)})^{-1}\right\| \left\|T|_{\Re(P)}x\right\| = \frac{1}{|\lambda|}\left\|T|_{\Re(P)}x\right\| \leq \frac{1}{|\lambda|}|\lambda|\,\|x\| = \|x\|\,.$$

This implies that $\frac{1}{\lambda}T|_{\Re(P)}$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T|_{\Re(P)}) = \{1\}$. Hence $T|_{\Re(P)} = \lambda I$ and λ is an eigenvalue of T. Therefore, $\Re(P) = \ker(T - \lambda I)$. Since $\ker(T - \lambda I) \subset \ker(T - \lambda I)^*$ by Theorem 2.4, it suffices to show that $\ker(T - \lambda I)^* \subset \ker(T - \lambda I)$. Since $\ker(T - \lambda I)$ is a reducing subspace of T by Theorem 2.4 and the restriction of a class $A(k)$ to its reducing subspace is also class $A(k)$ operator, we see that T is of the form $T = T' \oplus \lambda I$ on $\mathscr{H} = \ker(T - \lambda I) \oplus \ker(T - \lambda I)^\perp$, where T' is a class $A(k)$ operator with $\ker(T' - \lambda I) = \{0\}$. Since $\lambda \in \sigma(T) = \sigma(T') \cup \{\lambda\}$ is isolated, the only two cases occur. One is $\lambda \notin \sigma(T')$ and the other is that λ is an isolated point of $\sigma(T')$. The latter case, however, does not occur otherwise we have $\lambda \in \sigma_p(T')$ and this contradicts the fact that $\ker(T' - \lambda I) = \{0\}$. $\ker(T - \lambda I) = \ker(T - \lambda I)^*$ is immediate from the injectivity of $T' - \lambda I$ as an operator on $\ker(T - \lambda I)^\perp$.

Next, we show that P is self-adjoint. Since $\Re(P) = \ker(T - \lambda I) = \ker(T - \lambda I)^*$, we have $((T - zI)^*)^{-1}P = \overline{(z - \lambda)^{-1}}P$. Hence

$$\begin{aligned} P^*P &= -\frac{1}{2i\pi}\int_\gamma ((T - zI)^*)^{-1}P\, d\bar{z} \\ &= -\frac{1}{2i\pi}\int_\gamma \overline{(z - \lambda)^{-1}}P\, d\bar{z} \\ &= \overline{\left(\frac{1}{2i\pi}\int_\gamma \frac{1}{z - \lambda}\, d\bar{z}\right)}P \end{aligned}$$

$$= PP^*.$$

Therefore, the proof is achieved. $\square$

Example 3.2. There exists a class $A(k)$ operator T such that 0 is an isolated point of $\sigma(T)$, $\ker(T) \neq \ker(T^*)$ and the Riesz idempotent P with respect to 0 is not self-adjoint. To see this, let $0 < \alpha < 1$ and $A = \alpha \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ in Example 1.2. Then $T \in \mathcal{A}(k)$ with $k \geq \frac{-\log 2}{2\log \alpha}$ and 0 is an isolated point of $\sigma(T)$. Also $\ker(T) \neq \ker(T^*)$ and the Riesz idempotent P with respect to 0 is not self-adjoint.

Theorem 3.3. *Let $T \in \mathcal{Q}(A(k), m)$. Then*

$$H_0(T-\lambda) = \begin{cases} \ker(T-\lambda), & \text{if } \lambda \neq 0; \\ \ker(T^{m+1}), & \text{if } \lambda = 0. \end{cases}$$

Moreover, if $0 \neq \lambda$, then $H_0(T-\lambda) = \ker(T-\lambda) \subset \ker(T-\lambda)^*$.

Proof. Since T has Bishop's property (β) by Lemma 2.10 and $H_0(T-\lambda) = \chi_T(\{\lambda\})$ by Theorem 2.20 of [1], $H_0(T-\lambda)$ is closed and $\sigma(T|_{H_0(T-\lambda)}) \subset \{\lambda\}$ by Proposition 1.2.19 of [19]. Let $S = T|_{H_0(T-\lambda)}$. Then S is a $\mathcal{Q}(A(k), m)$ operator by Theorem 2.3. Hence, we divide into the cases:

Case I. If $\sigma(S) = \sigma(T|_{H_0(T-\lambda)}) = \emptyset$, then $H_0(T-\lambda) = \{0\}$, and so $\ker(T-\lambda) = \{0\}$.

Case II. If $\sigma(S) = \{\lambda\}$ and $\lambda \neq 0$, then $S = \lambda$ by Lemma 2.12, and $H_0(T-\lambda) = \ker(S-\lambda) \subset \ker(T-\lambda)$.

Case III. If $\sigma(S) = \{0\}$, then $S^{m+1} = 0$ by Lemma 2.12, and $H_0(T) = \ker(S^{m+1}) \subset \ker(T^{m+1})$.

Moreover, let $\lambda \neq 0$. In this case, $S = \lambda$. Hence S is normal and invertible, so $H_0(T-\lambda)$ reduces T by Theorem 2.4. Thus $H_0(T-\lambda) = \ker(T-\lambda) \subset (T-\lambda)^*$. $\square$

Theorem 3.4. *Let $T \in \mathcal{Q}(A(k), m)$. If $0 \neq \lambda \in iso\sigma(T)$ and P is the Riesz idempotent for λ, then P is self-adjoint and*

$$\Re(P) = \ker(T-\lambda) = \ker(T-\lambda)^*.$$

Moreover, if $\lambda = 0$, then $\Re(P) = H_0(T) = \ker(T^{m+1})$.

Proof. If T has a dense range, then T is a class $A(k)$ operator, so the result follows from Theorem 3.1. Therefore we may assume that $\overline{\Re(T^m)} \neq \mathscr{H}$. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\mathscr{H} = \overline{\Re(T^m)} \oplus \ker(T^{*m})$, where T_1 is a class $A(k)$, $T_3^m = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$. If $0 \neq \lambda \in iso\sigma(T)$, then $\lambda \in iso\sigma(T_1)$ because $\sigma(T) = \sigma(T_1) \cup \{0\}$. Let γ be the boundary of a closed disc $\mathbb{D}_\lambda = \{\mu \in \mathbb{C} : |\mu - \lambda| \leq r\}$ for which $0 \notin \mathbb{D}_\mu$ such that $\gamma \cap \sigma(T) = \{\lambda\}$. Then

$$P = \frac{1}{2\pi i} \int_\gamma \begin{pmatrix} \mu - T_1 & -T_2 \\ 0 & \mu - T_3 \end{pmatrix}^{-1} d\mu$$

$$= \frac{1}{2\pi i}\int_\gamma \begin{pmatrix} (\mu - T_1)^{-1} & (\mu - T_1)^{-1}T_2(\mu - T_3)^{-1} \\ 0 & (\mu - T_3)^{-1} \end{pmatrix} d\mu.$$

Let $P_1 = \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}d\mu$ be the Riesz idempotent of T_1 for μ. Since T_1 is a class $A(k)$, it follows from Theorem 3.1 that P_1 is self-adjoint and

$$\Re(P_1) = \ker(T_1 - \lambda) = \ker(T_1 - \lambda)^*.$$

We prove that

$$X = \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2(\mu - T_3)^{-1}d\mu = 0.$$

Since

$$(\mu - T_3)^{-1} = \frac{1}{\mu} + \frac{T_3}{\mu^2} + \frac{T_3^2}{\mu^3} + \cdots + \frac{T_3^{m-1}}{\mu^m},$$

we see that

$$\begin{aligned} X &= \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{1}{\mu}d\mu + \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{T_3}{\mu^2} + \cdots \\ &= X_0 + X_1 + \cdots + X_{m-1}. \end{aligned}$$

Since $\frac{1}{\mu} = \sum_{n=0}^\infty \frac{(-1)^n}{\lambda}\left(\frac{\mu-\lambda}{\lambda}\right)^n$, we have

$$\begin{aligned} X &= \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{1}{\mu}d\mu + \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{\mu-\lambda}{\mu^2} + \cdots \\ &= \frac{1}{\lambda}P_1T_2 - \frac{1}{\lambda^2}(T_1 - \lambda)P_1T_2 + \frac{1}{\lambda^3}(T_1 - \lambda)^2P_1T_2 - \cdots. \end{aligned}$$

We prove that

$$P_1T_2 = 0.$$

Let $y = P_1x$ for $x \in \overline{\Re(T)}$. Then $y \in \ker(T_1 - \lambda) = \ker(T_1 - \lambda)^*$. Therefore, from Theorem 2.4 we have

$$\begin{pmatrix} 0 \\ 0 \end{pmatrix} = (T - \lambda)\begin{pmatrix} y \\ 0 \end{pmatrix} = (T - \lambda)^*\begin{pmatrix} y \\ 0 \end{pmatrix} = \begin{pmatrix} (T_1 - \lambda)^*y \\ -T_2^*y \end{pmatrix}.$$

Thus $T_2^*y = T_2^*P_1x = 0$ for $x \in \overline{\Re(T)}$. This implies that $P_1T_2 = 0$ because P_1 is self-adjoint. Hence $X_0 = 0$. On the other hand, since $\frac{1}{\mu^2} = \frac{1}{\lambda^2} - \frac{2(\mu-\lambda)}{\lambda^3} + \frac{3(\mu-\lambda)^2}{\lambda^4} - \cdots$, we have

$$\begin{aligned} X_1 &= \frac{1}{2\pi i}\int_\gamma (\mu - T_1)^{-1}T_2\frac{T_3}{\mu^2}d\mu \\ &= \frac{1}{\lambda^2}P_1T_2T_3 - \frac{2}{\lambda^3}(T_1 - \lambda)P_1T_2T_3 + \frac{3}{\lambda^4}P_1T_2T_3 - \cdots \\ &= 0. \end{aligned}$$

Similarly we have $X_2 = X_3 = \cdots = X_{m-1} = 0$, and $X = 0$. Hence

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{3.1}$$

is self-adjoint as well as P_1. Now we claim that $\Re(P) = \ker(T-\lambda)$. We see from Equation (3.1) that

$$\Re(P) = \Re(P_1) \oplus \{0\} = \ker(T_1-\lambda) \oplus \{0\} = \ker(T_1-\lambda)^* \oplus \{0\}\,.$$

So, if $x \in \Re(P)$, then $x = \left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right)$, where $x_1 \in \ker(T_1-\lambda)$. Therefore,

$$(T-\lambda)x = \begin{pmatrix} T_1-\lambda & -T_2 \\ 0 & T_3-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus $\Re(P) \subset \ker(T-\lambda)$. Hence, since $\Re(P) \supseteq \ker(T-\lambda)$, we have that $\Re(P) = \ker(T-\lambda)$.

To end the proof, we must show that $\ker(T-\lambda)^* \subset \ker(T-\lambda)$. Let $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \ker(T-\lambda)^*$. Then

$$\begin{aligned}(T-\lambda)^*\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} (T_1-\lambda)^* & 0 \\ T_2^* & (T_3-\lambda)^3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ &= \begin{pmatrix} (T_1-\lambda)^*x_1 \\ T_2^*x_1 + (T_3-\lambda)^*x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.\end{aligned}$$

Therefore, $x_1 \in \ker(T_1-\lambda)^* = \ker(T_1-\lambda)$. Then $(T-\lambda)\left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$ implies that $(T-\lambda)^*\left(\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} (T_1-\lambda)^*x_1 \\ T_2^*x_1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right)$.

Thus we have that $T_2^*x_1 = 0$. This implies that $(T_3-\lambda)^*x_2 = 0$ and $x_2 = 0$ because T_3 is nilpotent. Therefore,

$$x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix} \in \ker(T_1-\lambda) \oplus \{0\} = \Re(P) = \ker(T-\lambda).$$

The proof of the case $\lambda = 0$ is straightforward from Theorem 3.3. So, the proof is achieved. □

4. Tensor product

Let $\mathscr{H}$ and $\mathscr{K}$ denote the Hilbert spaces. For given non-zero operators $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$, $T \otimes S$ denotes the tensor product on the product space $\mathscr{H} \otimes \mathscr{K}$. The normaloid property is invariant under tensor products [23]. $T \otimes S$ is normal if and only if T and S are normal [14, 25]. There exist paranormal operators T and S such that $T \otimes S$ is not paranormal [3]. In [15], I. H. Kim showed that for non-zero $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$, $T \otimes S$ is log-hyponormal if and only if T and S are log-hyponormal. This result was extended to p-quasi hyponormal operators, class A operators, quasi class A and quasi class (A,k) operators in [15], [11], [12] and [16], respectively. In this section, we prove an analogous result for $\mathcal{Q}(A(k),m)$ operators.

Remark 4.1. Let $T \in\ LB$ and $S \in \mathscr{L}(\mathscr{K})$ be non-zero operators, then we have

(i) $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$,
(ii) $|T \otimes S|^t = |T|^t \otimes |S|^t$ for any positive real t.

Lemma 4.2 ([25])**.** *Let $T_1, T_2 \in \mathscr{L}(\mathscr{H})$, $S_1, S_2 \in \mathscr{L}(\mathscr{K})$ be non-negative operators. If T_1 and S_1 are non-zero, then the following assertions are equivalent*:

(a) $T_1 \otimes S_1 \leq T_2 \otimes S_2$;

(b) *there exists $c > 0$ such that $T_1 \leq cT_2$ and $S_1 \leq c^{-1}S_2$.*

Lemma 4.3 (Hölder-McCarthy Inequality)**.** *Let $T \geq 0$. Then the following assertions hold.*

(i) $\langle T^r x, x\rangle \geq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for $r > 1$ and $x \in \mathscr{H}$.*

(ii) $\langle T^r x, x\rangle \leq \langle Tx, x\rangle^r \|x\|^{2(1-r)}$ *for $r \in [0,1]$ and $x \in \mathscr{H}$.*

Theorem 4.4. *Suppose that $T \in \mathscr{L}(\mathscr{H})$ and $S \in \mathscr{L}(\mathscr{K})$ are non-zero operators. Then $T \otimes S$ is a class $A(k)$ operator if and only T and S are class $A(k)$ operators.*

Proof. Assume that T and S are class $A(k)$ operators. Then

$$\begin{aligned}
&\big((T \otimes S)^*|T \otimes S|^{2k}(T \otimes S)\big)^{1/(k+1)} \\
=\ &\big((T^* \otimes S^*)(|T|^{2k} \otimes |S|^{2k})(T \otimes S)\big)^{1/(k+1)} \\
=\ &\big((T^*|T|^{2k}T) \otimes (S^*|S|^{2k}S)\big)^{1/(k+1)} \\
=\ &(T^*|T|^{2k}T)^{1/(k+1)} \otimes (S^*|S|^{2k}S)^{1/(k+1)} \\
\geq\ &|T|^2 \otimes |S|^2 = |T \otimes S|^2
\end{aligned}$$

which implies that $T \otimes S$ is a class $A(k)$ operator.

Conversely, assume that $T \otimes S$ is a class $A(k)$. We aim to show that T and S are class $A(k)$ operators. Without loss of generality, it is enough to show that T is a class $A(k)$ operator. Since $T \otimes S$ is a class $A(k)$ operator, we obtain

$$(T^*|T|^{2k}T)^{1/(k+1)} \otimes (S^*|S|^{2k}S)^{1/(k+1)} \geq |T|^2 \otimes |S|^2.$$

Hence by Lemma 4.2, there exists a positive real number c for which

$$|T|^2 \leq c(T^*|T|^{2k}T)^{1/(k+1)} \text{ and } |S|^2 \leq c^{-1}(S^*|S|^{2k}S)^{1/(k+1)}.$$

Consequently, for every $x \in \mathscr{H}$ and $y \in \mathscr{K}$ and by Hölder McCarthy Inequality, we have

$$\begin{aligned}
\|T\|^2 &= \sup_{\|x\|=1} \big\langle |T|^2 x, x\big\rangle \\
&\leq \sup_{\|x\|=1} \Big\langle c(T^*|T|^{2k}T)^{1/(k+1)}x, x\Big\rangle \\
&\leq c \sup_{\|x\|=1} \big\langle T^*|T|^{2k}Tx, x\big\rangle^{1/(k+1)} \\
&\leq c \sup_{\|x\|=1} \big\||T|^k Tx\big\|^{2/(k+1)} \\
&= c\,\big\||T|^k T\big\|^{2/(k+1)} = c\,\big\|T^{k+1}\big\|^{2/(k+1)} \leq c\,\|T\|^2
\end{aligned}$$

and

$$\begin{aligned}\|S\|^2 &= \sup_{\|y\|=1} \left\langle |S|^2 y, y \right\rangle \\ &\le \sup_{\|y\|=1} \left\langle c^{-1}(S^*|S|^{2k}S)^{1/(k+1)} y, y \right\rangle \\ &\le c^{-1} \sup_{\|y\|=1} \left\langle S^*|S|^{2k}Sy, y \right\rangle^{1/(k+1)} \\ &\le c^{-1} \sup_{\|y\|=1} \left\| |S|^k S y \right\|^{2/(k+1)} \\ &= c^{-1} \left\| |S|^k S \right\|^{2/(k+1)} \\ &= c^{-1} \left\| S^{k+1} \right\|^{2/(k+1)} \\ &\le c^{-1} \|S\|^2 .\end{aligned}$$

Thus, $c = 1$, and so T is a class $A(k)$ operator. □

Theorem 4.5. *Let $T, S \in \mathscr{L}(\mathscr{H})$ be non-zero operators. Then $T \otimes S \in \mathcal{Q}(A(k), m)$ if and only if one of the following holds*:

(i) $T \in \mathcal{Q}(A(k), m)$ *and* $S \in \mathcal{Q}(A(k), m)$.

(ii) $T^{m+1} = 0$ *or* $S^{m+1} = 0$.

Proof. By simple calculation we have $T \otimes S \in \mathcal{Q}(A(k), m)$ if and only if

$$\begin{aligned}&(T \otimes S)^{*m} \left(\left((T \otimes S)^* |T \otimes S|^{2k} (T \otimes S) \right)^{1/(k+1)} - |T \otimes S|^2 \right) (T \otimes S)^m \ge 0 \\ \Leftrightarrow\ & T^{*m}((T^*|T|^{2k}T)^{1/(k+1)} - |T|^2)T^m \otimes S^{*m}(S^*|S|^{2k}S)^{1/(k+1)}S^m \\ &+ T^{*m}|T|^2T^m \otimes S^{*m}((S^*|S|^{2k}S)^{1/(k+1)} - |S|^2)S^m \ge 0.\end{aligned}$$

Thus the sufficiency is easily proved. Conversely, suppose that $T \otimes S \in \mathcal{Q}(A(k), m)$. Then for $x \in \mathscr{H}$ and $y \in \mathscr{K}$ we have

(4.1)
$$\begin{aligned}&\left\langle T^{*m}((T^*|T|^{2k}T)^{1/(k+1)} - |T|^2)T^m x, x \right\rangle \left\langle S^{*m}(S^*|S|^{2k}S)^{1/(k+1)} S^m y, y \right\rangle \\ &+ \left\langle T^{*m}|T|^2T^m x, x \right\rangle \left\langle S^{*m}((S^*|S|^{2k}S)^{1/(k+1)} - |S|^2)S^m y, y \right\rangle \ge 0.\end{aligned}$$

It suffices to show that if the statement (ii) does not hold, the statement (i) holds. Thus, assume to the contrary that neither of T^{m+1} and S^{m+1} is the zero operator, and T is not in $\mathcal{Q}(A(k), m)$. Then there exists $x_0 \in \mathscr{H}$ such that

$$\begin{aligned}&\left\langle T^{*m}((T^*|T|^{2k}T)^{1/(k+1)} - |T|^2)T^m x_0, x_0 \right\rangle := \alpha < 0 \quad \text{and} \\ &\left\langle T^{m*}|T|^2T^m x_0, x_0 \right\rangle := \beta > 0.\end{aligned}$$

From (4.1) we have

$$(\alpha + \beta) \left\langle S^{*m}(S^*|S|^{2k}S)^{1/(k+1)} S^m y, y \right\rangle \ge \beta \left\langle S^{*m}|S|^2S^m y, y \right\rangle . \tag{4.2}$$

Thus $S \in \mathcal{Q}(A(k), m)$. By Hölder McCarthy Inequality, we have

$$\begin{aligned}\left\langle S^{*m}(S^*|S|^{2k}S)^{1/(k+1)}S^m y, y\right\rangle &= \left\langle (S^*|S|^{2k}S)^{1/(k+1)}S^m y, S^m y\right\rangle \\ &\leq \left\langle |S|^{2k}S^{m+1}y, S^{m+1}y\right\rangle^{1/(k+1)} \|S^m y\|^{2k/(k+1)} \\ &\leq \|S^m y\|^{2k/(k+1)} \left\||S|^k S^{m+1}y\right\|^{1/(k+1)} \\ &= \|S^m y\|^{\frac{2k}{k+1}} \left\|S^{k+m+1}y\right\|^{2/(k+1)}\end{aligned}$$

and

$$\left\langle S^{*m}|S|^2S^m y, y\right\rangle = \left\langle S^{m+1}y, S^{m+1}y\right\rangle = \left\|S^{m+1}y\right\|^2.$$

Therefore, we have

$$(\alpha+\beta)\|S^m y\|^{\frac{2k}{k+1}} \left\|S^{k+m+1}y\right\|^{2/(k+1)} \geq \beta \left\|S^{m+1}y\right\|^2. \tag{4.3}$$

On the other hand, since $S \in \mathcal{Q}(A(k), m)$, from Lemma 2.2 we have a decomposition of S as the following:

$$S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix} \quad \text{on} \quad \mathscr{H} = \overline{\Re(S^m)} \oplus \ker(S^{m*}),$$

where S_1 is a class $A(k)$ operator on $\overline{\Re(S^m)}$ and S_3 is a nilpotent with nilpotency m. By (4.3) we have

$$(\alpha+\beta)\|S_1^m\xi\|^{\frac{2k}{k+1}} \left\|S_1^{k+m+1}\xi\right\|^{2/(k+1)} \geq \beta \left\|S_1^{m+1}\xi\right\|^2 \quad \text{for all } \xi \in \overline{\Re(S^m)}. \tag{4.4}$$

Since S_1 is a class $A(k)$, S_1 is normaloid, and taking supremum on both sides of the inequality (4.4), we have

$$(\alpha+\beta)\|S_1\|^{2(m+1)} \geq \beta \|S_1\|^{2(m+1)}.$$

This inequality forces that $S_1 = 0$. Hence $S^{m+1}x = 0$ because $S^{m+1} = S_1 S^m$ for all $y \in \mathscr{K}$. This is a contradiction to that S^{m+1} is not a zero operator. Hence T must be in $\mathcal{Q}(A(k), m)$ operators. In a similar manner, we can prove that S is also a quasi-class $\mathcal{Q}(A(k), m)$ operator. Therefore, the proof of the theorem is finished. □

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