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GLOBAL ATTRACTOR FOR A SEMILINEAR PSEUDOPARABOLIC EQUATION WITH INFINITE DELAY

Dang Thi Phuong Thanh

Abstract. In this paper we consider a semilinear pseudoparabolic equation with polynomial nonlinearity and infinite delay. We first prove the existence and uniqueness of weak solutions by using the Galerkin method. Then, we prove the existence of a compact global attractor for the continuous semigroup associated to the equation. The existence and exponential stability of weak stationary solutions are also investigated.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$. In this paper, we consider the following semilinear pseudoparabolic equation with infinite delay

(1.1)
$$\begin{cases} \partial_t u(t,x)+A\partial_t u(t,x)+Au(t,x)+f(u(t,x))=g(u_t)+h(x), \ t>0, \ x\in\Omega, \\ u(s,x)=\phi(s,x), \quad s\in(-\infty,0], \ x\in\Omega, \end{cases}$$

where the (unbounded) linear operator A, the nonlinearity f, the external force h, the delay term g satisfy some certain conditions specified later, and $\phi(s)$ is the initial datum in the interval of time $(-\infty, 0]$. Here for a function u defined on $(-\infty, T)$, we denote by u_t the function defined on $(-\infty, 0]$ by the relation $u_t(s) = u(t+s), s \in (-\infty, 0]$.

In the special case $A = -\Delta$, the negative Laplacian, the equation (1.1) without the delay term is the so-called nonclassical diffusion equation, which was introduced as a model to describe some physical phenomena, such as non-Newtonian flows, soil mechanics and heat conduction theory (see, e.g., [1,19,24]). In the past years, the existence and long-time behavior of solutions to nonclassical diffusion equations has been studied extensively, in both autonomous case [16,17,20,21,25,27–30] and non-autonomous case [2,6,21,26,31]. On the other hand, there are situations in which the model is better described

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if some terms containing delays appear in the equations. These delays may appear, for instance, when one wants to control the system (in a certain sense) by applying a force which takes into account not only the present state, but the complete history of the solutions. However, to the best of our knowledge, all of existing results for nonclassical diffusion equations with delays are in the case of finite delay [7, 11, 32] and the infinite delay case has not been investigated before. This is the main motivation of our work.

As we know, the choice of phase spaces plays an important role in studying differential equations with infinite delay [13]. One possibility, and which we will use here, is to consider for any $\gamma > 0$, the space

$$C_\gamma(V) = \{\varphi \in C((-\infty,0];V) : \exists \lim_{s\to-\infty} e^{\gamma s}\varphi(s) \in V\}, \text{ with } V := D(A^{1/2}),$$

which is a Banach space with the norm

$$\|\varphi\|_{C_\gamma(V)} := \sup_{s\in(-\infty,0]} e^{\gamma s}\|\varphi(s)\|_V.$$

In order to study problem (1.1), we make the following assumptions:

(H1) A is a densely-defined self-adjoint positive linear operator with the domain $D(A) \subset L^2(\Omega)$ and with compact resolvent, and we furthermore assume that either $C_0^\infty(\Omega)$ or $C^\infty(\overline{\Omega})$ is contained and dense in $D(A^{1/2})$.

(H2) $g : C_\gamma(V) \to L^2(\Omega)$ satisfies the following conditions:

(g1) $g(0) = 0$,

(g2) There exists a constant $L_g > 0$ such that for all $\xi, \eta \in C_\gamma(V)$,

$$\|g(\xi) - g(\eta)\|_{L^2(\Omega)} \le L_g\|\xi - \eta\|_{C_\gamma(V)}.$$

(H3) $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable function satisfying

$$C_1|u|^p - C_0 \le f(u)u \le C_2|u|^p + C_0, \tag{1.2}$$

$$f'(u) \ge -\ell, \tag{1.3}$$

for some $p \ge 2$, where C_0, C_1, C_2 and ℓ are positive constants.

(H4) $h \in V' := D(A^{-1/2})$, the dual space of V.

Let us give some comments on the above assumptions. The operator A contains a large class of elliptic operators with suitable boundary conditions, for instance, the negative Laplace operator $-\Delta$ with homogeneous Dirichlet/Neumann boundary conditions (see other examples in [12]), and even some degenerate elliptic operators with homogeneous Dirichlet boundary condition such as the Caldiroli-Musina operator $-\mathrm{div}(\sigma(x)\nabla)$ in [10] or the $-\Delta_\lambda$-Laplace operator in [14]. The assumption on $D(A^{1/2})$ ensures the existence of a countable basis in $D(A^{1/2}) \cap L^p(\Omega)$, which is needed for the proof of the existence of weak solutions by using the Galerkin method. Here the nonlinearity is assumed to satisfy a dissipativity and growth condition of polynomial type; a typical example of the nonlinear term is an odd order polynomial with the positive leading coefficient. The condition (g1) of the delay term is not really a restriction, since otherwise, if $g(0) \in L^2(\Omega)$, we could redefine $\hat{h}(x) = h(x) + g(0)$ and $\hat{g}(\cdot) = g(\cdot) - g(0)$, then $\hat{h}$ and $\hat{g}$ will satisfy the required assumptions.

The appearance of the infinite delay term $g(u_t)$ and the term $A\partial_t u$ in equation (1.1) introduces some essential difficulty when proving the existence of solutions and existence of a global attractor. In particular, the corresponding dynamical system is only weakly dissipative. To overcome this difficulty, in this paper we try to combine the techniques dealing with infinite delays and techniques used for studying the nonclassical diffusion equations. More precisely, when prove the existence of solutions by using the Galerkin method, to pass to the limits in the nonlinear term and in the infinite delay term for approximate solutions, we combine the compactness method [15] and the energy method used in [18]. While to prove the asymptotic compactness of the associated semigroup, the most difficult step when proving the existence of a compact global attractor, we exploit the energy method in [18]. It is noticed that the existence and long-time behavior of solutions to abstract semilinear parabolic equations with delays, i.e., equation in (1.1) without the term $A\partial_t u$, has been extensively investigated in [3–5, 8, 9].

The paper is organized as follows. In Section 2, we prove the existence and uniqueness of weak solutions to problem (1.1) by using the Galerkin method. The existence of a compact global attractor for the continuous semigroup generated by weak solutions to the problem is proved in Section 3. To do this, we show the existence of a bounded absorbing set and then the asymptotic compactness of the semigroup. In the last section, we investigate the existence and exponential stability of stationary solutions to the problem.

Hereafter, for the sake of brevity, we denote $H = L^2(\Omega)$, $V = D(A^{1/2})$, $V' = D(A^{-1/2})$, $C_\gamma(V) = C_\gamma(D(A^{1/2}))$, with the corresponding norms $|\cdot|$, $\|\cdot\|$, $\|\cdot\|_*$, $\|\cdot\|_\gamma$; and $(\cdot,\cdot)$, $((\cdot,\cdot))$ are the scalar products in H, V, respectively. We use the notation $\langle\cdot,\cdot\rangle$ for the dual between V and V', and sometimes for the dual between $L^p(\Omega)$ and $L^{p'}(\Omega)$ (with $1/p + 1/p' = 1$).

Noting that by the assumption **(H1)**, the operator A has a discrete spectrum that only contains positive eigenvalues $\{\lambda_k\}_{k=1}^\infty$ satisfying

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots, \quad \lambda_k \to \infty, \quad \text{as } k \to \infty,$$

and the corresponding eigenfunctions $\{e_k\}_{k=1}^\infty$ compose an orthonormal basis of H such that

$$(e_j, e_k) = \delta_{jk} \text{ and } Ae_k = \lambda_k e_k, \quad k = 1, 2, \ldots$$

Hence we can define the fractional power spaces and operators as

$$X^\alpha = D(A^\alpha) = \{u = \sum_{k=1}^\infty c_k e_k \in H : \sum_{k=1}^\infty c_k^2 \lambda_k^{2\alpha} < \infty\},$$

$$A^\alpha u = \sum_{k=1}^{\infty} c_k \lambda_k^\alpha e_k, \text{ where } u = \sum_{k=1}^{\infty} c_k e_k.$$

It is known (see e.g. [12]) that if $\alpha > \beta$, then the space $D(A^\alpha)$ is compactly embedded into $D(A^\beta)$. In particular

$$V := D(A^{1/2}) \hookrightarrow L^2(\Omega) \hookrightarrow D(A^{-1/2}) =: V',$$

where the injections are dense and compact.

In what follows, we will frequently use the following inequality

$$\|u\|^2 \geq \lambda_1 |u|^2, \ \ \forall u \in V,$$

where $\lambda_1 > 0$ is the first eigenvalue of the operator A.

2. Existence and uniqueness of weak solutions

Definition 2.1. A weak solution on the interval $(0, T)$ of problem (1.1) with initial datum $\phi \in C_\gamma(V)$ is a function $u \in C((-\infty, T]; V) \cap L^p(0, T; L^p(\Omega))$ such that $u_0(\theta) = \phi(\theta)$ for all $\theta \leq 0$, $\frac{du}{dt} \in L^2(0, T; V)$, and

$$\frac{d}{dt}u(t) + Au(t) + A(\partial_t u(t)) + f(u(t)) = g(u_t) + h \text{ in } V'$$

in the distribution sense in $(0, T)$.

Theorem 2.1. *Under the assumptions* **(H1)-(H4)***, then for any $T > 0$ and $\phi \in C_\gamma(V)$ given, problem* (1.1) *has a unique weak solution u on the interval $(0, T)$.*

Proof. **(i) Uniqueness.** Let u and v be two solutions of problem (1.1) with the same initial datum $\phi \in C_\gamma(V)$. Putting $w = u - v$, we have

$$\partial_t w + A\partial_t w + Aw + f(u) - f(v) = g(u_t) - g(v_t). \tag{2.1}$$

Multiplying (2.1) by w and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}(|w|^2 + \|w\|^2) + \|w\|^2 + \int_\Omega (f(u) - f(v))(u - v)dx = (g(u_t) - g(v_t), w).$$

Using assumptions (1.3) and $(g2)$, we have

$$\frac{d}{dt}(|w|^2 + \|w\|^2) + 2\|w\|^2 \leq 2\ell |w|^2 + 2L_g \|w_t\|_\gamma |w|.$$

Noting that $w(\theta) = 0$ if $\theta \leq 0$, we have

$$\|w_s\|_\gamma = \sup_{\theta \leq 0} e^{\gamma\theta} \|w(s + \theta)\| \leq \sup_{\theta \in [-s, 0]} \|w(s + \theta)\| \ \text{ for } 0 \leq s \leq T,$$

and therefore for $t \in [0, T]$,

$$\begin{aligned} |w|^2 + \|w\|^2 &\leq 2\ell \int_0^t |w(s)|^2 ds + 2L_g \int_0^t \|w_s\|_\gamma |w(s)| ds \\ &\leq \frac{2\ell}{\lambda_1} \int_0^t \|w(s)\|^2 ds + \frac{2L_g}{\sqrt{\lambda_1}} \int_0^t \sup_{r \in [0, s]} \|w(r)\|^2 ds. \end{aligned}$$

Hence

$$\sup_{r\in[0,t]}\|w(r)\|^2 \le \Big(\frac{2\ell}{\lambda_1}+\frac{2L_g}{\sqrt{\lambda_1}}\Big)\int_0^t \sup_{r\in[0,s]}\|w(r)\|^2 ds,$$

whence the Gronwall lemma finishes the proof of uniqueness.

(ii) Existence. For the existence, we split the proof into several steps.

Step 1: *A Galerkin scheme.* We consider a basis $\{e_j\}_{j=1}^{\infty}\subset V\cap L^p(\Omega)$, which is orthonormal in H. The existence of such a basis follows from the assumption that either $C_0^\infty(\Omega)$ or $C^\infty(\overline{\Omega})$ is contained and dense in $V:=D(A^{1/2})$ (and the fact that they are dense in $L^p(\Omega)$).

We consider the approximate solution $u^n(t)$ in the form

$$u^n(t)=\sum_{j=1}^{n}\gamma_{nj}(t)e_j,$$

where the superscript n will be used instead of (n) for short since no confusion is possible with powers of u, and the coefficients γ_{nj} are required to satisfy the following system

(2.2)
$$\begin{cases}\dfrac{d}{dt}(u^n(t),e_j)+(A\partial_t u^n(t),e_j)+(Au^n(t),e_j)+\langle f(u^n),e_j\rangle=\langle h,e_j\rangle+(g(u_t^n),e_j),\\ (u^n(s),e_j)=(P_n\phi(s),e_j),\quad s\in(-\infty,0],\ 1\le j\le n.\end{cases}$$

The above system of ordinary functional differential equations with infinite delay fulfills the conditions for the existence of local solutions in [13, Theorem 1.1, p. 36], so the approximate solutions u^n exist.

Next, we will derive *a priori* estimates that ensure that the solutions do exist on the whole interval $[0,T]$.

Step 2: *A priori estimates.* Multiplying (2.2) by γ_{nj} and summing in j, we obtain

$$\frac{1}{2}\frac{d}{dt}(|u^n|^2+\|u^n\|^2)+\|u^n\|^2+\int_\Omega f(u^n)u^n dx=\int_\Omega g(u_t^n)u^n dx+\langle h,u^n\rangle.$$

Using hypothesis (1.2) and the Cauchy inequality, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|u^n|^2+\|u^n\|^2)+\|u^n\|^2+C_1\|u^n\|^p_{L^p(\Omega)}\\ &\le \frac{1}{2}\|u^n\|^2+\frac{1}{2}\|h\|_*^2+L_g\|u_t^n\|_\gamma|u^n|+C_0|\Omega|,\end{aligned}$$

and therefore

$$\frac{d}{dt}(|u^n|^2+\|u^n\|^2)+\|u^n\|^2+2C_1\|u^n\|^p_{L^p(\Omega)}\le\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_t^n\|_\gamma^2+2C_0|\Omega|.$$

Integrating from 0 to t, we obtain

$$|u^n(t)|^2+\|u^n(t)\|^2+\int_0^t\|u^n(s)\|^2ds+2C_1\int_0^t\|u^n(s)\|^p_{L^p(\Omega)}ds \tag{2.3}$$

$$\leq \left(|u^n(0)|^2 + \|u^n(0)\|^2\right) + \int_0^t \left(\|h\|_*^2 + \frac{2L_g}{\sqrt{\lambda_1}}\|u_s^n\|_\gamma^2 + 2C_0|\Omega|\right)ds$$
$$\leq \left(\frac{1}{\lambda_1} + 1\right)\|u^n(0)\|^2 + \int_0^t \left(\|h\|_*^2 + \frac{2L_g}{\sqrt{\lambda_1}}\|u_s^n\|_\gamma^2 + 2C_0|\Omega|\right)ds.$$

Furthermore,

$$\|u_t^n\|_\gamma^2 \leq \max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(t+\theta)\|^2;$$
$$\sup_{\theta\in[-t,0]}\Big[e^{2\gamma\theta}\big(\frac{1}{\lambda_1}+1\big)\|u^n(0)\|^2 + e^{2\gamma\theta}\int_0^{t+\theta}\left(\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s^n\|_\gamma^2 + 2C_0|\Omega|\right)ds\Big]\Big\}$$
$$\leq \max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(t+\theta)\|^2;$$
$$\big(\frac{1}{\lambda_1}+1\big)\|u^n(0)\|^2 + \int_0^t \left(\|h\|_*^2 + \frac{2L_g}{\sqrt{\lambda_1}}\|u_s^n\|_\gamma^2 + 2C_0|\Omega|\right)ds\Big\}.$$

Since

$$\sup_{\theta\in(-\infty,-t]} e^{\gamma\theta}\|\phi(\theta+t)\| = \sup_{\theta\leq 0} e^{\gamma(\theta-t)}\|\phi(\theta)\| = e^{-\gamma t}\|\phi\|_\gamma \leq \|\phi\|_\gamma,$$

and $\|u(0)\| \leq \|\phi\|_\gamma$, we deduce that

$$\|u_t^n\|_\gamma^2 \leq \big(\frac{1}{\lambda_1}+1\big)\|\phi\|_\gamma^2 + T(\|h\|_*^2 + 2C_0|\Omega|) + \frac{2L_g}{\sqrt{\lambda_1}}\int_0^t \|u_s^n\|_\gamma^2 ds.$$

Using the Gronwall lemma, we have

$$\|u_t^n\|_\gamma^2 \leq \left[\big(\frac{1}{\lambda_1}+1\big)\|\phi\|_\gamma^2 + T(\|h\|_*^2 + 2C_0|\Omega|)\right]\left(1 + \frac{2L_g t}{\sqrt{\lambda_1}} e^{\frac{2L_g}{\sqrt{\lambda_1}}t}\right).$$

Then we obtain the following estimates: There exists a constant C, depending on some constants of the problem (namely, λ_1, T, L_g and h) and $R > 0$, such that

$$\|u_t^n\|_\gamma^2 \leq C,\ \forall\, t\in[0,T],\ \|\phi\|_\gamma \leq R. \tag{2.4}$$

In particular, this implies that

$$\{u^n\} \text{ is bounded in } L^\infty(0,T;V).$$

From (2.3) and (2.4), we get

$$\|u^n\|_{L^p(0,T;L^p(\Omega))} \leq C. \tag{2.5}$$

Using (1.2) we get

$$\{f(u^n)\} \text{ is bounded in } L^{p'}(0,T;L^{p'}(\Omega)), \tag{2.6}$$

where p' is the conjugate of p.

Now, multiplying (1.1) by $\partial_t u$ and then integrating over Ω, we get

$$|\partial_t u^n|^2 + \|\partial_t u^n\|^2 + \frac{1}{2}\frac{d}{dt}\left(\|u^n\|^2 + 2\int_\Omega F(u^n)dx\right)$$

$$\leq \langle h, \partial_t u^n \rangle + L_g \|u_t^n\|_\gamma |\partial_t u^n|,$$

where $F(u) = \int_0^u f(s)ds$ is a primitive of $f(u)$. Hence

$$|\partial_t u^n|^2 + \|\partial_t u^n\|^2 + \frac{d}{dt}\left(\|u^n\|^2 + 2\int_\Omega F(u^n)dx\right) \leq \|h\|_*^2 + L_g^2 \|u_t^n\|_\gamma^2.$$

Integrating this inequality from 0 to t and using (2.4), (2.5), we deduce that

$$\{\partial_t u^n\} \text{ is bounded in } L^2(0,T;V).$$

Step 3: *Convergence in $C_\gamma(V)$ and existence of a weak solution.* We will prove that

$$u_t^n \to u_t \text{ in } C_\gamma(V),\ \forall t \in (-\infty, T],$$

by showing that

$$P_n\phi \to \phi \text{ in } C_\gamma(V), \tag{2.7}$$

$$u^n \to u \text{ in } C([0,T];V). \tag{2.8}$$

First, we check the convergence claimed in (2.7). Indeed, if not, there would exist $\varepsilon > 0$ and a subsequence, that we still denote the same, such that

$$e^{\gamma\theta_n}\|P_n\phi(\theta_n) - \phi(\theta_n)\| > \varepsilon. \tag{2.9}$$

One can assume that $\theta_n \to -\infty$, otherwise if $\theta_n \to \theta$, then $P_n(\theta_n) \to \phi(\theta)$, since

$$\|P_n\phi(\theta_n) - \phi(\theta)\| \leq \|P_n\phi(\theta_n) - P_n\phi(\theta)\| + \|P_n\phi(\theta) - \phi(\theta)\| \to 0 \text{ as } n \to +\infty.$$

But, with $\theta_n \to -\infty$ as $n \to +\infty$, denoting $\chi = \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta)$, we obtain

$$\begin{aligned} e^{\gamma\theta_n}\|P_n\phi(\theta_n) - \phi(\theta_n)\| &= \|P_n(e^{\gamma\theta_n}\phi(\theta_n)) - e^{\gamma\theta_n}\phi(\theta_n)\| \\ &\leq \|P_n(e^{\gamma\theta_n}\phi(\theta_n)) - P_n\chi\| + \|P_n\chi - \chi\| + \|\chi - e^{\gamma\theta_n}\phi(\theta_n)\| \to 0. \end{aligned}$$

This is a contradiction with (2.9), so (2.7) holds.

Now, we combine some well-known compactness results with an energy method to pass to the limits in a subsequence of $\{u^n\}$ to obtain a solution of (1.1).

From the estimates in Step 2, we deduce that there exist a subsequence (which we relabel the same) $\{u^n\}$, an element $u \in L^\infty(0,T;V)\cap L^p(0,T;L^p(\Omega))$ with $u' \in L^2(0,T;V)$, and $\xi \in L^2(0,T;H)$ such that

$$\begin{aligned} &u^n \rightharpoonup u \text{ weakly-star in } L^\infty(0,T;V),\\ &u^n \rightharpoonup u \text{ weakly in } L^p(0,T;L^p(\Omega)),\\ &\partial_t u^n \rightharpoonup \partial_t u \text{ weakly in } L^2(0,T;V),\\ &f(u^n) \rightharpoonup \zeta \text{ weakly in } L^{p'}(0,T;L^{p'}(\Omega)),\\ &g(u_t^n) \rightharpoonup \xi_g \text{ weakly in } L^2(0,T;H). \end{aligned} \tag{2.10}$$

Applying the Aubin-Lions lemma in [15], we can conclude that $u^n \to u$ strongly in $L^2(0,T;H)$, up to a subsequence. Hence $u^n \to u$ a.e. in $\Omega \times [0,T]$. Since f

is continuous, it follows that $f(u^n) \to f(u)$ a.e. in $\Omega \times [0,T]$. Thanks to (2.6) and Lemma 1.3 in [15, Chapter 1], one has

$$f(u^n) \rightharpoonup f(u) \text{ weakly in } L^{p'}(0,T;L^{p'}(\Omega)).$$

Hence $\zeta = f(u)$.

From the convergence of $\{u^n\}$ to u in $L^\infty(0,T;V)$, we deduce that

$$u^n(t) \to u(t) \text{ in } V \text{ a.e. } t \in (0,T).$$

Since

$$u^n(t) - u^n(s) = \int_s^t (u^n)'(r)dr \text{ in } H, \ \forall s,t \in [0,T],$$

from (2.10) we have that $\{u^n\}$ is equi-continuous on $[0,T]$ with values in H. By the compactness of the embedding $V \subset H$, from (2.4) and the equi-continuity in H, using the Ascoli-Arzela theorem we have

$$u^n \to u \text{ in } C([0,T];H). \tag{2.11}$$

Again from (2.10) we obtain that for any sequence $\{t_n\} \subset [0,T]$ with $t_n \to t$,

$$u^n(t_n) \rightharpoonup u(t) \text{ weakly in } V, \tag{2.12}$$

where we have used (2.11) in order to identify which is the weak limit.

Now, we are ready to prove (2.8) by a contradiction argument. If it would not be so, then taking into account that $u \in C([0,T];V)$, there would exist $\epsilon > 0$, a value $t_0 \in [0,T]$ and subsequences (relabeled the same) $\{u^n\}$ and $\{t_n\} \subset [0,T]$ with $\lim\limits_{n\to+\infty} t_n = t_0$ such that

$$\|u^n(t_n) - u(t_0)\| \geq \epsilon.$$

To prove that this is absurd, we will use an energy method. Observe that the following energy inequality holds for all u^n:

$$\begin{aligned} &\frac{1}{2}|u^n(t)|^2 + \frac{1}{2}\|u^n(t)\|^2 + \int_s^t \|u^n(r)\|^2 dr + \int_s^t (f(u^n(r)), u^n(r))dr \\ &\leq \int_s^t \langle h, u^n(r)\rangle dr + \frac{1}{2}|u^n(s)|^2 + \frac{1}{2}\|u^n(s)\|^2 + C_3(t-s), \ \forall \ s,t \in [0,T], \end{aligned} \tag{2.13}$$

where C_3 is a constant such that

$$\int_s^t |g(u^n_r)|^2 dr \leq C_3(t-s) \ \forall \ 0 \leq s < t \leq T.$$

On the other hand, by (2.4) and **(H2)**, there exists $\xi_g \in L^2(0,T;H)$ such that $\{g(u^n_t)\}$ converges weakly to ξ_g in $L^2(0,T;H)$. Thus, we can pass to the limits to deduce that u satisfies the following equality for all $v \in V \cap L^p(\Omega)$,

$$\frac{d}{dt}\big((u,v) + (\nabla u, \nabla v)\big) + (\nabla u, \nabla v) + \langle f(u), v\rangle = \langle h, v\rangle + (\xi_g, v).$$

Therefore, u satisfies the energy equality

$$|u(t)|^2 + \|u(t)\|^2 + 2\int_s^t \|u(r)\|^2 dr + 2\int_s^t \langle f(u(r)), u(r)\rangle dr$$
$$= |u(s)|^2 + \|u(s)\|^2 + 2\int_s^t \big(\langle h, u(r)\rangle + (\xi_g, u(r))\big)dr,\ \forall\, 0 \le s < t \le T,$$

and for the weak limit ξ_g we have the estimate

$$\int_s^t |\xi_g|^2 dr \le \liminf_{n\to+\infty} \int_s^t |g(u_r^n)|^2 dr \le C_3(t-s),\ \forall\, 0 \le s \le t \le T.$$

So, we have that u also satisfies inequality (2.13) with the same constant C_3. Now, consider two functions $J_n, J : [0,T] \to \mathbb{R}$ defined by

$$J_n(t) = \frac{1}{2}(|u^n(t)|^2 + \|u^n(t)\|^2) + \int_0^t \langle f(u^n(r)), u^n(r)\rangle dr - \int_0^t \langle h, u^n(r)\rangle dr - C_3 t,$$
$$J(t) = \frac{1}{2}(|u(t)|^2 + \|u(t)\|^2) + \int_0^t \langle f(u(r)), u(r)\rangle dr - \int_0^t \langle h, u(r)\rangle dr - C_3 t.$$

It is clear that J_n and J are non-increasing and continuous functions. Moreover, by the convergence of u^n to u a.e. in time with value in V, and weakly in $L^2(0,T;V)$, it holds that

$$J_n(t) \to J(t) \text{ for a.e. } t \in [0,T]. \tag{2.14}$$

Now, observe that the case $t_0 = 0$ follows directly from (2.13) with $s = 0$ and the definition of $u^n(0) = P_n\phi(0)$. So, we may assume that $t_0 > 0$. This is important, since we will approach this value t_0 from the left by a sequence $\{t'_k\}$, i.e., $\lim_{k\to+\infty} t'_k \nearrow t_0$. Since $u(\cdot)$ is continuous at t_0, there is k_ϵ such that

$$|J(t'_k) - J(t_0)| < \frac{\epsilon}{2},\ \forall\, k \ge k_\epsilon.$$

On the other hand, taking $n \ge n(k_\epsilon)$ such that $t_n > t'_{k_\epsilon}$, as J_n is non-increasing and for all t'_k the convergence (2.14) holds, one has

$$|J_n(t_n) - J(t_0)| \le |J_n(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| + |J(t'_{k_\epsilon}) - J(t_0)|,$$

and obviously, taking $n \ge n'(k_\epsilon)$, it is possible to obtain $|J_n(t'_{k_\epsilon}) - J(t'_{k_\epsilon})| < \frac{\epsilon}{2}$. Hence

$$\limsup_{n\to+\infty} \|u^n(t_n)\| \le \|u(t_0)\|. \tag{2.15}$$

Furthermore, from (2.12) we get

$$u^n(t_n) \rightharpoonup u(t_0) \text{ weakly in } V.$$

So, we have

$$\|u(t_0)\| \le \liminf_{n\to+\infty}\ \|u^n(t_n)\|. \tag{2.16}$$

Combining (2.16) and (2.15), we get

$$u^n(t_n) \to u(t_0) \text{ in } V.$$

Finally, we have to show that $g(u_t^n) \to g(u_t)$ in $L^2(0,T;H)$. We have

$$\begin{aligned}
&\|u_t^n - u_t\|_\gamma \\
=\ & \sup_{\theta\le 0} e^{\gamma\theta}\|u^n(t+\theta) - u(t+\theta)\| \\
=\ & \max\left\{\sup_{\theta\in(-\infty,-t]} e^{\gamma\theta}\|P_n\phi(\theta+t) - \phi(\theta+t)\|, \sup_{\theta\in[-t,0]} e^{\gamma\theta}\|u^n(t+\theta) - u(t+\theta)\|\right\} \\
\le\ & \max\left\{e^{-\gamma t}\|P_n\phi - \phi\|_\gamma, \max_{\theta\in[0,t]}\|u^n(\theta) - u(\theta)\|\right\} \to 0.
\end{aligned}$$

Hence, $u_t^n \to u_t$ in $C_\gamma(V)$, $\quad \forall\, t \le T$.

On the other hand, we identify the weak limit ξ from (2.10). So, we have that

$$g(u_t^n) \to g(u_t) \text{ in } L^2(0,T;H).$$

Therefore, u is a weak solution of problem (1.1). □

3. Existence of a global attractor

Thanks to Theorem 2.1, we can define a semigroup $S(t) : C_\gamma(V) \to C_\gamma(V)$, by the formula

$$S(t)\phi := u_t,$$

where $u(t)$ is the unique weak solution of (1.1) with the initial datum $\phi \in C_\gamma(V)$.

First, we prove the continuity of the semigroup $S(t)$.

Proposition 3.1. *Under the assumptions* **(H1)-(H4)***, the semigroup $S(t)$ is continuous on $C_\gamma(V)$.*

Proof. Denoting by u^i, for $i = 1, 2$, the corresponding solutions to (1.1) with initial data $\phi^i \in C_\gamma(V)$, respectively. Consider the equations satisfied by u^i for $i = 1, 2$, acting on the element $u^1 - u^2$ and taking the difference, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(|u^1(t) - u^2(t)|^2 + \|u^1(t) - u^2(t)\|^2\right) + \|u^1(t) - u^2(t)\|^2 \\
&+ \langle f(u^1) - f(u^2), u^1 - u^2\rangle = \big(g(u_t^1) - g(u_t^2), u^1 - u^2\big).
\end{aligned}$$

From (1.3) and $(g2)$, we get

$$\begin{aligned}
&\frac{d}{dt}\left(|u^1(t) - u^2(t)|^2 + \|u^1(t) - u^2(t)\|^2\right) + 2\|u^1(t) - u^2(t)\|^2 \\
&\quad - 2\ell|u^1(t) - u^2(t)|^2 \\
&\le 2L_g\|u_t^1 - u_t^2\|_\gamma|u^1(t) - u^2(t)|.
\end{aligned} \tag{3.1}$$

For all $s \in [0,t]$, one has

$$
\begin{aligned}
\|u_s^1 - u_s^2\|_\gamma &= \sup_{\theta \le 0} e^{\gamma\theta} \|u^1(s+\theta) - u^2(s+\theta)\| \\
&= \max\Big\{ \sup_{\theta \in (-\infty,-s]} e^{\gamma\theta} \|\phi^1(s+\theta) - \phi^2(s+\theta)\|; \\
&\qquad \sup_{\theta \in [-s,0]} e^{\gamma\theta} \|u^1(s+\theta) - u^2(s+\theta)\| \Big\} \\
&\le \max\Big\{ e^{-\gamma s} \|\phi^1 - \phi^2\|_\gamma; \max_{\theta \in [0,s]} \|u^1(\theta) - u^2(\theta)\| \Big\}.
\end{aligned} \tag{3.2}
$$

From (3.1) and (3.2), for all $t \in [0,T]$, we get

$$
\begin{aligned}
&|u^1(t) - u^2(t)|^2 + \|u^1(t) - u^2(t))\|^2 \\
&\le |\phi^1(0) - \phi^2(0)|^2 + \|\phi^1(0) - \phi^2(0)\|^2 + 2\ell \int_0^t |u^1(s) - u^2(s)|^2 ds \\
&\quad + 2L_g \|\phi^1 - \phi^2\|_\gamma \int_0^t e^{-\gamma s} |u^1(s) - u^2(s)| ds \\
&\quad + 2L_g \int_0^t |u^1(s) - u^2(s)| \max_{\theta \in [0,s]} \|u^1(\theta) - u^2(\theta)\| ds.
\end{aligned}
$$

If we now substitute t by $r \in [0,t]$ and consider the maximum when varying this r, from the above inequality we can conclude that

$$
\begin{aligned}
\max_{r \in [0,t]} \|u(r)^1 - u(r)^2\|^2 \le{}& \frac{\lambda_1 + 1}{\lambda_1} \|\phi^1(0) - \phi^2(0)\|^2 + \frac{L_g}{2\gamma} \|\phi^1 - \phi^2\|_\gamma^2 \\
&+ \frac{2\left(L_g(\sqrt{\lambda_1} + 1) + \ell\right)}{\lambda_1} \int_0^t \max_{r \in [0,s]} \|u^1(r) - u^2(r)\|^2 ds,
\end{aligned}
$$

where we have used the following estimates

$$
\begin{aligned}
&2L_g \|\phi^1 - \phi^2\|_\gamma \int_0^t e^{-\gamma s} |u^1(s) - u^2(s)| ds \\
&\le L_g \|\phi^1 - \phi^2\|_\gamma^2 \int_0^t e^{-2\gamma s} ds + L_g \int_0^t |u^1(s) - u^2(s)|^2 ds \\
&\le \frac{L_g}{2\gamma} \|\phi^1 - \phi^2\|_\gamma^2 + \frac{L_g}{\lambda_1} \int_0^t \|u^1(s) - u^2(s)\|^2 ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&2L_g \int_0^t |u^1(s) - u^2(s)| \max_{\theta \in [0,s]} \|u^1(\theta) - u^2(\theta)\| ds \\
&\le \frac{2L_g}{\sqrt{\lambda_1}} \int_0^t \max_{r \in [0,s]} \|u^1(r) - u^2(r)\|^2 ds.
\end{aligned}
$$

Hence, by the Gronwall lemma, we obtain

$$\max_{r\in[0,t]} \|u^1(r)-u^2(r)\|^2 \le \left(\frac{\lambda_1+1}{\lambda_1}\|\phi^1(0)-\phi^2(0)\|^2+\frac{L_g}{2\gamma}\|\phi^1-\phi^2\|_\gamma^2\right) \times \left[1+\frac{2(L_g(\sqrt{\lambda_1}+1)+\ell)t}{\lambda_1}e^{\frac{2(L_g(\sqrt{\lambda_1}+1)+\ell)}{\lambda_1}t}\right].$$

Combining with (3.2), we get

$$\|u_t^1-u_t^2\|_\gamma^2 \le \left(\frac{\lambda_1+1}{\lambda_1}+\frac{L_g}{2\gamma}\right)\|\phi^1-\phi^2\|_\gamma^2\left[1+\frac{2(L_g(\sqrt{\lambda_1}+1)+\ell)t}{\lambda_1}e^{\frac{2(L_g(\sqrt{\lambda_1}+1)+\ell)}{\lambda_1}t}\right].$$

This completes the proof. □

Lemma 3.2. *Let the assumptions* **(H1)-(H4)** *hold and let*

$$\frac{2L_g}{\sqrt{\lambda_1}} < \frac{\lambda_1}{1+\lambda_1} < 2\gamma.$$

Then the ball

$$\mathcal{B} = \left\{ v\in C_\gamma(V) : \|v\|_\gamma \le \sqrt{\frac{4\|h\|_*^2+4C_0|\Omega|}{\frac{\lambda_1}{1+\lambda_1}-\frac{2L_g}{\sqrt{\lambda_1}}}} \right\}$$

is a bounded absorbing set in $C_\gamma(V)$ for the semigroup $S(t)$.

Proof. Multiplying the first equation of (1.1) by u, we obtain

$$\frac{1}{2}\frac{d}{dt}(|u(t)|^2+\|u(t)\|^2)+\|u(t)\|^2+\langle f(u),u(t)\rangle = (g(u_t),u(t))+\langle h,u(t)\rangle.$$

Using (1.2), (g2) and the Cauchy inequality, we have

$$C_1\|u\|_{L^p(\Omega)}^p - C_0|\Omega| \le \langle f(u),u(t)\rangle,$$
$$(g(u_t),u(t)) \le |g(u_t)||u(t)| \le L_g\|u_t\|_\gamma|u(t)|,$$
$$\langle h,u(t)\rangle \le \|h\|_*^2+\frac{1}{4}\|u(t)\|^2.$$

Hence

$$\frac{d}{dt}\Big(|u(t)|^2+\|u(t)\|^2\Big)+\frac{3}{2}\|u(t)\|^2+2C_1\|u\|_{L^p(\Omega)}^p \le 2\|h\|_*^2+2L_g\|u_t\|_\gamma|u(t)|+2C_0|\Omega|.$$

Because $|u(t)| \le \frac{1}{\sqrt{\lambda_1}}\|u(t)\| \le \frac{1}{\sqrt{\lambda_1}}\|u_t\|_\gamma$, we conclude that

$$\frac{d}{dt}\Big(|u(t)|^2+\|u(t)\|^2\Big)+\frac{3}{2}\|u(t)\|^2+2C_1\|u\|_{L^p(\Omega)}^p \le 2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_t\|_\gamma^2+2C_0|\Omega|.$$

Multiplying this inequality by $e^{\frac{\lambda_1}{1+\lambda_1}s}$ and integrating from 0 to t, we obtain

$$\begin{aligned}(3.3)\qquad & e^{\frac{\lambda_1}{1+\lambda_1}t}(|u(t)|^2+\|u(t)\|^2)-\int_0^t \frac{\lambda_1}{1+\lambda_1}e^{\frac{\lambda_1}{1+\lambda_1}s}(|u(s)|^2+\|u(s)\|^2)ds\\ &+\int_0^t e^{\frac{\lambda_1}{1+\lambda_1}s}\Big(\frac{3}{2}\|u(s)\|^2+2C_1\|u(s)\|^p_{L^p(\Omega)}\Big)ds\\ &\le |u(0)|^2+\|u(0)\|^2+\int_0^t e^{\frac{\lambda_1}{1+\lambda_1}s}\bigg(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\bigg)ds.\end{aligned}$$

Multiplying (3.3) by $e^{-\frac{\lambda_1}{1+\lambda_1}t}$ and noting that $|u|^2+\|u\|^2\le\frac{1+\lambda_1}{\lambda_1}\|u\|^2$, we obtain

(3.4)

$$\begin{aligned}&|u(t)|^2+\|u(t)\|^2+\int_0^t e^{-\frac{\lambda_1}{1+\lambda_1}(t-s)}\Big(\frac{1}{2}\|u(s)\|^2+2C_1\|u(s)\|^p_{L^p(\Omega)}\Big)ds\\ &\le e^{-\frac{\lambda_1}{1+\lambda_1}t}(|u(0)|^2+\|u(0)\|^2)+\int_0^t e^{-\frac{\lambda_1}{1+\lambda_1}(t-s)}\bigg(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\bigg)ds,\end{aligned}$$

and therefore

$$\begin{aligned}\|u(t)\|^2\le{}&\frac{1+\lambda_1}{\lambda_1}e^{-\frac{\lambda_1}{1+\lambda_1}t}\|u(0)\|^2\\ &+\int_0^t e^{-\frac{\lambda_1}{1+\lambda_1}(t-s)}\Big(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\Big)ds.\end{aligned}$$

Furthermore,

$$\begin{aligned}\|u_t\|_\gamma^2 ={}& \sup_{\theta\le 0} e^{2\gamma\theta}\|u(t+\theta)\|^2\\ \le{}& \max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|u(t+\theta)\|^2, \sup_{\theta\in[-t,0]} e^{2\gamma\theta}\|u(t+\theta)\|^2\Big\}\\ \le{}& \max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(\theta+t)\|^2; \sup_{\theta\in[-t,0]}\Big[\frac{1+\lambda_1}{\lambda_1}e^{2\gamma\theta-\frac{\lambda_1}{1+\lambda_1}(t+\theta)}\|u(0)\|^2\\ &+e^{2\gamma\theta}\int_0^{t+\theta} e^{-\frac{\lambda_1}{1+\lambda_1}(t+\theta-s)}\Big(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\Big)ds\Big]\Big\}.\end{aligned}$$

By the assumption $2\gamma>\frac{\lambda_1}{1+\lambda_1}$, we get

$$\begin{aligned}&\sup_{\theta\in[-t,0]} e^{2\gamma\theta}\int_0^{t+\theta} e^{-\frac{\lambda_1}{1+\lambda_1}(t+\theta-s)}\Big(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\Big)ds\\ &\le \sup_{\theta\in[-t,0]}\int_0^{t+\theta} e^{-\frac{\lambda_1}{1+\lambda_1}(t-s)}\Big(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\Big)ds\Big]\Big\}.\end{aligned}$$

Hence

$$\|u_t\|_\gamma^2\le \max\Big\{\sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(\theta+t)\|^2; \sup_{\theta\in[-t,0]}\Big[\frac{1+\lambda_1}{\lambda_1}e^{-\frac{\lambda_1}{1+\lambda_1}t}\|u(0)\|^2$$

$$+\int_0^{t+\theta} e^{-\frac{\lambda_1}{1+\lambda_1}(t-s)}\Big(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\Big)ds\Big]\Big\}.$$

Since

$$\sup_{\theta\in(-\infty,-t]} e^{\gamma\theta}\|\phi(\theta+t)\| = \sup_{\theta\le 0} e^{\gamma(\theta-t)}\|\phi(\theta)\| = e^{-\gamma t}\|\phi\|_\gamma$$

and $\|u(0)\|\le\|\phi\|_\gamma$, we deduce that

$$\|u_t\|_\gamma^2 \le \frac{1+\lambda_1}{\lambda_1}e^{-\frac{\lambda_1}{1+\lambda_1}t}\|\phi\|_\gamma^2+\int_0^t e^{-\frac{\lambda_1}{1+\lambda_1}(t-s)}\Big(2\|h\|_*^2+\frac{2L_g}{\sqrt{\lambda_1}}\|u_s\|_\gamma^2+2C_0|\Omega|\Big)ds.$$

By the Gronwall lemma, we have

(3.5)

$$\begin{aligned}\|u_t\|_\gamma^2 &\le \frac{1+\lambda_1}{\lambda_1}e^{-(\frac{\lambda_1}{1+\lambda_1}-\frac{2L_g}{\sqrt{\lambda_1}})t}\|\phi\|_\gamma^2+\int_0^t e^{-(\frac{\lambda_1}{1+\lambda_1}-\frac{2L_g}{\sqrt{\lambda_1}})(t-s)}(2\|h\|_*^2+2C_0|\Omega|)ds\\ &= \frac{1+\lambda_1}{\lambda_1}e^{-(\frac{\lambda_1}{1+\lambda_1}-\frac{2L_g}{\sqrt{\lambda_1}})t}\|\phi\|_\gamma^2+\frac{2\|h\|_*^2+2C_0|\Omega|}{\frac{\lambda_1}{1+\lambda_1}-\frac{2L_g}{\sqrt{\lambda_1}}}.\end{aligned}$$

Since $\frac{2L_g}{\sqrt{\lambda_1}}<\frac{\lambda_1}{1+\lambda_1}$, this inequality implies that the set $\mathcal{B}$ defined above is a bounded absorbing set in $C_\gamma(V)$ for the semigroup $S(t)$. □

To show the existence of a global attractor, it remains to prove the asymptotic compactness of the semigroup $S(t)$.

Lemma 3.3. *Under the assumptions of Lemma 3.2, the semigroup $S(t)$ is asymptotically compact.*

Proof. Let B be a bounded set in $C_\gamma(V)$ and $u^n(\cdot)$ be a sequence of solutions in $[0,+\infty)$ with initial data $\phi^n\in B$. Consider the sequence $\xi^n=S(t_n)\phi^n$, where $t_n\to+\infty$ as $n\to+\infty$. We will prove that this sequence is relatively compact in $C_\gamma(V)$.

Step 1: Consider an arbitrary value $T>0$. We will prove that $\xi^n|_{[-T,0]}$ is relatively compact in $C([-T,0];V)$. It follows from (3.5) that there exists n_0 such that $t_n\ge T$ for all $n>n_0$ and

$$\|\xi^n\|_\gamma\le R, \tag{3.6}$$

where $R=\sqrt{\frac{4\|h\|_*^2+4C_0|\Omega|}{\frac{\lambda_1}{1+\lambda_1}-\frac{2L_g}{\sqrt{\lambda_1}}}}$.

Let $y^n(\cdot)=u^n_{t_n-T}(\cdot)=u^n(\cdot+t_n-T)$. Then for each $n\ge1$ such that $t_n\ge T$, the function y^n is a solution on $[0,T]$ of a similar problem to (1.1), namely,

$$\frac{d}{dt}y^n(t)+A\partial_t y^n(t)+Ay^n(t)+f(y^n(t))=g(y^n_t)+h, \tag{3.7}$$

with $y^n_0=u^n_{t_n-T}$, $y^n_T=\xi^n$. Then y^n_0 satisfies the estimate in (3.6) for all $n>n_0$. Using arguments as in the proof of Theorem 2.1, we can prove that $\{y^n\}$ is bounded in $L^\infty(0,T;V)\cap L^p(0,T;L^p(\Omega))$, and that $\{(y^n)'\}_n$ is bounded in $L^2(0,T;V)$. Thus, up to a subsequence (relabeled the same), for some function $y(\cdot)$ we have

$$\begin{aligned}
&y^n \rightharpoonup y \quad \text{weakly star in } L^\infty(0,T;V),\\
&(y^n)' \rightharpoonup y \quad \text{weakly in } L^2(0,T;V),\\
&f(y^n) \rightharpoonup f(y) \quad \text{weakly in } L^{p'}(0,T;L^{p'}(\Omega)).
\end{aligned}$$

Applying the Aubin-Lions lemma (see [15]), we can assume that $y^n \to y$ strongly in $L^2(0,T;H)$. Hence $y^n(t) \to y(t)$ for a.e. $t \in (0,T)$.

Moreover, reasoning as in the proof of Theorem 2.1, we obtain

$$y^n(t_n) \rightharpoonup y(t_0) \text{ weakly in } V \text{ if } t_n \to t_0 \in [0,T].$$

Also, by (H2) we have

$$\int_0^t |g(y^n_s)|^2 ds \le Ct, \ \forall\, 0 \le t \le T,$$

where $C > 0$ does not depend either on n or t. Since $g(y^n_\cdot) \rightharpoonup \xi$ in $L^2(0,T;V)$, we get

$$\int_s^t |\xi|^2 dr \le \liminf_{n\to+\infty} \int_s^t |g(y^n_r)|^2 dr \le C(t-s), \ \forall\, 0 \le s \le t \le T.$$

Thus, we can pass to the limits and prove that y is a solution of a similar problem to (1.1), that is

$$\frac{d}{dt}(y(t),v) + \frac{d}{dt}((y(t),v)) + ((y(t),v)) + \int_\Omega \langle f(y(t)), v\rangle dx = (\xi, v) + \langle h, v\rangle,$$

for all $v \in L^\infty(0,T;V) \cap L^p(0,T;L^p(\Omega))$. Since

$$\int_s^t \int_\Omega g(z_r) z_r dx dr \le \frac{1}{2\lambda_1}\int_s^t |g(z_r)|^2 dr + \frac{\lambda_1}{2}\int_s^t |z(r)|^2 dr,$$

we obtain the energy inequality

$$\begin{aligned}
&|z(t)|^2 + \|z(t)\|^2 + \int_s^t \|z(r)\|^2 dr + 2\int_0^t \langle f(z(r)), z(r)\rangle dr\\
&= |z(s)|^2 + \|z(s)\|^2 + 2\int_s^t \langle h, z(r)\rangle dr + 2C(t-s), \ \forall\, 0 \le s \le t \le T,
\end{aligned}$$

where $z = y^n$ or $z = y$.

Now, consider two functions $J_n, J : [0,T] \to \mathbb{R}$ defined by

$$\begin{aligned}
J_n(t) &= \frac{1}{2}(|y^n(t)|^2 + \|y^n(t)\|^2) + \int_0^t \langle f(y^n(r)), y^n(r)\rangle dr\\
&\quad - \int_0^t \langle h, y^n(r)\rangle dr - Ct,\\
J(t) &= \frac{1}{2}(|y(t)|^2 + \|y(t)\|^2) + \int_0^t \langle f(y(r)), y(r)\rangle dr - \int_0^t \langle h, y(r)\rangle dr - Ct.
\end{aligned}$$

It is clear that J_n and J are non-increasing and continuous functions.

Since $y^n(t)$ converges to $y(t)$ for a.e. $t \in (0,T)$, we obtain that

$$J_n(t) \to J(t) \text{ for a.e. } t \in [0,T].$$

Analogously as we did in Step 4 in the proof of Theorem 2.1, for a fixed $t_0 > 0$, using a sequence $\{\tilde{t}_k\}$ with $\tilde{t}_k \nearrow t_0$, we are able to establish the convergence of the norms

$$\lim_{n\to\infty} \|y^n(t_n)\| = \|y(t_0)\|.$$

And therefore, jointly with the weak convergence already proved, we deduce that $y^n \to y$ in $C([\delta,T];V)$, for any $\delta \geq 0$.

Now, as we had $T > 0$, and $y^n \to y$ in $C([0,T];V)$, we obtain that $\xi^n \to \psi$ in $C([-T,0];V)$, where $\psi(s) = y(s+T)$, for $s \in [-T,0]$. Repeating the same procedure for $2\overline{T}, 3\overline{T}$, etc., for a diagonal subsequence (relabeled the same) we can obtain a continuous function $\psi : (-\infty,0] \to V$ and a subsequence such that $\xi^n \to \psi$ in $C([-T,0];V)$ on every interval $[-T,0]$.

Moreover, for a fixed $T > 0$, we also have

$$\|\psi(s)\| \leq R, \ \forall\, s \in [-T,0], \ \forall\, T > 0.$$

Step 2: We claim that ξ_n converges to ψ in $C_\gamma(V)$. Indeed, we have to prove that for every $\epsilon > 0$ there exists n_ϵ such that

$$\sup_{s\in(-\infty,0]} \|\xi^n(s) - \psi(s)\|^2 e^{2\gamma s} \leq \epsilon, \quad \forall\, n \geq n_\epsilon. \tag{3.8}$$

Fix $T_\epsilon > 0$ such that $e^{-2\gamma T_\epsilon} R^2 \leq \frac{\epsilon}{4}$.

In Step 1, we proved that $\xi^n \to \psi$ in $C([-T_\epsilon,0];V)$, so there exists $n_\epsilon = n_\epsilon(T_\epsilon)$ such that for all $n \geq n_\epsilon$, we have

$$\sup_{s\in[-T_\epsilon,0]} \|\xi^n(s) - \psi(s)\|^2 e^{2\gamma s} \leq \epsilon, \ \forall\, t_n \geq T_\epsilon.$$

(This is possible since the convergence of ξ^n to ψ holds in compact intervals of time.) So, in order to prove (3.8) we only have to check that

$$\sup_{s\in(-\infty,T_\epsilon)} \|\xi^n(s) - \psi(s)\|^2 e^{2\gamma s} \leq \epsilon, \ \forall\, n \geq n_\epsilon.$$

Because of (3.6) and the choice of T_ϵ, we can check that for all $k \in \mathbb{N}\cup\{0\}$ and $s \in [-(T_\epsilon+k+1), -(T_\epsilon+k)]$, it holds that

$$\begin{aligned}
\sup_{s\in[-(T_\epsilon+k+1),-(T_\epsilon+k)]} e^{2\gamma s}\|\psi(s)\|^2 &\leq \sup_{s\in[-1,0]} e^{2\gamma(s-T_\epsilon-k)}\|\psi(s-T_\epsilon-k)\|^2 \\
&\leq e^{-2\gamma(T_\epsilon+k)} R^2 \\
&\leq \frac{\epsilon}{4}.
\end{aligned}$$

Thus, it suffices to prove the following

$$\sup_{s\in(-\infty,-T_\epsilon]} e^{2\gamma s}\|\xi^n(s)\|^2 \leq \epsilon/4, \quad \forall\, n \geq n_\epsilon.$$

We have

$$\xi^n(s) = \begin{cases} \phi^n(s+t_n), & \text{if } s \in (-\infty, -t_n], \\ u^n(s+t_n), & \text{if } s \in [-t_n, 0]. \end{cases}$$

So, the proof is finished if we prove that

$$\max\left\{ \sup_{s\in(-\infty,-t_n]} e^{2\gamma s}\|\phi^n(s+t_n)\|^2, \sup_{s\in[-t_n,-T_\epsilon]} e^{2\gamma s}\|u^n(s+t_n)\|^2 \right\} \le \epsilon/4.$$

The first term above can be estimated as follows

$$\begin{aligned} \sup_{s\le -t_n} e^{2\gamma s}\|\phi^n(s+t_n)\|^2 &= \sup_{s\le -t_n} e^{2\gamma(s+t_n)}e^{-2\gamma t_n}\|\phi^n(s+t_n)\|^2 \\ &= e^{-2\gamma t_n}\|\phi^n\|_\gamma^2 \\ &\le \epsilon/4, \end{aligned}$$

thanks to the choice of n_ϵ. And finally, for the second term, we have

$$\begin{aligned} \sup_{s\in[-t_n,-T_\epsilon]} e^{2\gamma s}\|u^n(s+t_n)\|^2 &= \sup_{s\in[-t_n+T_\epsilon,0]} e^{2\gamma(s-T_\epsilon)}\|u^n(t_n-T_\epsilon+s)\|^2 \\ &\le e^{-2\gamma T_\epsilon}||u^n_{t_n-T_\epsilon}||_\gamma^2 \\ &\le e^{-2\gamma T_\epsilon}R^2 \\ &\le \epsilon/4, \end{aligned}$$

where we have used (3.7) with $T = T_\epsilon$. □

From Lemmas 3.2 and 3.3, by the classical abstract results on existence of global attractors (see e.g. Theorem 1.1 in [23]), we get the main result of this section.

Theorem 3.4. *Under the assumptions of Lemma 3.2, the semigroup $S(t)$ has a compact global attractor in the space $C_\gamma(V)$.*

4. Existence and stability of stationary solutions

A stationary solution to problem (1.1) is an element $u^* \in V \cap L^p(\Omega)$ such that

$$((u^*, v)) + \langle f(u^*), v\rangle = (g(u^*), v) + \langle h, v\rangle, \tag{4.1}$$

for all test functions $v \in V \cap L^p(\Omega)$.

Theorem 4.1. *Suppose that* **(H1)-(H4)** *hold. Then*

(a) *There exists at least one stationary solution to* (1.1);

(b) *If the following condition holds*

$$\lambda_1 - \ell - L_g\sqrt{\lambda_1} > 0, \tag{4.2}$$

then the stationary solution of (1.1) *is unique.*

Proof. Let $\{e_j : j \geq 1\}$ be a basis of $V \cap L^p(\Omega)$. For each integer $n \geq 1$, let us denote $V_n = \operatorname{span}\{e_1, \dots, e_n\}$ and we would like to define an approximate solution u^n of (1.1) by

$$u^n = \sum_{j=1}^{n} \gamma_{nj} e_j, \tag{4.3}$$

where

$$((u^n, e_j)) + \langle f(u^n), e_j \rangle = (g(u^n), e_j) + \langle h, e_j \rangle, \quad j = 1, \dots, n. \tag{4.4}$$

To prove the existence of u^n, we define operators $R_n : V_n \to V_n$ by

$$((R_n u, v)) := ((u, v)) + \langle f(u), v \rangle - (g(u), v) - \langle h, v \rangle, \quad \forall\, u, v \in V_n.$$

For all $u \in V_n$, we have

$$\begin{aligned} ((R_n u, u)) &\geq \|u\|^2 + C_1 \|u\|^p_{L^p(\Omega)} - C_0|\Omega| - \|h\|_* \|u\| - L_g \|u\| |u| \\ &\geq \frac{1}{2}\|u\|^2 + C_1 \|u\|^p_{L^p(\Omega)} - C_0|\Omega| - \|h\|_*^2 - L_g^2 |u|^2. \end{aligned}$$

Using Young's inequality, we get

$$L_g^2 |u|^2 \leq \frac{C_1}{2} \|u\|^p_{L^p(\Omega)} + \frac{p-2}{p} \Big(\frac{C_1 p}{4}\Big)^{\frac{-2}{p-2}} L_g^{\frac{2p}{p-2}} |\Omega|.$$

So, we have

$$\begin{aligned} ((R_n u, u)) \geq \frac{1}{2}\|u\|^2 &+ \frac{C_1}{2} \|u\|^p_{L^p(\Omega)} - C_0|\Omega| - \|h\|_*^2 \\ &- \frac{p-2}{p} \Big(\frac{C_1 p}{4}\Big)^{\frac{-2}{p-2}} L_g^{\frac{2p}{p-2}} |\Omega|. \end{aligned} \tag{4.5}$$

Thus, if we take

$$\beta > 2C_0|\Omega| + 2\|h\|_*^2 + \frac{2p-4}{p} \Big(\frac{C_1 p}{4}\Big)^{\frac{-2}{p-2}} L_g^{\frac{2p}{p-2}} |\Omega|,$$

we obtain $((R_n u, u)) \geq 0\ \forall u \in V_n$ with $\|u\|^2 = \beta$. Consequently, by a corollary of the Brouwer fixed point (see e.g. [22, Chapter 2, Lemma 1.4]), for each $n \geq 1$, there exists $u^n \in V_n$ such that $R_n(u_n) = 0$, with $\|u_n\| \leq \beta$.

Multiplying (4.4) by γ_{nj} and adding corresponding equalities for $j = 1, \dots, n$, we have

$$\begin{aligned} \|u^n\|^2 &= (g(u^n), u^n) + \langle h, u^n \rangle - \langle f(u^n), u^n \rangle \\ &\leq \frac{p-2}{p} \Big(\frac{C_1 p}{4}\Big)^{\frac{-2}{p-2}} L_g^{\frac{2p}{p-2}} |\Omega| + C_0|\Omega| + \|h\|_*^2 + \frac{1}{2}\|u^n\|^2 - \frac{C_1}{2} \|u^n\|^p_{L^p(\Omega)}. \end{aligned}$$

Hence, we obtain *a priori* estimate

$$\|u^n\|^2 + C_1 \|u^n\|^p_{L^p(\Omega)} \leq 2C_0|\Omega| + 2\|h\|_*^2 + \frac{2p-4}{p} \Big(\frac{C_1 p}{4}\Big)^{\frac{-2}{p-2}} L_g^{\frac{2p}{p-2}} |\Omega|.$$

Since the sequence $\{u^n\}$ is bounded in $V \cap L^p(\Omega)$, there exist some u^* in $V \cap L^p(\Omega)$ and a subsequence (still denoted the same) such that

$$u^n \rightharpoonup u^* \text{ weakly in } V \cap L^p(\Omega).$$

On the other hand, using (1.2) we get

$$\{f(u^n)\} \text{ is bounded in } L^{p'}(\Omega), \tag{4.6}$$

where p' is the conjugate of p. Thus,

$$f(u^n) \rightharpoonup \xi \text{ weakly in } L^{p'}(\Omega).$$

Using the compactness of the embedding $V \hookrightarrow H$, we can assume that $u^n \to u^*$ strongly in H. Hence $u^n \to u^*$ a.e. in Ω. Since f is continuous, it follows that $f(u^n) \to f(u^*)$ a.e. in Ω. Thanks to (4.6) and Lemma 1.3 in [15, Chapter 1], one has

$$f(u^n) \rightharpoonup f(u^*) \text{ weakly in } L^{p'}(\Omega).$$

Thus, $\xi = f(u^*)$. On the other hand, using (g2) we can check that

$$g(u^n) \rightharpoonup g(u^*) \text{ weakly in } H.$$

Thus, we conclude that u^* is a stationary solution of (1.1).

Finally, we verify the uniqueness. Let u^* and v^* be two stationary solutions to (1.1). Denote $w = u^* - v^*$, we have

$$Aw + f(u^*) - f(v^*) = g(u^*) - g(v^*) \text{ in } (V \cap L^p(\Omega))' = V' + L^{p'}(\Omega).$$

Hence, choosing the test function $v = u^* - v^*$, we get

$$\|u^* - v^*\|^2 + \langle f(u^*) - f(v^*), u^* - v^* \rangle = (g(u^*) - g(v^*), u^* - v^*).$$

Using (g2) and (1.3), we get

$$\|u^* - v^*\|^2 \leq \ell \lambda_1^{-1} \|u^* - v^*\|^2 + L_g \lambda_1^{-1/2} \|u^* - v^*\|^2.$$

Thus,

$$(1 - \ell \lambda_1^{-1} - L_g \lambda_1^{-1/2}) \|u^* - v^*\|^2 \leq 0,$$

and since $\lambda_1 - \ell - L_g \sqrt{\lambda_1} > 0$, this completes the proof. □

Theorem 4.2. *Assume* **(H1)-(H4)** *and* (4.2) *are satisfied. Then there exists a value* $0 < \lambda < 2\gamma$ *such that for the solution* $u(\cdot)$ *of* (1.1) *with* $\phi \in C_\gamma(V)$*, and* $w(t) = u(t) - u^*$*, with* u^* *is the unique stationary solution given by Theorem 4.1, the following estimates hold for all* $t \geq 0$:

(4.7)

$$\|w(t)\|^2 \leq e^{-\lambda t} \left(|w(0)|^2 + \|w(0)\|^2 + \frac{2L_g^2}{\lambda_1(2\gamma - \lambda)} \|\phi - u^*\|_\gamma^2 \right),$$

(4.8)

$$\|w_t\|_\gamma^2 \leq \max \left\{ e^{-2\gamma t} \|\phi - u^*\|_\gamma^2, e^{-\lambda t} \left(|w(0)|^2 + \|w(0)\|^2 + \frac{2L_g^2}{\lambda_1(2\gamma - \lambda)} \|\phi - u^*\|_\gamma^2 \right) \right\}.$$

Proof. For $w(t) = u(t) - u^*$, we have

$$\begin{aligned}&(\partial_t w(t), w(t)) + (A\partial_t w(t), w(t)) + (Aw(t), w(t)) + \langle f(u(t)) - f(u^*), w(t)\rangle \\ &= (g(u_t) - g(u^*), w(t)).\end{aligned}$$

Using (1.3), $(H2)$, and introducing an exponential term $e^{\lambda t}$ with a positive value λ to be fixed, we obtain

$$\begin{aligned}&\frac{d}{dt}\bigg(e^{\lambda t}(|w(t)|^2 + \|w(t)\|^2)\bigg) + 2e^{\lambda t}\|w(t)\|^2 \\ &\le e^{\lambda t}\bigg(\lambda(|w(t)|^2 + \|w(t)\|^2) + 2\ell|w(t)|^2 + 2L_g\|w_t\|_\gamma |w(t)|\bigg).\end{aligned}$$

Hence, using Young's inequality with $\delta > 0$ to be fixed later, we conclude that

$$\begin{aligned}&\frac{d}{dt}\bigg(e^{\lambda t}(|w(t)|^2 + \|w(t)\|^2)\bigg) \\ &\le e^{\lambda t}\bigg(\lambda(\lambda_1^{-1} + 1) + 2\ell\lambda_1^{-1} + \delta L_g\lambda_1^{-1} - 2\bigg)\|w(t)\|^2 + \frac{L_g}{\delta}e^{\lambda t}\|w_t\|_\gamma^2.\end{aligned}$$

Therefore, integrating from 0 to t, we have

$$\begin{aligned}&e^{\lambda t}(|w(t)|^2 + \|w(t)\|^2) \\ &\le |w(0)|^2 + \|w(0)\|^2 + \frac{L_g}{\delta}\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds \\ &\quad + \bigg(\lambda(\lambda_1^{-1} + 1) + 2\ell\lambda_1^{-1} + \delta L_g\lambda_1^{-1} - 2\bigg)\int_0^t e^{\lambda s}\|w(s)\|^2 ds.\end{aligned} \tag{4.9}$$

In order to control the term $\int_0^t e^{\lambda s}\|w_s\|_\gamma^2 ds$, we proceed as follows

$$\begin{aligned}&\int_0^t e^{\lambda s}\sup_{\theta\le 0} e^{2\gamma\theta}\|w(s+\theta)\|^2 ds \\ &= \int_0^t e^{\lambda s}\max\bigg\{\sup_{\theta\le -s} e^{2\gamma\theta}\|w(s+\theta)\|^2, \sup_{\theta\in[-s,0]} e^{2\gamma\theta}\|w(s+\theta)\|^2\bigg\} ds \\ &= \int_0^t \max\bigg\{e^{-(2\gamma-\lambda)s}\|\phi - u^*\|_\gamma^2, \sup_{\theta\in[-s,0]} e^{(2\gamma-\lambda)\theta}e^{\lambda(s+\theta)}\|w(s+\theta)\|^2\bigg\} ds.\end{aligned}$$

So, if $\lambda < 2\gamma$, using the above equality in (4.9), we obtain

$$\begin{aligned}&e^{\lambda t}(|w(t)|^2 + \|w(t)\|^2) \\ &\le |w(0)|^2 + \|w(0)\|^2 + \frac{L_g}{\delta}\|\phi - u^*\|_\gamma^2 \int_0^t e^{(\lambda-2\gamma)s} ds \\ &\quad + \bigg(\lambda(\lambda_1^{-1} + 1) + 2\ell\lambda_1^{-1} + \delta L_g\lambda_1^{-1} - 2 + \frac{L_g}{\delta}\bigg)\int_0^t \max_{r\in[0,s]}(e^{\lambda r}\|w(r)\|^2) ds.\end{aligned}$$

Observe that the (optimal) choice of $\delta = \sqrt{\lambda_1}$ makes that $\delta L_g \lambda_1^{-1} - 2 + \frac{L_g}{\delta}$ is minimal and the coefficient of the last integral becomes

$$\lambda(\lambda_1^{-1} + 1) + 2\ell\lambda_1^{-1} + 2L_g\lambda_1^{-1/2} - 2. \tag{4.10}$$

Using the hypothesis $\lambda_1 - \ell - L_g\sqrt{\lambda_1} > 0$, we can choose $\lambda \in (0, 2\gamma)$ such that (4.10) is negative. So, we can deduce that

$$e^{\lambda t}(|w(t)|^2+\|w(t)\|^2) \leq (|w(0)|^2+\|w(0)\|^2)+\frac{L_g}{\sqrt{\lambda_1}(2\gamma-\lambda)}(1-e^{(\lambda-2\gamma)t})\|\phi-u^*\|_\gamma^2,$$

whence (4.7) follows.

Finally, (4.8) can be deduced as follows

$$\begin{aligned}\|w_t\|_\gamma^2 &\leq \sup_{\theta\leq 0} e^{2\gamma\theta}\|w(t+\theta)\|^2 \\ &= \max\left\{ \sup_{\theta\in(-\infty,-t]} e^{2\gamma\theta}\|\phi(t+\theta) - u^*\|^2, \max_{\theta\in[-t,0]} e^{2\gamma\theta}\|w(t+\theta)\|^2 \right\} \\ &= \max\left\{ e^{-2\gamma t}\|\phi - u^*\|^2, \max_{\theta\in[-t,0]} e^{2\gamma\theta}\|w(t+\theta)\|^2 \right\}\end{aligned}$$

and the second term can be estimated using (4.8) and the fact that $e^{(2\gamma-\lambda)\theta} \leq 1$ when $\theta \leq 0$. $\square$

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Dang Thi Phuong Thanh
Department of Mathematics
Hung Vuong University
Nong Trang, Viet Tri, Phu Tho, Vietnam
E-mail address: thanhdp83@gmail.com