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GENERALIZED RELATIVE ORDER-DEPENDENT GROWTHS OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of their generalized relative orders and generalized relative lower orders.

1. Introduction

A single valued function of one complex variable which is analytic in the finite complex plane is called an integral (entire) function. For example $\exp z$, $\sin z$, $\cos z$ etc. are examples of entire functions. In the value distribution theory one studies how an entire function assumes some values and the influence of assuming certain values in some specific manner on a function. In 1926 Rolf Nevanlinna initiated the value distribution theory of entire functions. This value distribution theory is a prominent branch of complex analysis and is the prime concern of the paper. Perhaps the Fundamental Theorem of Classical Algebra which states that "If f is a polynomial of degree n with real or complex coefficients, then the equation $f(z) = 0$ has at least one root" is the most well known value distribution theorem.

The value distribution theory deals with various aspects of the behavior of entire functions one of which is the study of comparative growth properties. For any entire function f, $M(r, f)$, a function of r is defined as follows:

$$M_f(r) = \max_{|z|=r} |f(z)|.$$

Similarly for another entire function g, $M_g(r)$ is defined. The ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli.

An entire function f has an everywhere convergent power series expansion as

$$f = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n + \cdots$$

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The maximum term $\mu_f(r)$ of f can be defined in the following way:

$$\mu_f(r) = \max_{n \geq 0} (|a_n| r^n).$$

In fact $\mu_f(r)$ is much weaker than $M_f(r)$ in some sense. So from another angle of view $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ is also called the growth of f with respect to g where $\mu_g(r)$ denotes the maximum term of entire g.

However, the *order* of an entire function f which is generally used in computational purpose is defined in terms of the growth of f with respect to the $\exp z$ function as

$$\rho_f = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log \log M_{\exp z}(r)} = \limsup_{r \to \infty} \frac{\log \log M_f(r)}{\log r}.$$

L. Bernal ([1], [2]) introduced the *relative order* between two entire functions to avoid comparing growth just with $\exp z$. Extending the notion of *relative order* as cit.op. Lahiri and Banerjee [11] introduced the definition of *generalized relative order*.

For entire functions, the notions of the growth indicators such as *order* is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of *relative orders* and consequently the *generalized relative orders* of entire functions and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their *relative orders* are the prime concern of this paper. In fact some light has already been thrown on such type of works by Datta et al. in [4], [5], [6], [7], [9], and [10]. Actually in this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of their *generalized relative orders* (respectively *generalized relative lower orders*).

2. Notation and preliminary remarks

Our notations are standard within the theory of Nevanlinna's value distribution of entire functions and therefore we do not explain those in detail as available in [16]. In the sequel the following two notations are used:

$$\log^{[k]} x = \log\left(\log^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \dots;$$
$$\log^{[0]} x = x$$

and

$$\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right) \text{ for } k = 1, 2, 3, \dots;$$
$$\exp^{[0]} x = x.$$

Taking this into account the *order* (respectively, *lower order*) of an entire function f is given by

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]} M_f(r)}{\log r}\quad\left(\text{respectively } \lambda_f = \frac{\log^{[2]} M_f(r)}{\log r}\right).$$

Let us recall that Sato [12] defined the *generalized order* and *generalized lower order* of an entire function f, respectively, as follows:

$$\rho_f^{[l]} = \limsup_{r\to\infty}\frac{\log^{[l]} M_f(r)}{\log r}\quad\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty}\frac{\log^{[l]} M_f(r)}{\log r}\right),$$

where l is any positive integer. These definitions extended the order ρ_f and lower order λ_f of an entire function f since these correspond to the particular case $\rho_f^{[2]} = \rho_f$ and $\lambda_f^{[2]} = \lambda_f$.

Since for $0 \le r < R$,

$$\mu_f(r) \le M_f(r) \le \frac{R}{R-r}\mu_f(R)\ \ (\text{cf. [14]})$$

it is easy to see that

$$\rho_f = \limsup_{r\to\infty}\frac{\log^{[2]} \mu_f(r)}{\log r}\quad\left(\text{respectively } \lambda_f = \liminf_{r\to\infty}\frac{\log^{[2]} \mu_f(r)}{\log r}\right)$$

and

$$\rho_f^{[l]} = \limsup_{r\to\infty}\frac{\log^{[l]} \mu_f(r)}{\log r}\quad\left(\text{respectively } \lambda_f^{[l]} = \liminf_{r\to\infty}\frac{\log^{[l]} \mu_f(r)}{\log r}\right).$$

Given a non-constant entire function f defined in the open complex plane $\mathbb{C}$, its maximum modulus function M_f is strictly increasing and continuous. Hence there exists its inverse function $M_f^{-1} : (|f(0)|,\infty) \to (0,\infty)$ with $\lim_{s\to\infty} M_f^{-1}(s) = \infty$.

Then Bernal ([1], [2]) introduced the definition of relative order of f with respect to g, denoted by $\rho_g(f)$ as follows:

$$\begin{aligned}\rho_g(f) &= \inf\{\mu > 0 : M_f(r) < M_g(r^\mu) \ \text{for all } r > r_0(\mu) > 0\}\\ &= \limsup_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log r}.\end{aligned}$$

This definition coincides with the classical one [15] if $g = \exp z$. Similarly, one can define the relative lower order of f with respect to g denoted by $\lambda_g(f)$ as

$$\lambda_g(f) = \liminf_{r\to\infty}\frac{\log M_g^{-1}M_f(r)}{\log r}.$$

In the case of relative order, it therefore seems reasonable to state suitably an alternative definition of relative order of entire function in terms of its maximum terms. Datta and Maji [10] introduced such a definition in the following way:

Definition 1 ([10]). The relative order $\rho_g(f)$ and the relative lower order $\lambda_g(f)$ of an entire function f with respect to another entire function g are defined as follows:

$$\rho_g(f) = \limsup_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r} \text{ and } \lambda_g(f) = \liminf_{r\to\infty} \frac{\log \mu_g^{-1}\mu_f(r)}{\log r}.$$

Lahiri and Banerjee [11] gave a more generalized concept of relative order in the following way:

Definition 2 ([11]). If $l \geq 1$ is a positive integer, then the l-th generalized relative order of f with respect to g, denoted by $\rho_g^{[l]}(f)$ is defined by

$$\rho_g^{[l]}(f) = \inf\left\{\mu > 0 : M_f(r) < M_g\left(\exp^{[l-1]} r^{\mu}\right) \text{ for all } r > r_0(\mu) > 0\right\}$$
$$= \limsup_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

Clearly $\rho_g^1(f) = \rho_g(f)$ and $\rho_{\exp z}^1(f) = \rho_f$.

Likewise one can define the generalized relative lower order of f with respect to g denoted by $\lambda_g^{[l]}(f)$ as

$$\lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} M_g^{-1} M_f(r)}{\log r}.$$

In terms of maximum terms of entire functions, Definition 2 can be reformulated as:

Definition 3. For any positive integer $l \geq 1$, the growth indicators $\rho_g^{[l]}(f)$ and $\lambda_g^{[l]}(f)$ of an entire function f are defined as:

$$\rho_g^{[l]}(f) = \limsup_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1}\mu_f(r)}{\log r} \text{ and } \lambda_g^{[l]}(f) = \liminf_{r\to\infty} \frac{\log^{[l]} \mu_g^{-1}\mu_f(r)}{\log r}.$$

In fact, the equivalence of Definition 2 and Definition 3 has been established in [8].

3. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 ([3]). *If f and g are any two entire functions, then for all sufficiently large values of r,*

$$M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right) - |g(0)|\right) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

Lemma 2 ([13]). *Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha - 1} \mu_f\left(\frac{\alpha R}{R - r} \mu_g(R)\right).$$

Lemma 3 ([13])**.** *If f and g are any two entire functions with $g(0)=0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r)\geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right)-|g(0)|\right).$$

Lemma 4 ([2])**.** *Suppose f is an entire function and $\alpha>1$, $0<\beta<\alpha$. Then for all sufficiently large r,*

$$M_f(\alpha r)\geq \beta M_f(r).$$

Lemma 5 ([10])**.** *If f be an entire and $\alpha>1$, $0<\beta<\alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r)\geq \beta\mu_f(r).$$

4. Main results

In this section we present the main results of the paper.

Theorem 1. *Let f and h be any two entire functions such that $0<\lambda_h^{[p]}(f)\leq \rho_h^{[p]}(f)<\infty$ and g be an entire function with $\lambda_g^{[q]}>0$ where p,q are any integers with $p>1$ and $q>2$. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}{\left[\log^{[p]}M_h^{-1}M_f\left(\left\{\exp^{[q-2]}r\right\}^{\delta}\right)\right]^{1+\alpha}}=\infty.$$

Proof. If α be such that $1+\alpha\leq 0$, then the theorem is trivial. So we suppose that $1+\alpha>0$. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from the first part of Lemma 1 for all sufficiently large values of r that

$$\log^{[p]}M_h^{-1}M_{f\circ g}(r)\geq\left(\lambda_h^{[p]}(f)-\varepsilon\right)\frac{1}{16}+\left(\lambda_h^{[p]}(f)-\varepsilon\right)\log M_g\left(\frac{r}{2}\right)$$

$$\text{i.e., }\log^{[p]}M_h^{-1}M_{f\circ g}(r)\geq\left(\lambda_h^{[p]}(f)-\varepsilon\right)\frac{1}{16}+\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]}\left(\frac{r}{2}\right)^{\lambda_g^{[q]}-\varepsilon},\tag{1}$$

where we choose ε in such a way that $0<\varepsilon<\min(\lambda_h^{[p]}(f),\lambda_g^{[q]})$.
Again from the definition of $\rho_h^{[p]}(f)$, it follows for all sufficiently large values of r that

$$\left[\log^{[p]}M_h^{-1}M_f\left(\left\{\exp^{[q-2]}r\right\}^{\delta}\right)\right]^{1+\alpha}\leq\left(\rho_h^{[p]}(f)+\varepsilon\right)^{1+\alpha}\delta^{1+\alpha}\left(\exp^{[q-3]}r\right)^{1+\alpha}.\tag{2}$$

Now from (1) and (2), it follows for all sufficiently large values of r that

$$\frac{\log^{[p]} M_h^{-1} M_{f \circ g}(r)}{\left[\log^{[p]} M_h^{-1} M_f\left(\left\{\exp^{[q-2]} r\right\}^\delta\right)\right]^{1+\alpha}}$$
$$\geqslant \frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\frac{1}{16}+\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]}\left(\frac{r}{2}\right)^{\lambda_g^{[q]}-\varepsilon}}{\left(\rho_h^{[p]}(f)+\varepsilon\right)^{1+\alpha}\delta^{1+\alpha}\left(\exp^{[q-3]} r\right)^{1+\alpha}}.$$

Since $\frac{\exp^{[q-2]}\left(\frac{r}{2}\right)^{\lambda_g^{[q]}-\varepsilon}}{\left(\exp^{[q-3]} r\right)^{1+\alpha}} \to \infty$ as $r \to \infty$, the theorem follows from above. □

In the line of Theorem 1, one may state the following theorem without its proof:

Theorem 2. *Let f, g, h and k be any four entire functions with $\lambda_h^{[p]}(f) > 0, \lambda_g^{[q]} > 0$ and $\rho_k^{[m]}(g) < \infty$ where p, q, m are any integers with $p > 1$, $q > 2$ and $m > 1$. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1} M_{f \circ g}(r)}{\left[\log^{[m]} M_k^{-1} M_g\left(\left\{\exp^{[q-2]} r\right\}^\delta\right)\right]^{1+\alpha}} = \infty.$$

In the line of Theorem 1 and Theorem 2, the following two theorems can be proved by using Lemma 3 and Definition 3 and hence their proofs are omitted.

Theorem 3. *Let f and h be any two entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and g be an entire function with $\lambda_g^{[q]} > 0$ where p, q are any two integers with $p > 1$ and $q > 2$. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(r)}{\left[\log^{[p]} \mu_h^{-1} \mu_f\left(\left\{\exp^{[q-2]} r\right\}^\delta\right)\right]^{1+\alpha}} = \infty.$$

Theorem 4. *Let f, g, h and k be any four entire functions with $\lambda_h^{[p]}(f) > 0, \lambda_g^{[q]} > 0$ and $\rho_k^{[m]}(g) < \infty$ where p, q, m are any integers with $p > 1$, $q > 2$ and $m > 1$. Then for every positive constant δ and every real number α,*

$$\lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(r)}{\left[\log^{[m]} \mu_k^{-1} \mu_g\left(\left\{\exp^{[q-2]} r\right\}^\delta\right)\right]^{1+\alpha}} = \infty.$$

Remark 1. Theorem 1 and Theorem 3 are still valid with "limit superior" instead of "limit" if we replace the condition "$0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$" by "$0 < \lambda_h^{[p]}(f) < \infty$".

Remark 2. In Theorem 2 and Theorem 4 if we take the condition $\lambda_k^{[m]}(g) < \infty$ instead of $\rho_k^{[m]}(g) < \infty$, then also Theorem 2 and Theorem 4 remain true with "limit superior" in place of "limit".

Theorem 5. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\rho_g^{[q]} < \infty$ where p, q are any integers with $p > 1$ and $q > 2$. Then for every positive constant δ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f(\exp^{[q-1]} r^{\delta})} = 0 \text{ if } \delta > (1+\alpha)\rho_g^{[q]}.$$

Proof. If $1+\alpha \leq 0$, then the theorem is obvious. We consider $1+\alpha > 0$. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from the second part of Lemma 1 for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f) + \varepsilon\right) \log M_g(r)$$

$$\text{(3)} \qquad i.e., \ \log^{[p]} M_h^{-1} M_{f\circ g}(r) \leq \left(\rho_h^{[p]}(f) + \varepsilon\right) \exp^{[q-2]} r^{\rho_g^{[q]}+\varepsilon}.$$

Again from the definition of generalized relative lower order, we get for all sufficiently large values of r that

$$\text{(4)} \qquad \log^{[p]} M_h^{-1} M_f(\exp^{[q-1]} r^{\delta}) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right) \exp^{[q-2]} r^{\delta}.$$

Therefore for all sufficiently large values of r we get from (3) and (4) that

$$\text{(5)} \quad \frac{\left\{\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f(\exp^{[q-1]} r^{\mu})} \leq \frac{\left(\rho_h^{[p]}(f) + \varepsilon\right)^{1+\alpha} \cdot \exp^{[q-2]} r^{\left(\rho_g^{[q]}+\varepsilon\right)(1+\alpha)}}{\left(\lambda_h^{[p]}(f) - \varepsilon\right) \exp^{[q-2]} r^{\delta}},$$

where we choose $0 < \varepsilon < \min\left\{\lambda_h^{[p]}(f), \frac{\delta}{1+\alpha} - \rho_g^{[q]}\right\}$. So from (5) we obtain that

$$\lim_{r\to\infty} \frac{\left\{\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} M_h^{-1} M_f(\exp^{[q-1]} r^{\delta})} = 0.$$

This proves the theorem. □

In view of Theorem 5 the following theorem can be carried out.

Theorem 6. *Let f, g, h and k be any three entire functions with $\rho_h^{[p]}(f) < \infty$, $\rho_g^{[q]} < \infty$ and $\lambda_k^{[m]}(g) > 0$ where p, q, m are any three integers with $p > 1$, $q > 2$ and $m > 1$. Then for every positive constant δ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p]} M_h^{-1} M_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[m]} M_k^{-1} M_g(\exp^{[q-1]} r^{\delta})} = 0 \text{ if } \delta > (1+\alpha)\rho_g^{[q]}.$$

The proof is omitted.

Theorem 7. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\rho_g^{[q]} < \infty$ where p, q are any integers with $p > 1$ and $q > 2$. Then for every positive constant δ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q-1]} r^{\delta})} = 0 \text{ if } \delta > (1+\alpha)\rho_g^{[q]}.$$

Theorem 8. *Let f, g, h and k be any three entire functions with $\rho_h^{[p]}(f) < \infty$, $\rho_g^{[q]} < \infty$ and $\lambda_k^{[m]}(g) > 0$ where p, q, m are any integers with $p > 1$, $q > 2$ and $m > 1$. Then for every positive constant δ and each $\alpha \in (-\infty, \infty)$,*

$$\lim_{r\to\infty} \frac{\left\{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(r)\right\}^{1+\alpha}}{\log^{[m]} \mu_k^{-1}\mu_g(\exp^{[q-1]} r^{\delta})} = 0 \text{ if } \delta > (1+\alpha)\rho_g^{[q]}.$$

We omit the proof of Theorem 7 and Theorem 8 because those can be carried out in the line of Theorem 5 and Theorem 6 respectively and with the help of Lemma 2, Lemma 5 and Definition 3.

Remark 3. In Theorem 5 and Theorem 7 if we take the condition $0 < \rho_h^{[p]}(f) < \infty$ instead of $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$, then also Theorem 5 and Theorem 7 remain true with "limit inferior" in place of "limit".

Remark 4. In Theorem 6 and Theorem 8 if we take the condition $\rho_k^{[m]}(g) > 0$ instead of $\lambda_k^{[m]}(g) > 0$, the theorem remains true with "limit" replaced by "limit inferior".

Theorem 9. *Let f, g and h be any three entire functions such that $\rho_h^{[p]}(f) < \infty$ and $\lambda_h^{[p]}(f \circ g) = \infty$ where p is any integer > 1. Then for every $A\,(> 0)$,*

$$\lim_{r\to\infty} \frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f(r^A)} = \infty.$$

Proof. If possible, let there exist a constant β such that for a sequence of values of r tending to infinity,

$$\log^{[p]} M_h^{-1}M_{f\circ g}(r) \leq \beta \cdot \log^{[p]} M_h^{-1}M_f\left(r^A\right). \tag{6}$$

Again from the definition of $\rho_h^{[p]}(f)$, it follows for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1}M_f\left(r^A\right) \leq \left(\rho_h^{[p]}(f) + \varepsilon\right) \cdot A \cdot \log r. \tag{7}$$

Now from (6) and (7), we get for a sequence of values of r tending to infinity that

$$\log^{[p]} M_h^{-1}M_{f\circ g}(r) \leq \beta \cdot \left(\rho_h^{[p]}(f) + \varepsilon\right) \cdot A \cdot \log r$$

$$\text{i.e., } \lambda_h^{[p]}(f \circ g) \leq \beta \cdot A\left(\rho_h^{[p]}(f) + \varepsilon\right),$$

which contradicts the condition $\lambda_h^{[p]}(f \circ g) = \infty$.

So for all sufficiently large values of r we get that

$$\log^{[p]} M_h^{-1} M_{f \circ g}(r) \geq \beta \cdot \log^{[p]} M_h^{-1} M_f\left(r^A\right),$$

from which the theorem follows. □

In the line of Theorem 9 the following theorem can also be proved:

Theorem 10. *Let f, g and h be any three entire functions such that $\rho_h^{[p]}(f) < \infty$ and $\lambda_h^{[p]}(f \circ g) = \infty$ where p is any integer > 1. Then for every $A\,(> 0)$,*

$$\lim_{r \to \infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(r)}{\log^{[p]} \mu_h^{-1} \mu_f(r^A)} = \infty.$$

Remark 5. Theorem 9 and Theorem 10 are also valid with "limit superior" instead of "limit" if $\lambda_h^{[p]}(f \circ g) = \infty$ is replaced by $\rho_h^{[p]}(f \circ g) = \infty$ and the other conditions remain the same.

Corollary 1. *Under the assumptions of Theorem 9 and Theorem 10,*

$$\lim_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_f(r^A)} = \infty \quad \textit{and} \quad \lim_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_f(r^A)} = \infty$$

respectively hold.

Proof. By Theorem 9 we obtain for all sufficiently large values of r and for $K > 1$,

$$\log^{[p]} M_h^{-1} M_{f \circ g}(r) \geq K \log^{[p]} M_h^{-1} M_f\left(r^A\right)$$

$$\textit{i.e.,}\ \log^{[p-1]} M_h^{-1} M_{f \circ g}(r) \geq \left\{\log^{[p-1]} M_h^{-1} M_f\left(r^A\right)\right\}^K,$$

from which the first part of the corollary follows.

Similarly using Theorem 10, we obtain the second part of the corollary. □

Corollary 2. *Under the assumptions of Remark 5,*

$$\limsup_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_f(r^A)} = \infty \quad \textit{and} \quad \limsup_{r \to \infty} \frac{\mu_h^{-1} \mu_{f \circ g}(r)}{\mu_h^{-1} \mu_f(r^A)} = \infty$$

respectively hold.

The proof is omitted.

Analogously one may also state the following theorems and corollaries without their proofs as those may be carried out in the line of Remark 5, Theorem 9,Theorem 10, Corollary 1 and Corollary 2 respectively.

Theorem 11. *If f, g and k be any three entire functions such that $\rho_k^{[m]}(g) < \infty$ and $\rho_k^{[m]}(f \circ g) = \infty$ where m is any integer > 1. Then for every $B\,(> 0)$,*

$$\limsup_{r \to \infty} \frac{\log^{[m]} M_k^{-1} M_{f \circ g}(r)}{\log^{[m]} M_k^{-1} M_g(r^B)} = \infty.$$

Theorem 12. *If f, g and k be any three entire functions such that $\rho_k^{[m]}(g) < \infty$ and $\rho_k^{[m]}(f \circ g) = \infty$ where m is any integer > 1. Then for every $B\,(> 0)$,*

$$\limsup_{r\to\infty} \frac{\log^{[m]} \mu_k^{-1} \mu_{f\circ g}(r)}{\log^{[m]} \mu_k^{-1} \mu_g(r^B)} = \infty.$$

Corollary 3. *Theorem 11 and Theorem 12 are also valid with "limit" instead of "limit superior" if $\rho_k^{[m]}(f \circ g) = \infty$ is replaced by $\lambda_k^{[m]}(f \circ g) = \infty$ and the other conditions remain the same.*

Corollary 4. *Under the assumptions of Theorem 11 and Theorem 12,*

$$\limsup_{r\to\infty} \frac{M_k^{-1} M_{f\circ g}(r)}{M_k^{-1} M_g(r^B)} = \infty \quad \textit{and} \quad \limsup_{r\to\infty} \frac{\mu_k^{-1} \mu_{f\circ g}(r)}{\mu_k^{-1} \mu_g(r^B)} = \infty$$

respectively hold.

Corollary 5. *Under the assumptions of Corollary 3,*

$$\lim_{r\to\infty} \frac{M_k^{-1} M_{f\circ g}(r)}{M_k^{-1} M_g(r^B)} = \infty \quad \textit{and} \quad \lim_{r\to\infty} \frac{\mu_k^{-1} \mu_{f\circ g}(r)}{\mu_k^{-1} \mu_g(r^B)} = \infty$$

respectively hold.

Theorem 13. *Let f and h be any two entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ for any integer $p > 1$. Also suppose g be an entire function with $0 < \delta < \rho_g^{[q]} \leq \infty$ where q is any integer > 2. Then for a sequence of values of r tending to infinity,*

$$M_h^{-1} M_{f\circ g}(r) > M_h^{-1} M_f(\exp^{[q-1]} r^\delta).$$

Proof. As $M_h^{-1}(r)$ is an increasing function of r, in view of Lemma 1 we get for a sequence of values of r tending to infinity that

(8)

$$\log^{[p]} M_h^{-1} M_{f\circ g}(r) \geq \left(\lambda_h^{[p]}(f) - \varepsilon\right) \frac{1}{16} + \left(\lambda_h^{[p]}(f) - \varepsilon\right) \exp^{[q-2]} \left(\frac{r}{2}\right)^{\rho_g^{[q]} - \varepsilon}.$$

Again from the definition of $\rho_h^{[p]}(f)$, we obtain for all sufficiently large values of r that

$$\log^{[p]} M_h^{-1} M_f(\exp^{[q-1]} r^\delta) \leq \left(\rho_h^{[p]}(f) + \varepsilon\right) \cdot \exp^{[q-2]} r^\delta. \tag{9}$$

Now from (8) and (9), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} M_h^{-1} M_{f\circ g}(r)}{\log^{[p]} M_h^{-1} M_f(\exp^{[q-1]} r^\delta)} \geq \frac{\left(\lambda_h^{[p]}(f) - \varepsilon\right) \frac{1}{16} + \left(\lambda_h^{[p]}(f) - \varepsilon\right) \exp^{[q-2]} \left(\frac{r}{2}\right)^{\rho_g^{[q]} - \varepsilon}}{\left(\rho_h^{[p]}(f) + \varepsilon\right) \cdot \exp^{[q-2]} r^\delta}. \tag{10}$$

As $\delta < \rho_g^{[q]}$ we can choose $\varepsilon (> 0)$ in such a way that

$$\delta < \rho_g^{[q]} - \varepsilon. \tag{11}$$

Thus, from (10) and (11), we get that

$$\limsup_{r\to\infty}\frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[p]} M_h^{-1}M_f(\exp^{[q-1]} r^{\delta})}=\infty. \tag{12}$$

From (12), we obtain for a sequence of values of r tending to infinity and $K>1$ that

$$M_h^{-1}M_{f\circ g}(r)>M_h^{-1}M_f(\exp^{[q-1]} r^{\delta}).$$

This proves the theorem. □

Theorem 14. *Let f and h be any two entire functions with $0<\lambda_h^{[p]}(f)\le\rho_h^{[p]}(f)<\infty$ for any integer $p>1$. Also suppose g and k be any two entire functions such that $\rho_k^{[m]}(g)<\infty$ and $0<\delta<\rho_g^{[q]}$ where q,m are integers with $q>1$ and $m>2$. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} M_h^{-1}M_{f\circ g}(r)>\log^{[m-1]} M_k^{-1}M_g(\exp^{[q-1]} r^{\delta}).$$

Proof. Let $0<\delta<\delta_0<\rho_g^{[q]}$. Then in view of Theorem 13, for a sequence of values of r tending to infinity, we get that

$$\log^{[p]} M_h^{-1}M_{f\circ g}(r)>\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]} r^{\delta_0}. \tag{13}$$

Again from the definition of $\rho_k^{[m]}(g)$, we obtain for all sufficiently large values of r that

$$\log^{[m]} M_k^{-1}M_g(\exp^{[q-1]} r^{\delta})\le\left(\rho_k^{[m]}(g)+\varepsilon\right)\exp^{[q-2]} r^{\delta}. \tag{14}$$

So combining (13) and (14), we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[m]} M_k^{-1}M_g(\exp^{[q-1]} r^{\mu})}\geqslant\frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]} r^{\delta_0}}{\left(\rho_k^{[m]}(g)+\varepsilon\right)\exp^{[q-2]} r^{\delta}}. \tag{15}$$

Since $\delta_0>\delta$, from (15), it follows that

$$\limsup_{r\to\infty}\frac{\log^{[p]} M_h^{-1}M_{f\circ g}(r)}{\log^{[m]} M_k^{-1}M_g(\exp^{[q-1]} r^{\delta})}=\infty. \tag{16}$$

Thus the theorem follows from (16). □

In the line of Theorem 13 and Theorem 14, the following two theorems can be proved by using Lemma 3 and Definition 3 and therefore their proofs are omitted.

Theorem 15. *Let f and h be any two entire functions such that $0<\lambda_h^{[p]}(f)\le\rho_h^{[p]}(f)<\infty$ for any integer $p>1$. Also suppose g be an entire function with*

$0<\delta<\rho_g^{[q]}\leq\infty$ where q is any integer >2. Then for a sequence of values of r tending to infinity,

$$\mu_h^{-1}\mu_{f\circ g}(r)>\mu_h^{-1}\mu_f(\exp^{[q-1]}r^{\delta}).$$

Theorem 16. *Let f and h be any two entire functions with $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ for any integer $p>1$. Also suppose g and k be any two entire functions such that $\rho_k^{[m]}(g)<\infty$ and $0<\delta<\rho_g^{[q]}$ where q,m are integers with $q>1$ and $m>2$. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}(r)>\log^{[m-1]}\mu_k^{-1}\mu_g(\exp^{[q-1]}r^{\delta}).$$

Theorem 17. *Let f, g and h be any three entire functions such that $0<\lambda_h^{[p]}(f)\leq\rho_h^{[p]}(f)<\infty$ and $\lambda_g^{[q]}<\delta<\infty$ where p,q are any integers with $p>1$ and $q>2$. Then for a sequence of values of r tending to infinity,*

$$M_h^{-1}M_{f\circ g}(r)<M_h^{-1}M_f(\exp^{[q-1]}r^{\delta}).$$

Proof. Since $M_h^{-1}(r)$ is an increasing function of r, it follows from the second part of Lemma 1 for a sequence of values of r tending to infinity that

$$\log^{[p]}M_h^{-1}M_{f\circ g}(r)\leq\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^{\lambda_g^{[q]}+\varepsilon}. \tag{17}$$

Now, from (4) and (17), it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}M_h^{-1}M_f(\exp^{[q-1]}r^{\delta})}{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}\geq\frac{\left(\lambda_h^{[p]}(f)-\varepsilon\right)\exp^{[q-2]}r^{\delta}}{\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^{\lambda_g^{[q]}+\varepsilon}}. \tag{18}$$

As $\lambda_g^{[q]}<\delta$ we can choose $\varepsilon\,(>0)$ in such a way that

$$\lambda_g^{[q]}+\varepsilon<\delta<\rho_g^{[q]}. \tag{19}$$

Thus, from (18) and (19), we obtain that

$$\limsup_{r\to\infty}\frac{\log^{[p]}M_h^{-1}M_f(\exp^{[q-1]}r^{\delta})}{\log^{[p]}M_h^{-1}M_{f\circ g}(r)}=\infty. \tag{20}$$

From (20), we obtain for a sequence of values of r tending to infinity and also for $K>1$

$$M_h^{-1}M_f(\exp^{[q-1]}r^{\delta})>M_h^{-1}M_{f\circ g}(r).$$

Thus the theorem follows. □

In the line of Theorem 17, we may state the following theorem without its proof:

Theorem 18. *Let f, g, h and k be any four entire functions with $\lambda_k^{[m]}(g) > 0$ and $\rho_h^{[p]}(f) < \infty$ where p, m are any integers with $p > 1$ and $q > 1$. Also suppose that $\lambda_g^{[q]} < \delta < \infty$ where q is any integer > 2, then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} M_h^{-1} M_{f \circ g}(r) < \log^{[m-1]} M_k^{-1} M_g(\exp^{[q-1]} r^{\delta}).$$

In the line of Theorem 17 and Theorem 18 the following two theorems can also be proved:

Theorem 19. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\lambda_g^{[q]} < \delta < \infty$ where p, q are any integers with $p > 1$ and $q > 2$. Then for a sequence of values of r tending to infinity,*

$$\mu_h^{-1} \mu_{f \circ g}(r) < \mu_h^{-1} \mu_f(\exp^{[q-1]} r^{\delta}).$$

Theorem 20. *Let f, g, h and k be any four entire functions with $\lambda_k^{[m]}(g) > 0$ and $\rho_h^{[p]}(f) < \infty$ where p, m are any integers with $p > 1$ and $q > 1$. Also suppose that $\lambda_g^{[q]} < \delta < \infty$ where q is any integer > 2. Then for a sequence of values of r tending to infinity,*

$$\log^{[p-1]} \mu_h^{-1} \mu_{f \circ g}(r) < \log^{[m-1]} \mu_k^{-1} \mu_g(\exp^{[q-1]} r^{\delta}).$$

As an application of Theorem 13 and Theorem 17, we may state the following theorem:

Theorem 21. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\lambda_g^{[q]} < \delta < \rho_g^{[q]}$ where p, q are any integers with $p > 1$ and $q > 2$. Then*

$$\liminf_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_f(\exp^{[q-1]} r^{\delta})} \leq 1 \leq \limsup_{r \to \infty} \frac{M_h^{-1} M_{f \circ g}(r)}{M_h^{-1} M_f(\exp^{[q-1]} r^{\delta})}.$$

The proof is omitted.

In view of Theorem 14 and Theorem 18, the following theorem can be carried out:

Theorem 22. *Let f, g, h and k be any four entire functions with $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$, $0 < \lambda_k^{[m]}(g) \leq \rho_k^{[m]}(g) < \infty$ and $0 < \lambda_g^{[m]} < \delta < \rho_g^{[m]} < \infty$ where p, q, m are any three integers with $p > 1$, $q > 2$ and $m > 1$. Then*

$$\liminf_{r \to \infty} \frac{\log^{[p-1]} M_h^{-1} M_{f \circ g}(r)}{\log^{[m-1]} M_k^{-1} M_g(\exp^{[q-1]} r^{\delta})} \leq 1 \leq \limsup_{r \to \infty} \frac{\log^{[p-1]} M_h^{-1} M_{f \circ g}(r)}{\log^{[m-1]} M_k^{-1} M_g(\exp^{[q-1]} r^{\delta})}.$$

Analogously one may also state the following two theorems without their proofs as those may be carried out in the line of Theorem 15, Theorem 19 and Theorem 16, Theorem 20 respectively.

Theorem 23. *Let f, g and h be any three entire functions such that $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$ and $\lambda_g^{[q]} < \delta < \rho_g^{[q]}$ where p, q are any integers with $p > 1$ and $q > 2$. Then*

$$\liminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(\exp^{[q-1]} r^\delta)} \leq 1 \leq \limsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(r)}{\mu_h^{-1}\mu_f(\exp^{[q-1]} r^\delta)}.$$

Theorem 24. *Let f, g, h and k be any four entire functions with $0 < \lambda_h^{[p]}(f) \leq \rho_h^{[p]}(f) < \infty$, $0 < \lambda_k^{[m]}(g) \leq \rho_k^{[m]}(g) < \infty$ and $0 < \lambda_g^{[m]} < \delta < \rho_g^{[m]} < \infty$ where p, q, m are any integers with $p > 1$, $q > 2$ and $m > 1$. Then*

$$\liminf_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m-1]} \mu_k^{-1}\mu_g(\exp^{[q-1]} r^\delta)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}(r)}{\log^{[m-1]} \mu_k^{-1}\mu_g(\exp^{[q-1]} r^\delta)}.$$

Theorem 25. *Let f, g, h, k, l and b be any six entire functions such that $\lambda_b^{[m]}(l) > 0$, $\rho_h^{[p]}(f) < \infty$ and $\rho_g^{[q]} < \lambda_k^{[n]}$ where p, q, m, n are all positive integers with $p \geq 1$, $m \geq 1$ and $n \geq q \geq 2$. Then*

$$\text{(i)} \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty \text{ if } p = m$$

$$\text{(ii)} \lim_{r\to\infty} \frac{M_b^{-1}M_{l\circ k}(r)}{\log^{[p-m]} M_h^{-1}M_{f\circ g}(r)} = \infty \text{ if } p > m$$

and

$$\text{(iii)} \lim_{r\to\infty} \frac{\log^{[m-p]} M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} = \infty \text{ if } p < m.$$

Proof. Since $M_b^{-1}(r)$ is an increasing function of r, it follows from Lemma 1 for all sufficiently large values of r that

$$\log^{[m]} M_b^{-1}M_{l\circ k}(r) \geq \left(\lambda_b^{[m]}(l) - \varepsilon\right)\frac{1}{16} + \left(\lambda_b^{[m]}(l) - \varepsilon\right)\log M_k\left(\frac{r}{2}\right)$$

$$i.e., \ \log^{[m]} M_b^{-1}M_{l\circ k}(r) \geq \left(\lambda_b^{[m]}(l) - \varepsilon\right)\frac{1}{16} + \left(\lambda_b^{[m]}(l) - \varepsilon\right)\exp^{[n-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[n]}-\varepsilon}. \tag{21}$$

Since $\rho_g^{[q]} < \lambda_k^{[n]}$, we can choose $\varepsilon\,(> 0)$ in such a manner that

$$\rho_g^{[q]} + \varepsilon < \lambda_k^{[n]} - \varepsilon. \tag{22}$$

Case I. Let $p = m$.
Therefore combining (3) and (21) and in view of (22) we get for all sufficiently

large values of r that

$$\frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)} \geqslant \frac{\exp^{[m]}\left[\left(\lambda_b^{[m]}(l)-\varepsilon\right)\frac{1}{16}+\left(\lambda_b^{[m]}(l)-\varepsilon\right)\exp^{[n-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[n]}-\varepsilon}\right]}{\exp^{[m]}\left[\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^{\rho_g^{[q]}+\varepsilon}\right]}$$

$$i.e.,\ \lim_{r\to\infty}\frac{M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)}=\infty.$$

Thus the first part of the theorem follows from above.
Case II. Let $p > m$.
Now combining (3) and (21) and in view of (22), we obtain for all sufficiently large values of r that

$$\frac{M_b^{-1}M_{l\circ k}(r)}{\log^{[p-m]}M_h^{-1}M_{f\circ g}(r)}$$
$$\geqslant \frac{\exp^{[m]}\left[\left(\lambda_b^{[m]}(l)-\varepsilon\right)\frac{1}{16}+\left(\lambda_b^{[m]}(l)-\varepsilon\right)\exp^{[n-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[n]}-\varepsilon}\right]}{\exp^{[m]}\left[\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^{\rho_g^{[q]}+\varepsilon}\right]}$$

$$i.e.,\ \lim_{r\to\infty}\frac{M_b^{-1}M_{l\circ k}(r)}{\log^{[p-m]}M_h^{-1}M_{f\circ g}(r)}=\infty,$$

which is the second part of the theorem.
Case III. Let $p < m$.
Now combining (3) and (21) and in view of (22), we obtain for all sufficiently large values of r that

$$\frac{\log^{[m-p]}M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)}$$
$$\geqslant \frac{\exp^{[p]}\left[\left(\lambda_b^{[m]}(l)-\varepsilon\right)\frac{1}{16}+\left(\lambda_b^{[m]}(l)-\varepsilon\right)\exp^{[n-2]}\left(\frac{r}{2}\right)^{\lambda_k^{[n]}-\varepsilon}\right]}{\exp^{[p]}\left[\left(\rho_h^{[p]}(f)+\varepsilon\right)\exp^{[q-2]}r^{\rho_g^{[q]}+\varepsilon}\right]}$$

$$i.e.,\ \lim_{r\to\infty}\frac{\log^{[m-p]}M_b^{-1}M_{l\circ k}(r)}{M_h^{-1}M_{f\circ g}(r)}=\infty,$$

Thus the third part of the theorem is established.
Thus the theorem follows. □

Theorem 26. *Let f, g, h, k, l and b be any six entire functions such that $\lambda_b^{[m]}(l) > 0$, $\rho_h^{[p]}(f) < \infty$ and $\rho_g^{[q]} < \lambda_k^{[n]}$ where p, q, m, n are all positive integers with $p \geq 1$, $m \geq 1$ and $n \geq q \geq 2$. Then*

$$\text{(i)}\ \lim_{r\to\infty}\frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r)}=\infty\ \textit{if}\ p=m$$

$$\text{(ii)} \lim_{r\to\infty} \frac{\mu_b^{-1}\mu_{l\circ k}(r)}{\log^{[p-m]} \mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ } if \text{ } p > m$$

and

$$\text{(iii)} \lim_{r\to\infty} \frac{\log^{[m-p]} \mu_b^{-1}\mu_{l\circ k}(r)}{\mu_h^{-1}\mu_{f\circ g}(r)} = \infty \text{ } if \text{ } p < m \text{ }.$$

We omit the proof of Theorem 26 because it can be carried out in the line of Theorem 25 and with the help of Lemma 2, Lemma 3 and Lemma 5.

Remark 6. If we consider $\rho_g^{[q]} < \rho_k^{[n]}$ instead of $\rho_g^{[q]} < \lambda_k^{[n]}$ in Theorem 25 and Theorem 26 and the other conditions remain the same, the conclusion of Theorem 25 and Theorem 26 remain valid with "limit superior" replaced by "limit".

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