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CERTAIN GENERALIZED OSTROWSKI TYPE INEQUALITIES FOR LOCAL FRACTIONAL INTEGRALS

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Abstract. We give a function associated with generalized Ostrowski type inequality and its integral representation for local fractional calculus. Then, using this function and its integral representation, we establish several inequalities of generalized Ostrowski type for twice local fractional differentiable functions. We also consider some special cases of the main results which are further applied to a concrete function to yield two interesting inequalities associated with two generalized means.

1. Introduction and preliminaries

Throughout this paper, let $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ be the sets of real and positive real numbers, rational numbers, integers and positive integers, respectively, and

$$\mathbb{J} := \mathbb{R} \setminus \mathbb{Q} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

In order to describe the definition of the local fractional derivative and local fractional integral, recently, one has introduced to define the following sets (see, *e.g.*, [14, 15]): For $0 < \alpha \leq 1$,

(i) the α-type set of integers $\mathbb{Z}^\alpha$ is defined by

$$\mathbb{Z}^\alpha := \{0^\alpha\} \cup \{\pm m^\alpha \,:\, m \in \mathbb{N}\}\,;$$

(ii) the α-type set of rational numbers $\mathbb{Q}^\alpha$ is defined by

$$\mathbb{Q}^\alpha := \{q^\alpha \,:\, q \in \mathbb{Q}\} = \left\{ \left(\frac{m}{n}\right)^\alpha \,:\, m \in \mathbb{Z},\, n \in \mathbb{N} \right\};$$

(iii) the α-type set of irrational numbers $\mathbb{J}^\alpha$ is defined by

$$\mathbb{J}^\alpha := \{r^\alpha \,:\, r \in \mathbb{J}\} = \left\{ r^\alpha \neq \left(\frac{m}{n}\right)^\alpha \,:\, m \in \mathbb{Z},\, n \in \mathbb{N} \right\};$$

(iv) the α-type set of real line numbers $\mathbb{R}^\alpha$ is defined by $\mathbb{R}^\alpha := \mathbb{Q}^\alpha \cup \mathbb{J}^\alpha$.

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Throughout this paper, whenever the α-type set $\mathbb{R}^\alpha$ of real line numbers is involved, the α is assumed to be tacitly $0 < \alpha \leq 1$.

One has also defined two binary operations the addition $+$ and the multiplication $\cdot$ (which is conventionally omitted) on the α-type set $\mathbb{R}^\alpha$ of real line numbers as follows (see, *e.g.*, [14, 15]): For a^α, $b^\alpha \in \mathbb{R}^\alpha$,

$$a^\alpha + b^\alpha := (a+b)^\alpha \quad \text{and} \quad a^\alpha \cdot b^\alpha = a^\alpha b^\alpha := (ab)^\alpha. \tag{1.1}$$

Then one finds that

- $(\mathbb{R}^\alpha, +)$ is a commutative group: For a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$,
  - (A_1) $a^\alpha + b^\alpha \in \mathbb{R}^\alpha$;
  - (A_2) $a^\alpha + b^\alpha = b^\alpha + a^\alpha$;
  - (A_3) $a^\alpha + (b^\alpha + c^\alpha) = (a^\alpha + b^\alpha) + c^\alpha$;
  - (A_4) 0^α is the identity for $(\mathbb{R}^\alpha, +)$: For any $a^\alpha \in \mathbb{R}^\alpha$, $a^\alpha + 0^\alpha = 0^\alpha + a^\alpha = a^\alpha$;
  - (A_5) For each $a^\alpha \in \mathbb{R}^\alpha$, $(-a)^\alpha$ is the inverse element of a^α for $(\mathbb{R}^\alpha, +)$:
    $a^\alpha + (-a)^\alpha = (a + (-a))^\alpha = 0^\alpha$;
- $(\mathbb{R}^\alpha \setminus \{0^\alpha\}, \cdot)$ is a commutative group: For a^α, b^α, $c^\alpha \in \mathbb{R}^\alpha$,
  - (M_1) $a^\alpha b^\alpha \in \mathbb{R}^\alpha$;
  - (M_2) $a^\alpha\, b^\alpha = b^\alpha\, a^\alpha$;
  - (M_3) $a^\alpha\,(b^\alpha c^\alpha) = (a^\alpha\, b^\alpha)\, c^\alpha$;
  - (M_4) 1^α is the identity for $(\mathbb{R}^\alpha, \cdot)$: For any $a^\alpha \in \mathbb{R}^\alpha$, $a^\alpha 1^\alpha = 1^\alpha a^\alpha = a^\alpha$;
  - (M_5) For each $a^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, $(1/a)^\alpha$ is the inverse element of a^α for $(\mathbb{R}^\alpha, \cdot)$:
    $a^\alpha (1/a)^\alpha = (a(1/a))^\alpha = 1^\alpha$;
- Distributive law holds: $a^\alpha\,(b^\alpha + c^\alpha) = a^\alpha b^\alpha + a^\alpha c^\alpha$.

Furthermore we observe some additional properties for $(\mathbb{R}^\alpha, +, \cdot)$ which are stated in the following proposition.

Proposition 1. *Each of the following statements holds true*:

(i) *Like the usual real number system* $(\mathbb{R}, +, \cdot)$, $(\mathbb{R}^\alpha, +, \cdot)$ *is a field*;
(ii) *The additive identity* 0^α *and the multiplicative identity* 1^α *are unique, respectively*;
(iii) *The additive inverse element and the multiplicative inverse element are unique, respectively*;
(iv) *For each* $a^\alpha \in \mathbb{R}^\alpha$, *its inverse element* $(-a)^\alpha$ *may be written as* $-a^\alpha$; *for each* $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, *its inverse element* $(1/b)^\alpha$ *may be written as* $1^\alpha/b^\alpha$ *but not as* $1/b^\alpha$;
(v) *If the order* $<$ *is defined on* $(\mathbb{R}^\alpha, +, \cdot)$ *as follows*: $a^\alpha < b^\alpha$ *in* $\mathbb{R}^\alpha$ *if and only if* $a < b$ *in* $\mathbb{R}$, *then* $(\mathbb{R}^\alpha, +, \cdot, <)$ *is an ordered field like* $(\mathbb{R}, +, \cdot, <)$.

Proof. We prove only (iv). For each $a^\alpha \in \mathbb{R}^\alpha$ and each $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, $a^\alpha + (-a^\alpha) = 0^\alpha$ and $b^\alpha\,(1^\alpha/b^\alpha) = 1^\alpha$. Then, in view of uniqueness of inverse elements in (iii), we may identify $(-a)^\alpha$ and $(1/b)^\alpha$ with $-a^\alpha$ and $1^\alpha/b^\alpha$, respectively. Yet, if $1^\alpha/b^\alpha = 1/b^\alpha$ for $b^\alpha \in \mathbb{R}^\alpha \setminus \{0^\alpha\}$, then, after multiplying b^α on each side, we have $1^\alpha = 1$. If $1^\alpha = 1$ is right, then we can write

$$2^\alpha = (1+1)^\alpha = 1^\alpha + 1^\alpha = 1 + 1 = 2,$$

which is obviously impossible when $0 < \alpha < 1$. $\square$

In order to introduce the local fractional calculus on $\mathbb{R}^\alpha$, we begin with the concept of the local fractional continuity as in Definition 1.

Definition 1. A non-differentiable function $f : \mathbb{R} \to \mathbb{R}^\alpha$, $x \mapsto f(x)$, is called to be local fractional continuous at x_0 if for any $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that

$$|f(x) - f(x_0)| < \varepsilon^\alpha$$

holds for $|x - x_0| < \delta$. If a function f is local continuous on the interval (a, b), we denote $f \in C_\alpha(a, b)$.

Among several attempts to have defined local fractional derivative and local fractional integral (see [14, Section 2.1]), we choose to recall the following definitions of local fractional calculus (see, *e.g.*, [3, 14, 15]):

Definition 2. The local fractional derivative of $f(x)$ of order α at $x = x_0$ is defined by

$$f^{(\alpha)}(x_0) = {}_{x_0}D_x^\alpha f(x) = \left.\frac{d^\alpha f(x)}{dx^\alpha}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^\alpha\left(f(x) - f(x_0)\right)}{(x - x_0)^\alpha},$$

where $\Delta^\alpha\left(f(x) - f(x_0)\right) \cong \Gamma(\alpha+1)\left(f(x) - f(x_0)\right)$ and Γ is the familiar Gamma function (see, e.g., [12, Section 1.1]).

Let $f^{(\alpha)}(x) = D_x^\alpha f(x)$. If there exists $f^{(k+1)\alpha}(x) = \overbrace{D_x^\alpha \cdots D_x^\alpha}^{k+1 \text{ times}} f(x)$ for any $x \in I \subseteq \mathbb{R}$, then we denote $f \in D_{(k+1)\alpha}(I)$ $(k \in \mathbb{N}_0)$.

Definition 3. Let $f \in C_\alpha[a, b]$. Also let $P = \{t_0, \ldots, t_N\}$ $(N \in \mathbb{N})$ be a partition of the interval $[a, b]$ which satisfies $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$. Further, for this partition P, let $\Delta t := \max\limits_{0\le j\le N-1} \Delta t_j$ where $\Delta t_j := t_{j+1} - t_j$, $j = 0, \ldots, N-1$. Then the local fractional integral of f on the interval $[a, b]$ of order α (denoted by ${}_aI_b^{(\alpha)} f$) is defined by

$$ {}_aI_b^{(\alpha)} f = \frac{1}{\Gamma(\alpha+1)} \int_a^b f(t)(dt)^\alpha := \frac{1}{\Gamma(\alpha+1)} \lim_{\Delta t\to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t_j)^\alpha, \tag{1.2}$$

provided the limit exists (in fact, this limit exists if $f \in C_\alpha[a, b]$).

Here, it follows that ${}_aI_b^{(\alpha)} f = 0$ if $a = b$ and ${}_aI_b^{(\alpha)} f = -{}_bI_a^{(\alpha)} f$ if $a < b$.

If ${}_aI_x^{(\alpha)} g$ exists for any $x \in [a, b]$ and a function $g : [a, b] \to \mathbb{R}^\alpha$, then we denote $g \in I_x^{(\alpha)}[a, b]$.

Remark 2. Let f and g be two bounded real-valued functions defined on the interval $[a, b]$. Let, for simplicity, us use the notations in Definition 3. Then the Riemann-Stieltjes integral of f on $[a, b]$ with respect to the function g (denoted by $\int_a^b f\,dg$) can be defined by

$$\int_a^b f\,dg := \lim_{\Delta t \to 0} \sum_{j=0}^{N-1} f(t_j)\ (g(t_{j+1}) - g(t_j)), \tag{1.3}$$

provided the limit exists.

Here, if we choose the function $g(t) = t^\alpha$, then, by noticing

$$(\Delta t_j)^\alpha = (t_{j+1})^\alpha - (t_j)^\alpha = g(t_{j+1}) - g(t_j),$$

where the first equality follows from the operation in $\mathbb{R}^\alpha$, in a sense of the formality, the local fractional integral in (1.2) may be considered as a special case of the Riemann-Stieltjes integral in (1.3), except for the auxiliary constant $1/\Gamma(\alpha + 1)$.

For the present investigation, we recall some properties for the local fractional calculus given in Lemma 3 (see, *e.g.*, [3, 14, 15]).

Lemma 3. *The following formulas hold true*:

(1) (α*-local fractional derivative*)

$$\frac{d^\alpha x^{k\alpha}}{dx^\alpha} = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k-1)\,\alpha)} x^{(k-1)\alpha} \quad (k \in \mathbb{R});$$

(2) (*Local fractional integration is anti-differentiation*)
Suppose that $f = g^{(\alpha)} \in C_\alpha\,[a, b]$. *Then we have*

$${}_aI_b^{(\alpha)}\,f = g(b) - g(a);$$

(3) (*Local fractional integration by parts*)
Suppose that $f,\, g \in D_\alpha\,[a, b]$ *and* $f^{(\alpha)},\ g^{(\alpha)} \in C_\alpha\,[a, b]$. *Then we have*

$${}_aI_b^{(\alpha)}\left(f g^{(\alpha)}\right) = f\,g|_a^b - {}_aI_b^{(\alpha)}\left(f^{(\alpha)}\,g\right);$$

(4) (*Local fractional definite integrals of* $x^{k\alpha}$)

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b x^{k\alpha} (dx)^\alpha = \frac{\Gamma(1 + k\alpha)}{\Gamma(1 + (k+1)\,\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha}\right) \quad (k \in \mathbb{R}).$$

For more and detailed properties about local fractional calculus, one may refer to such works as (for example) [14, 15, 16, 17, 18, 19].

A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$, is said to be convex on the interval I if the following inequality

$$f\,(tx + (1-t)\,y) \leq t f\,(x) + (1-t)\,f\,(y) \tag{1.4}$$

holds for every $x,\, y \in I$ and $t \in [0, 1]$.

The following inequality gives an estimate of the (integral) mean value of convex functions: Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}, \tag{1.5}$$

which is known as Hermite-Hadamard inequality.

Mo *et al.* [7] introduced to investigate the following generalized convex function:

Definition 4. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^\alpha$. If the following inequality

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\alpha f(x_1) + (1-\lambda)^\alpha f(x_2) \tag{1.6}$$

holds for any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, then f is said to be a generalized convex function on I. If the inequality in (1.6) is reversed, then f is called a generalized concave function on I.

Example 1. Two generalized convex functions are recalled (see [7, p. 3]): The one elementary function is

$$f(x) = x^{\alpha p} \quad (x \ge 0;\ p > 1). \tag{1.7}$$

The Mittag-Leffler function $E_\alpha(x)$ is defined by

$$E_\alpha(x) := \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(1+k\alpha)} \quad \left(x \in \mathbb{R};\ \alpha \in \mathbb{R}^+\right) \tag{1.8}$$

(for various generalizations, see, *e.g.*, [2]). It is easy to see that $E_1(x) = \exp(x)$, since $\Gamma(n+1) = n!$ $(n \in \mathbb{N}_0)$. Hence the Mittag-Leffler function in (1.8) is often referred to as a generalized exponential.

Here the other one is given by

$$g(x) := E_\alpha(x^\alpha) = \sum_{k=0}^{\infty} \frac{x^{k\alpha}}{\Gamma(1+k\alpha)} \quad (x \in \mathbb{R};\ 0 < \alpha \le 1), \tag{1.9}$$

which may be considered as the exponential function in the α-type set $\mathbb{R}^\alpha$ (see, *e.g.*, [14, Section 1.14]).

Mo *et al.* [7, Theorem 14] proved the following generalized Hermite-Hadamard inequality for local fractional integral: Let $f \in I_x^{(\alpha)}[a,b]$ be a generalized convex function on $[a,b]$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(1+\alpha)}{(b-a)^\alpha}\, {}_aI_b^{(\alpha)} f \le \frac{f(a)+f(b)}{2^\alpha}. \tag{1.10}$$

Remark 4. Some of the classical inequalities for integral means can be derived from (1.10) by appropriately choosing the involved function f. Both inequalities in (1.10) hold in the reverse direction if f is generalized concave. For some new results which generalize, improve and extend the inequalities (1.10), one may refer to such works as (for example) [1, 3, 5, 6, 9, 10, 11, 13] and references therein.

Recently an attention has been paid on an interesting Hölder's inequality for the local fractional integral, which was established by Yang [15], as in following lemma:

Lemma 5. *Let $f, g \in C_\alpha[a,b]$ and p, $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\begin{aligned} &\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)g(x)| \, (dx)^\alpha \\ &\leq \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |f(x)|^p \, (dx)^\alpha \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(\alpha+1)} \int_a^b |g(x)|^q \, (dx)^\alpha \right)^{\frac{1}{q}}. \end{aligned} \tag{1.11}$$

Ostrowski [8] established an integral inequality which is now classical and given in Theorem 6.

Theorem 6. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $f' : [a,b] \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty = \sup_{t\in(a,b)} |f'(t)| < \infty$. Then the following inequality holds true*:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \, \|f'\|_\infty \tag{1.12}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

The inequality (1.12) has been paid considerable attention by mathematicians and other researchers due mainly to its wide and various applications in such areas as (for example) numerical analysis and the theory of certain special means (see, *e.g.*, [3]). Very recently, Sarikaya and Budak [9] obtained a generalized Ostrowski inequality for local fractional integral, which is recalled in Theorem 7.

Theorem 7. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I° is the interior of I) such that $f^{(\alpha)} \in D_\alpha(I^\circ)$ and $f^{(2\alpha)} \in C_\alpha[a,b]$ for $a, b \in I^\circ$ with $a < b$. Also assume that*

$$\|f^{(\alpha)}\|_\infty := \sup_{t\in[a,b]} |f^{(\alpha)}(t)| < \infty.$$

Then, for all $x \in [a,b]$,

$$\begin{aligned} &\left| f(x) - \frac{\Gamma(1+\alpha)}{(b-a)^\alpha} \, {}_aI_b^{(\alpha)} f \right| \\ &\leq 2^\alpha \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \left[\frac{1}{4^\alpha} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\alpha} \right] (b-a)^\alpha \, \|f^{(\alpha)}\|_\infty. \end{aligned} \tag{1.13}$$

Here, in this paper, we give a function and its integral representation associated with local fractional calculus. Then, using this function and its integral representation, we establish several inequalities of generalized Ostrowski type (see, *e.g.*, [3, Theorem 9]) for twice local fractional differentiable functions. We also consider some special cases of the main results which are further applied to a concrete function to yield two interesting inequalities associated with the generalized means in (3.1) and (3.2).

2. Main results

In order to establish further inequalities of generalized Ostrowski type for twice local fractional differentiable functions, we begin by introducing a function and its integral representation for twice local fractional differentiable functions asserted by the following lemma.

Lemma 8. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I° is the interior of I) such that f, $f^{(\alpha)} \in D_\alpha(I^\circ)$ and $f^{(2\alpha)} \in C_\alpha[a,b]$ for a, $b \in I^\circ$ with $a < b$. Then the following equality holds true: For any $x \in [\frac{a+b}{2}, b]$,*

$$\mathcal{L}(\alpha; a, b; x) = \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} \int_0^1 k(t) f^{(2\alpha)}\left(ta + (1-t)b\right)(dt)^\alpha, \tag{2.1}$$

where

$$\begin{aligned} \mathcal{L}(\alpha; a, b; x) :=& \frac{1}{(b-a)^\alpha}\ {}_aI_b^{(\alpha)} f \\ &- \frac{1}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\left[f(x) + f(a+b-x)\right] \\ &+ \frac{1}{2^\alpha\Gamma(1+\alpha)}\left(x - \frac{a+3b}{4}\right)^\alpha \left[f^{(\alpha)}(x) - f^{(\alpha)}(a+b-x)\right] \end{aligned} \tag{2.2}$$

and

$$k(t) := \begin{cases} t^{2\alpha} & \text{if} \quad 0 \le t \le \frac{b-x}{b-a}, \\ (t-\frac{1}{2})^{2\alpha} & \text{if} \quad \frac{b-x}{b-a} < t < \frac{x-a}{b-a}, \\ (t-1)^{2\alpha} & \text{if} \quad \frac{x-a}{b-a} \le t \le 1. \end{cases} \tag{2.3}$$

Proof. Let $\mathcal{I}$ be the following integral:

$$\mathcal{I} := \frac{1}{\Gamma(1+\alpha)} \int_0^1 k(t) f^{(2\alpha)}\left(ta + (1-t)b\right)(dt)^\alpha.$$

Then, in view of $k(t)$, we have

$$\mathcal{I} = \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \tag{2.4}$$

where

$$\mathcal{I}_1 := \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} t^{2\alpha} f^{(2\alpha)}\left(ta + (1-t)b\right)(dt)^\alpha,$$

$$\mathcal{I}_2 := \frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2}\right)^{2\alpha} f^{(2\alpha)}\,(ta+(1-t)b)\,(dt)^{\alpha}$$

and

$$\mathcal{I}_3 := \frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^{1} (t-1)^{2\alpha} f^{(2\alpha)}\,(ta+(1-t)b)\,(dt)^{\alpha}.$$

Applying the local fractional integration by parts, we have

$$\begin{aligned}\mathcal{I}_1 =\; & \frac{t^{2\alpha}}{(a-b)^{\alpha}} f^{(\alpha)}\,(ta+(1-t)b)\,\Big|_0^{\frac{b-x}{b-a}} \\ & - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(a-b)^{\alpha}} \int_0^{\frac{b-x}{b-a}} t^{\alpha} f^{(\alpha)}\,(ta+(1-t)b)\,(dt)^{\alpha}.\end{aligned}$$

Again, applying the local fractional integration by parts, we obtain

$$\begin{aligned}\mathcal{I}_1 =\; & \frac{1}{(a-b)^{\alpha}} \left(\frac{b-x}{b-a}\right)^{2\alpha} f^{(\alpha)}(x) \\ & - \frac{\Gamma(1+2\alpha)}{\Gamma(1+\alpha)(a-b)^{\alpha}} \left[\frac{t^{\alpha}}{(a-b)^{\alpha}} f\,(ta+(1-t)b)\,\Big|_0^{\frac{b-x}{b-a}} \right. \\ & \left. - \frac{\Gamma(1+\alpha)}{(a-b)^{\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} f\,(ta+(1-t)b)\,(dt)^{\alpha} \right].\end{aligned}$$

Then, after a little simplification, we get

$$\begin{aligned}\mathcal{I}_1 =\; & -\frac{(b-x)^{2\alpha}}{(b-a)^{3\alpha}} f^{(\alpha)}(x) - \frac{\Gamma(1+2\alpha)(b-x)^{\alpha}}{\Gamma(1+\alpha)(b-a)^{3\alpha}} f(x) \\ & + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} f\,(ta+(1-t)b)\,(dt)^{\alpha}.\end{aligned} \tag{2.5}$$

Similarly we have

$$\begin{aligned}\mathcal{I}_2 =\; & \frac{(a+b-2x)^{2\alpha}}{4^{\alpha}(b-a)^{3\alpha}} \left[f^{(\alpha)}(x) - f^{(\alpha)}(a+b-x)\right] \\ & + \frac{\Gamma(1+2\alpha)(a+b-2x)^{\alpha}}{2^{\alpha}\Gamma(1+\alpha)(b-a)^{3\alpha}} \left[f(x)+f(a+b-x)\right] \\ & + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} f\,(ta+(1-t)b)\,(dt)^{\alpha}\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\mathcal{I}_3 =\; & \frac{(b-x)^{(2\alpha)}}{(b-a)^{3\alpha}} f^{(\alpha)}(a+b-x) - \frac{\Gamma(1+2\alpha)(b-x)^{\alpha}}{\Gamma(1+\alpha)(b-a)^{3\alpha}} f(a+b-x) \\ & + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^{1} f\,(ta+(1-t)b)\,(dt)^{\alpha}.\end{aligned} \tag{2.7}$$

We find from (2.5), (2.6) and (2.7) that

$$
\begin{aligned}
\mathcal{I} = {} & \frac{1}{(b-a)^{2\alpha}} \left(x - \frac{a+3b}{4} \right)^{\alpha} \left(f^{(\alpha)}(x) - f^{(\alpha)}(a+b-x) \right) \\
& - \frac{\Gamma(1+2\alpha)}{2^{\alpha}\Gamma(1+\alpha)(b-a)^{2\alpha}} \left(f(x) + f(a+b-x) \right) \\
& + \frac{\Gamma(1+2\alpha)}{(b-a)^{2\alpha}} \frac{1}{\Gamma(1+\alpha)} \int_0^1 f(ta+(1-t)b)(dt)^{\alpha}.
\end{aligned} \tag{2.8}
$$

Finally, changing the variable $u = ta + (1-t)b$ ($t \in [0,1]$) in (2.8) and multiplying each side of the resulting identity by $(b-a)^{2\alpha}/\Gamma(1+2\alpha)$, we can get the desired equality (2.1). $\square$

Theorem 9. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ (I° is the interior of I) such that f, $f^{(\alpha)} \in D_{\alpha}(I^{\circ})$ and $f^{(2\alpha)} \in C_{\alpha}[a,b]$ for $a, b \in I^{\circ}$ with $a < b$. Also assume that*

$$
\|f^{(2\alpha)}\|_{\infty} := \sup_{t\in[a,b]} |f^{(2\alpha)}(t)| < \infty.
$$

Then the following inequality holds true: For any $x \in [\frac{a+b}{2}, b]$,

$$
|\mathcal{L}(\alpha; a, b; x)| \le \frac{\left\| f^{(2\alpha)} \right\|_{\infty}}{\Gamma(1+3\alpha)\,(b-a)^{\alpha}} \left[2^{\alpha}\,(b-x)^{3\alpha} + \frac{(2x-a-b)^{3\alpha}}{4^{\alpha}} \right], \tag{2.9}
$$

where $\mathcal{L}(\alpha; a, b; x)$ is given as in (2.2).

Proof. Let $\mathcal{L} := \mathcal{L}(\alpha; a, b; x)$ in (2.2). Then we have

$$
\begin{aligned}
|\mathcal{L}| &\le \frac{(b-a)^{2\alpha}}{\Gamma(1+\alpha)\,\Gamma(1+2\alpha)} \int_0^1 |k(t)| \left| f^{(2\alpha)}(ta+(1-t)b) \right| (dt)^{\alpha} \\
&\le \frac{(b-a)^{2\alpha} \left\| f^{(2\alpha)} \right\|_{\infty}}{\Gamma(1+\alpha)\,\Gamma(1+2\alpha)} \int_0^1 |k(x,t)| (dt)^{\alpha}.
\end{aligned}
$$

Using (2.3), we get

$$
\begin{aligned}
|\mathcal{L}| \le {} & \frac{(b-a)^{2\alpha} \left\| f^{(2\alpha)} \right\|_{\infty}}{\Gamma(1+2\alpha)} \left\{ \frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} t^{2\alpha} (dt)^{\alpha} \right. \\
& \left. + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2} \right)^{2\alpha} (dt)^{\alpha} + \frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^{1} (t-1)^{2\alpha} (dt)^{\alpha} \right\}.
\end{aligned} \tag{2.10}
$$

Using Lemma 3, we have

$$
\frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} t^{2\alpha} (dt)^{\alpha} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \Big(\frac{b-x}{b-a} \Big)^{3\alpha}, \tag{2.11}
$$

$$
\frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2} \right)^{2\alpha} (dt)^{\alpha} = \frac{\Gamma(1+2\alpha)}{4^{\alpha}\Gamma(1+3\alpha)} \Big(\frac{2x-a-b}{b-a} \Big)^{3\alpha} \tag{2.12}
$$

and

$$\frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^{1} (t-1)^{2\alpha} (dt)^{\alpha} = \frac{1}{\Gamma(1+\alpha)} \int_{0}^{\frac{b-x}{b-a}} u^{2\alpha} (du)^{\alpha} = \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{b-x}{b-a}\right)^{3\alpha}. \tag{2.13}$$

Finally, substituting (2.11), (2.12) and (2.13) into (2.10) is immediately seen to yield the desired inequality (2.9). □

Setting $x = (a+b)/2$ in Theorem 9 gives an interesting inequality associated with the local fractional integral asserted by the following corollary.

Corollary 1. *Under the assumptions of Theorem 9, the following inequality holds true*:

$$\left| \frac{1}{(b-a)^{\alpha}} {}_a I_b^{\alpha} f - \frac{2^{\alpha}}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} f\left(\frac{a+b}{2}\right) \right| \leq \frac{\left\| f^{(2\alpha)} \right\|_{\infty} (b-a)^{2\alpha}}{4^{\alpha}\Gamma(1+3\alpha)}. \tag{2.14}$$

Theorem 10. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$ (I° is the interior of I) such that f, $f^{(\alpha)} \in D_{\alpha}(I^{\circ})$ and $f^{(2\alpha)} \in C_{\alpha}[a, b]$ for a, $b \in I^{\circ}$ with $a < b$. If $\left|f^{(2\alpha)}\right|$ is generalized convex, then the following inequality holds true*: *For any $x \in [\frac{a+b}{2}, b]$,*

$$|\mathcal{L}(\alpha; a, b; x)| \leq \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \left[\mathcal{K}_{\alpha}(x; a, b) + \mathcal{L}_{\alpha}(x; a, b) + \mathcal{M}_{\alpha}(x; a, b)\right], \tag{2.15}$$

where $\mathcal{L}(\alpha; a, b; x)$ is given as in (2.2), *and*

$$\mathcal{K}_{\alpha}(x; a, b) := \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \left(\frac{b-x}{b-a}\right)^{4\alpha} \left|f^{(2\alpha)}(a)\right| + \left[\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{b-x}{b-a}\right)^{3\alpha} - \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \left(\frac{b-x}{b-a}\right)^{4\alpha}\right] \left|f^{(2\alpha)}(b)\right|, \tag{2.16}$$

$$\mathcal{L}_{\alpha}(x; a, b) := \mathrm{C}_{\alpha}(x; a, b) \left|f^{(2\alpha)}(a)\right| - \mathrm{D}_{\alpha}(x; a, b) \left|f^{(2\alpha)}(b)\right|, \tag{2.17}$$

$$\begin{aligned} \mathrm{C}_{\alpha}(x; a, b) := & \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)} \left(\frac{(x-a)^{4\alpha} - (b-x)^{4\alpha}}{(b-a)^{4\alpha}}\right) \\ & - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \left(\frac{(x-a)^{3\alpha} - (b-x)^{3\alpha}}{(b-a)^{3\alpha}}\right) \\ & + \frac{\Gamma(1+\alpha)}{4^{\alpha}\Gamma(1+2\alpha)} \left(\frac{(x-a)^{2\alpha} - (b-x)^{2\alpha}}{(b-a)^{2\alpha}}\right) \end{aligned}$$

and

$$\begin{aligned}\mathrm{D}_\alpha(x;a,b) :=& \frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{(x-a)^{4\alpha}-(b-x)^{4\alpha}}{(b-a)^{4\alpha}}\right)\\ &-2^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{(x-a)^{3\alpha}-(b-x)^{3\alpha}}{(b-a)^{3\alpha}}\right)\\ &+\left(\frac{5}{4}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{(x-a)^{2\alpha}-(b-x)^{2\alpha}}{(b-a)^{2\alpha}}\right)\\ &-\frac{1}{4^\alpha\,\Gamma(1+\alpha)}\left(\frac{(2x-b-a)^\alpha}{(b-a)^\alpha}\right),\end{aligned}$$

and

$$\begin{aligned}\mathcal{M}_\alpha(x;a,b) :=& \left[\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{(b-a)^{4\alpha}-(x-a)^{4\alpha}}{(b-a)^{4\alpha}}\right)\right.\\ &-\frac{2^\alpha\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left(\frac{(b-a)^{3\alpha}-(x-a)^{3\alpha}}{(b-a)^{3\alpha}}\right)\\ &\left.+\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\frac{b+a-2x}{b-a}\right)^\alpha\right]\left|f^{(2\alpha)}(a)\right|\\ &+\frac{\Gamma(1+3\alpha)}{\Gamma(1+4\alpha)}\left(\frac{b-x}{b-a}\right)^{4\alpha}\left|f^{(2\alpha)}(b)\right|.\end{aligned} \tag{2.18}$$

Proof. As in the proof of Theorem 10, let $\mathcal{L} := \mathcal{L}(\alpha;a,b;x)$ in (2.2). Then, in view of $k(t)$, we have

$$\begin{aligned}|\mathcal{L}| &\le \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)}\frac{1}{\Gamma(1+\alpha)}\int_0^1 |k(t)|\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^\alpha\\ &\le \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)}\Big(\mathcal{H}_1+\mathcal{H}_2+\mathcal{H}_3\Big),\end{aligned} \tag{2.19}$$

where

$$\mathcal{H}_1 := \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{b-x}{b-a}} t^{2\alpha}\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^\alpha,$$

$$\mathcal{H}_2 := \frac{1}{\Gamma(1+\alpha)}\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t-\frac{1}{2}\right)^{2\alpha}\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^\alpha$$

and

$$\mathcal{H}_3 := \frac{1}{\Gamma(1+\alpha)}\int_{\frac{x-a}{b-a}}^1 (t-1)^{2\alpha}\left|f^{(2\alpha)}(ta+(1-t)b)\right|(dt)^\alpha.$$

By using the generalized convexity of $\left|f^{(2\alpha)}\right|$ (see Definition 4) and applying Lemma 3 to compute local fractional integrals of the involved powers, we have

$$\begin{aligned}\mathcal{H}_1 &\le \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{b-x}{b-a}}\left(t^{3\alpha}\left|f^{(2\alpha)}(a)\right|+t^{2\alpha}(1-t)^\alpha\left|f^{(2\alpha)}(b)\right|\right)(dt)^\alpha\\ &= \mathcal{K}_\alpha(x;a,b),\end{aligned} \tag{2.20}$$

$$\begin{aligned} \mathcal{H}_2 \leq \frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} & \left[t^\alpha \left(t-\frac{1}{2}\right)^{2\alpha} \left|f^{(2\alpha)}(a)\right| \right. \\ & \left. + (1-t)^\alpha \left(t-\frac{1}{2}\right)^{2\alpha} \left|f^{(2\alpha)}(a)\right| \right] (dt)^\alpha = \mathcal{L}_\alpha(x;a,b) \end{aligned} \tag{2.21}$$

and

$$\begin{aligned} \mathcal{H}_3 \leq \frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^{1} & \left[t^\alpha (t-1)^{2\alpha} \left|f^{(2\alpha)}(a)\right| \right. \\ & \left. + (1-t)^\alpha (t-1)^{2\alpha} \left|f^{(2\alpha)}(b)\right| \right] (dt)^\alpha = \mathcal{M}_\alpha(x;a,b). \end{aligned} \tag{2.22}$$

Finally, by substituting (2.20), (2.21) and (2.22) in (2.19), we can get the desired inequality. □

Theorem 11. *Let $I \subseteq \mathbb{R}$ be an interval, $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^\alpha$ (I° is the interior of I) such that f, $f^{(\alpha)} \in D_\alpha(I^\circ)$ and $f^{(2\alpha)} \in C_\alpha[a,b]$ for a, $b \in I^\circ$ with $a < b$. Also let p, $q \in \mathbb{R}$ with p, $q > 1$ and*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $\left|f^{(2\alpha)}\right|^q$ is generalized convex, then the following inequality holds true: For any $x \in \left[\frac{a+b}{2}, b\right]$,

$$\begin{aligned} & |\mathcal{L}(\alpha;a,b;x)| \\ & \leq \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \frac{1}{\{2^\alpha\,\Gamma(1+\alpha)\}^{\frac{1}{q}}} \left\{ \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)} \right\}^{\frac{1}{p}} \mathcal{J}_\alpha(x;a,b;p,q), \end{aligned} \tag{2.23}$$

where $\mathcal{L}(\alpha;a,b;x)$ is given as in (2.2), and $\mathcal{J}_\alpha(x;a,b;p,q) := \sum_{j=1}^{3} \mathcal{J}_{\alpha,j}(x;a,b;p,q)$ with

$$\mathcal{J}_{\alpha,1}(x;a,b;p,q) := \left(\frac{b-x}{b-a}\right)^{(2+1/p)\alpha} \left(\left|f^{(2\alpha)}(b)\right|^q + \left|f^{(2\alpha)}(x)\right|^q \right)^{\frac{1}{q}},$$

$$\begin{aligned} \mathcal{J}_{\alpha,2}(x;a,b;p,q) := & \left\{ \left(\frac{2x-b-a}{2(b-a)}\right)^{(2p+1)\alpha} - \left(\frac{a+b-2x}{2(b-a)}\right)^{(2p+1)\alpha} \right\}^{\frac{1}{p}} \\ & \times \left(\left|f^{(2\alpha)}(x)\right|^q + \left|f^{(2\alpha)}(a+b-x)\right|^q \right)^{\frac{1}{q}} \end{aligned}$$

and

$$\mathcal{J}_{\alpha,3}(x;a,b;p,q) := \left(\frac{b-x}{b-a}\right)^{(2+1/p)\alpha} \left(\left|f^{(2\alpha)}(a)\right|^q + \left|f^{(2\alpha)}(a+b-x)\right|^q \right)^{\frac{1}{q}}.$$

Proof. Let $\mathcal{L} := \mathcal{L}(\alpha; a, b; x)$ in (2.2). Then we find from (2.19) that

$$
\begin{aligned}
|\mathcal{L}| &\leq \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \frac{1}{\Gamma(1+\alpha)} \int_0^1 |k(t)| \left| f^{(2\alpha)}(ta + (1-t)b) \right| (dt)^\alpha \\
&\leq \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \Big(\mathcal{H}_1 + \mathcal{H}_2 + \mathcal{H}_3 \Big),
\end{aligned}
\tag{2.24}
$$

where $\mathcal{H}_i$ $(i = 1, 2, 3)$ are given as in (2.19).

By applying the Hölder's inequality for the local fractional integral in Lemma 5 to $\mathcal{H}_i$ $(i = 1, 2, 3)$, we have

$$
\begin{aligned}
\mathcal{H}_1 &\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} t^{2p\alpha} (dt)^\alpha \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} \left| f^{(2\alpha)}(ta + (1-t)b) \right|^q (dt)^\alpha \right)^{\frac{1}{q}},
\end{aligned}
\tag{2.25}
$$

$$
\begin{aligned}
\mathcal{H}_2 &\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2} \right)^{2p\alpha} (dt)^\alpha \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left| f^{(2\alpha)}(ta + (1-t)b) \right|^q (dt)^\alpha \right)^{\frac{1}{q}}
\end{aligned}
\tag{2.26}
$$

and

$$
\begin{aligned}
\mathcal{H}_3 &\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^1 (t-1)^{2p\alpha} (dt)^\alpha \right)^{\frac{1}{p}} \\
&\quad \times \left(\frac{1}{\Gamma(1+\alpha)} \int_{\frac{x-a}{b-a}}^1 \left| f^{(2\alpha)}(ta + (1-t)b) \right|^q (dt)^\alpha \right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.27}
$$

Here, using Lemma 3, we obtain

$$
\frac{1}{\Gamma(1+\alpha)} \int_0^{\frac{b-x}{b-a}} t^{2p\alpha} (dt)^\alpha = \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)} \left(\frac{b-x}{b-a} \right)^{(2p+1)\alpha},
\tag{2.28}
$$

$$
\begin{aligned}
&\frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}} \left(t - \frac{1}{2} \right)^{2p\alpha} (dt)^\alpha = \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)} \\
&\quad \times \left[\left(\frac{2x-b-a}{2(b-a)} \right)^{(2p+1)\alpha} - \left(\frac{a+b-2x}{2(b-a)} \right)^{(2p+1)\alpha} \right]
\end{aligned}
\tag{2.29}
$$

and

$$\begin{aligned}\frac{1}{\Gamma(1+\alpha)}\int_{\frac{x-a}{b-a}}^{1}(t-1)^{2p\alpha}(dt)^{\alpha}&=\frac{1}{\Gamma(1+\alpha)}\int_{\frac{x-a}{b-a}}^{1}(1-t)^{2p\alpha}(dt)^{\alpha}\\&=\frac{1}{\Gamma(1+\alpha)}\int_{0}^{\frac{b-x}{b-a}}u^{2p\alpha}(du)^{\alpha}\\&=\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\left(\frac{b-x}{b-a}\right)^{(2p+1)\alpha}.\end{aligned}\tag{2.30}$$

Also, since $\left|f^{(2\alpha)}\right|^{q}$ is generalized convex on $[a,b]$, by the generalized Hermite-Hadamard inequality in (1.10), we have

$$\begin{aligned}\int_{0}^{\frac{b-x}{b-a}}\left|f^{(2\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}&=\frac{1}{(b-a)^{\alpha}}\int_{x}^{b}\left|f^{(2\alpha)}(u)\right|^{q}(du)^{\alpha}\\&\leq\frac{\left|f^{(2\alpha)}(b)\right|^{q}+\left|f^{(2\alpha)}(x)\right|^{q}}{2^{\alpha}},\end{aligned}\tag{2.31}$$

$$\begin{aligned}&\int_{\frac{b-x}{b-a}}^{\frac{x-a}{b-a}}\left|f^{(2\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}\\&\leq\frac{\left|f^{(2\alpha)}(x)\right|^{q}+\left|f^{(2\alpha)}(a+b-x)\right|^{q}}{2^{\alpha}}\end{aligned}\tag{2.32}$$

and

$$\begin{aligned}&\int_{\frac{x-a}{b-a}}^{1}\left|f^{(2\alpha)}(ta+(1-t)b)\right|^{q}(dt)^{\alpha}\\&\leq\frac{\left|f^{(2\alpha)}(a+b-x)\right|^{q}+\left|f^{(2\alpha)}(a)\right|^{q}}{2^{\alpha}}.\end{aligned}\tag{2.33}$$

Finally, setting the equalities (2.28)-(2.30) and the inequalities (2.31)-(2.33) in the inequalities (2.25)-(2.27), and substituting the resulting inequalities into (2.24), we can obtain the desired inequality (2.23). □

Setting $x=b$ in Theorem 11 yields an inequality involving a local fractional integral asserted by Corollary 2. Here we need to recall the following inequality (see, *e.g.*, [4, p. 54]):

$$\begin{gathered}\sum_{k=1}^{n}(u_k+v_k)^{s}\leq\sum_{k=1}^{n}(u_k)^{s}+\sum_{k=1}^{n}(v_k)^{s}\\(n\in\mathbb{N};\ 0\leq s\leq 1;\ u_k,\,v_k\geq 0\ (1\leq k\leq n)).\end{gathered}\tag{2.34}$$

Corollary 2. *Under the assumptions of Theorem 11, the following inequality holds true*:

(2.35)
$$\left| \frac{1}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f - \frac{f(a)+f(b)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(\frac{b-a}{8}\right)^{\alpha} \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\Gamma(1+\alpha)} \right|$$
$$\leq \frac{(b-a)^{2\alpha}}{4^{\alpha}\Gamma(1+2\alpha)} \left\{ \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)} \right\}^{\frac{1}{p}} \left\{ \frac{|f^{(2\alpha)}(a)|+|f^{(2\alpha)}(b)|}{(2^{\alpha}\Gamma(1+\alpha))^{\frac{1}{q}}} \right\}.$$

Proof. Let $\mathcal{L}_1$ be the left-hand side of the inequality in (2.35). Then, setting $x=b$ in Theorem 11, we obtain

$$\begin{aligned} \mathcal{L}_1 \leq{} & \frac{(b-a)^{2\alpha}}{\Gamma(1+2\alpha)} \left\{ \frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)} \right\}^{\frac{1}{p}} \\ & \times \frac{[1^{\alpha}-(-1)^{\alpha}]^{\frac{1}{p}}}{2^{(2+\frac{1}{p})\alpha}} \left\{ \frac{|f^{(2\alpha)}(a)|+|f^{(2\alpha)}(b)|}{(2^{\alpha}\Gamma(1+\alpha))^{\frac{1}{q}}} \right\}. \end{aligned} \tag{2.36}$$

Now, applying the inequality (2.34) to each of the last two terms in (2.36) is easily seen to give the desired inequality (2.35). □

3. Application to some special means

In order to apply some results in Section 2, we first recall the following generalized means (see, *e.g.*, [3]):

$$A(a,b) := \frac{a^{\alpha}+b^{\alpha}}{2^{\alpha}} \tag{3.1}$$

and

$$L_n(a,b) := \left[\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n+1)\alpha)} \left(\frac{b^{(n+1)\alpha}-a^{(n+1)\alpha}}{(b-a)^{\alpha}} \right) \right]^{\frac{1}{n}} \tag{3.2}$$

$$\left(n \in \mathbb{Z} \setminus \{-1,0\};\ a,\, b \in \mathbb{R} \text{ with } a \neq b\right).$$

Now consider a function $f : I \subseteq \mathbb{R}^{+} \to \mathbb{R}^{\alpha}$ defined by

$$f(t) = t^{n\,\alpha} \quad (n \in \mathbb{Z} \setminus \{-1,0\}).$$

Then it is easy to see the following relations: For a, $b \in I$ with $a < b$,

$$f\left(\frac{a+b}{2}\right) = [A(a,b)]^{n} \tag{3.3}$$

and

$$\frac{1}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} f(t) = [L_n(a,b)]^{n}\,. \tag{3.4}$$

Here applying (3.3) and (3.4) to the inequalities (2.14) and (2.35), we obtain the following inequalities associated with $A(a,b)$ in (3.1) and $L_n(a,b)$ in (3.2), respectively:

$$\left|[L_n(a,b)]^n - \frac{2^\alpha\,[A(a,b)]^n}{\Gamma(1+\alpha)\Gamma(1+2\alpha)}\right| \le \frac{\left\|f^{(2\alpha)}\right\|_\infty (b-a)^{2\alpha}}{4^\alpha\Gamma(1+3\alpha)} \tag{3.5}$$

and

(3.6)

$$\begin{aligned}&\left|[L_n(a,b)]^n - \frac{2^\alpha A(a^n,b^n)}{\Gamma(1+\alpha)\Gamma(1+2\alpha)} + \left(\frac{b-a}{8}\right)^\alpha \frac{f^{(\alpha)}(b)-f^{(\alpha)}(a)}{\Gamma(1+\alpha)}\right|\\ &\le \frac{(b-a)^{2\alpha}}{2^\alpha\Gamma(1+2\alpha)}\,\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-2)\alpha)}\left\{\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\alpha)}\right\}^{\frac{1}{p}}\frac{A\left(a^{(n-2)},b^{(n-2)}\right)}{(2^\alpha\Gamma(1+\alpha))^{\frac{1}{q}}}.\end{aligned}$$

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