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EXTENDED CESÀRO OPERATORS BETWEEN α-BLOCH SPACES AND Q_K SPACES

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Abstract. Many scholars studied the boundedness of Cesàro operators between Q_K spaces and Bloch spaces of holomorphic functions in the unit disc in the complex plane, however, they did not describe the compactness. Let $0 < \alpha < +\infty$, $K(r)$ be right continuous nondecreasing functions on $(0, +\infty)$ and satisfy

$$\int_0^{\frac{1}{e}} K(\log \frac{1}{r}) r dr < +\infty.$$

Suppose g is a holomorphic function in the unit disk. In this paper, some sufficient and necessary conditions for the extended Cesàro operators T_g between α-Bloch spaces and Q_K spaces in the unit disc to be bounded and compact are obtained.

1. Introduction and motivation

Let $\mathbb{D}$ be the open unit disk in the complex plane and $H(\mathbb{D})$ the class of all analytic functions on $\mathbb{D}$. The α-Bloch space $\mathcal{B}^\alpha$ on $\mathbb{D}$ is the space of all analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{\mathcal{B}^\alpha} = |f(0)| + \sup_{z\in\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)| < \infty.$$

Under the above norm, $\mathcal{B}^\alpha$ is a Banach space. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which $(1 - |z|^2)^\alpha |f'(z)| \to 0$ as $|z| \to 1$. This space is called the little Bloch space.

Let $dA(z)$ be the Euclidean area element on $\mathbb{D}$. Throughout this paper, we assume that $K : [0, \infty) \to [0, \infty)$ is a nondecreasing and right-continuous function. A function $f \in H(\mathbb{D})$ is said to belong to Q_K space (see [2]) if

$$\|f\|_K^2 = \sup_{a\in\mathbb{D}} \iint_{\mathbb{D}} |f'(z)| K(g(z, a)) dA(z) < \infty,$$

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where $g(z,a)$ is the Green function with logarithmic singularity at a, that is, $g(z,a) = \log(1/|\varphi_a(z)|)$ (φ_a is a conformal automorphism defined by $\varphi_a(z) = (a-z)/(1-\overline{a}z)$ for $a \in \mathbb{D}$). Q_K is a Banach space under the norm

$$\|f\|_{Q_K} = |f(0)| + \|f\|_K.$$

From [2], we know that $Q_K \subseteq \mathcal{B}$ if

$$\int_0^{\frac{1}{e}} K(-\log r) r dr < \infty. \tag{1.1}$$

Suppose $g \in H(\mathbb{D})$. We define the extended Cesàro operator as

$$(T_g f)(z) = \int_0^z f(t) g'(t) dt,$$

where $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$.

Recall that a linear operator $T : X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that $\|T(f)\|_Y \leq M\|f\|_X$ for all maps $f \in X$. And $X \to Y$ is compact if it takes bounded sets in X to sets in Y which have compact closure. For Banach spaces X and Y of $H(\mathbb{D})$, T is compact from X to Y if and only if for each bounded sequence $\{x_n\}$ in X, the sequence $\{Tx_n\} \in Y$ contains a subsequence converging to some limit in Y.

Many scholars studied the boundedness of Cesàro operators between Q_K spaces and Bloch spaces of holomorphic functions in the unit disc in the complex plane. In [8], Li and Wulan characterized the boundedness of Cesàro operators on Q_K. Xiao [18] characterized the boundedness of Cesàro operators on $\mathcal{B}^\alpha$. In this paper, we are motivated by boundedness and compactness of the composition operators between Q_p and $\mathcal{B}^\alpha$ established in Theorem 2.2.1 of [19] to get some criteria for boundedness and compactness of T_g acting between Q_K and $\mathcal{B}^\alpha$. Throughout this paper, let C denote a constant which can denote different values at different places and $\mathbb{B}_X$ denote the unit ball of the given Banach space $(X, \mathbb{B}_X)$.

2. The boundedness

Lemma 2.1 ([10])**.** *Let $\alpha > 0$, for $f \in \mathcal{B}^\alpha$, then*

1) $\|f_t\|_\alpha \leq \|f\|_{\mathcal{B}^\alpha} (0 < t < 1)$, *where* $f_t(z) = f(tz)$;
2) $|f(z)| \leq \frac{2-\alpha}{1-\alpha}\|f\|_{\mathcal{B}^\alpha}$, *where* $0 < \alpha < 1$;
3) $|f(z)| \leq (\frac{1}{\log 2} \log \frac{2}{1-|z|^2})\|f\|_{\mathcal{B}^\alpha}$, *where* $\alpha = 1$;
4) $|f(z)| \leq (1 + \frac{2^{\alpha-1}}{\alpha-1})\frac{1}{(1-|z|^2)^{\alpha-1}}\|f\|_{\mathcal{B}^\alpha}$, *where* $\alpha > 1$;
5) $|f''(z)| \leq \frac{C}{(1-|z|^2)^{\alpha+1}}\|f\|_{\mathcal{B}^\alpha}$, *where C is a constant.*

Lemma 2.2 ([7])**.** *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing and right-continuous function. Then for every $f \in Q_K$,*

$$|f(z)| \leq \log \frac{1}{1-|z|}\|f\|_{Q_K}.$$

Lemma 2.3 ([7]). *Let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing and right-continuous function. If*

$$\int_0^{\frac{1}{e}} K(-\log r)\frac{r}{1-r^2}dr < \infty. \tag{2.1}$$

holds, we have $\log(1-z) \in Q_K$.

Theorem 2.4. *Let $g \in H(\mathbb{D})$ and $0 < \alpha < 1$, then the following statements are equivalent.*

1) $T_g : \mathcal{B}^\alpha \to Q_K$ *is bounded*;
2) $T_g : \mathcal{B}_0^\alpha \to Q_K$ *is bounded*;
3) $g \in Q_K$.

Proof. 1) $\Rightarrow$ 2) It is obvious.

2) $\Rightarrow$ 3) Assume that $T_g : \mathcal{B}_0^\alpha \to Q_K$ is bounded, where $0 < \alpha < 1$ and $g \in H(\mathbb{D})$. Then for $\forall f \in \mathcal{B}_0^\alpha$, we have

$$\|T_g(f)\|_{Q_K} \le C\|f\|_{\mathcal{B}_0^\alpha}.$$

Taking the function $f(z) = 1 \in \mathcal{B}_0^\alpha$, then

$$\begin{aligned}\infty > \|T_g\|_{\mathcal{B}_0^\alpha \to Q_K} \ge \|T_g(f)\|_{Q_K} &\ge \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |(T_g f)'(z)|^2 K(\mathrm{g}(z,a))dA(z)\\ &= \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^2 K(\mathrm{g}(z,a))dA(z).\end{aligned}$$

Hence $g \in Q_K$.

3) $\Rightarrow$ 1) Suppose $g \in Q_K$. By Lemma 2.1, for $\forall f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |(T_g f)'(z)|^2 K(\mathrm{g}(z,a))dA(z) &\le \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2|g'(z)|^2 K(\mathrm{g}(z,a))dA(z)\\ &\le C\|f\|^2_{\mathcal{B}^\alpha}\|g\|_{Q_K} < \infty.\end{aligned}$$

It follows that $T_g : \mathcal{B}^\alpha \to Q_K$ is bounded. □

Theorem 2.5. *Let $g \in H(\mathbb{D})$ and $\alpha = 1$, then the following statements are equivalent.*

1) $T_g : \mathcal{B} \to Q_K$ *is bounded*;
2) $T_g : \mathcal{B}_0 \to Q_K$ *is bounded*;
3) $M = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\log\frac{2}{1-|z|^2})^2|g'(z)|^2 K(\mathrm{g}(z,a))dA(z) < \infty$.

Proof. 1) $\Rightarrow$ 2) It is obvious.

2) $\Rightarrow$ 3) Assume that $T_g : \mathcal{B}_0 \to Q_K$ is bounded, where $\alpha = 1$ and $g \in H(\mathbb{D})$. Then for $\forall f \in \mathcal{B}_0$, we have

$$\|T_g(f)\|_{Q_K} \le C\|f\|_{\mathcal{B}_0^\alpha}.$$

Taking the function $f(z) = \log\frac{2}{1-e^{-i\theta}z}$, where $\theta \in [0,2\pi)$, then $f \in \mathcal{B}_0$ and

$$\infty > \|T_g\|_{\mathcal{B}_0 \to Q_K} \ge \|T_g(f)\|_{Q_K} \ge \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |(T_g f)'(z)|^2 K(\mathrm{g}(z,a))dA(z)$$

$$= \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |g'(z) \log \frac{2}{1-e^{-i\theta}z}|^2 K(\mathrm{g}(z,a)) dA(z).$$

Let $z = re^{i\theta}(r = |z|, \theta \in [0, 2\pi))$, then we get

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |g'(z) \log \frac{2}{1-|z|}|^2 K(\mathrm{g}(z,a)) dA(z) \le \|T_g f\|_{Q_K} < \infty.$$

It follows

$$M = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\log \frac{2}{1-|z|^2})^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \infty.$$

3) $\Rightarrow$ 1) Suppose 3) holds. By Lemma 2.1, for $\forall f \in \mathcal{B}$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |(T_g f)'(z)|^2 K(\mathrm{g}(z,a)) dA(z) \\
\le\ &\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |f(z)|^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) \\
\le\ &C\|f\|_{\mathcal{B}}^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (\log \frac{2}{1-|z|^2})^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \infty.
\end{aligned}$$

It follows that $T_g : \mathcal{B} \to Q_K$ is bounded. □

Theorem 2.6. *Let $g \in H(\mathbb{D})$ and $\alpha > 1$, then the following statements are equivalent.*

1) $T_g : \mathcal{B}^\alpha \to Q_K$ *is bounded*;
2) $T_g : \mathcal{B}_0^\alpha \to Q_K$ *is bounded*;
3) $M = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1-|z|^2)^{2-2\alpha} |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \infty$.

Proof. 1) $\Rightarrow$ 2) It is obvious.

2) $\Rightarrow$ 3) Assume that $T_g : \mathcal{B}_0^\alpha \to Q_K$ is bounded, where $\alpha > 1$ and $g \in H(\mathbb{D})$. Then for $\forall f \in \mathcal{B}_0^\alpha$, we have

$$\|T_g(f)\|_{Q_K} \le C\|f\|_{\mathcal{B}_0^\alpha}.$$

Taking the function $f(z) = (1-e^{-i\theta}z)^{1-\alpha}$, where $\theta \in [0, 2\pi)$, then $f \in \mathcal{B}_0^\alpha$ and

$$\begin{aligned}
\infty > \|T_g\|_{\mathcal{B}_0^\alpha \to Q_K} \ge \|T_g(f)\|_{Q_K} &\ge \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |(T_g f)'(z)|^2 K(\mathrm{g}(z,a)) dA(z) \\
&= \sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)(1-e^{-i\theta}z)^{1-\alpha}|^2 K(\mathrm{g}(z,a)) dA(z).
\end{aligned}$$

Let $z = re^{i\theta}(r = |z|, \theta \in [0, 2\pi))$, then we get

$$\sup_{z\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)(1-|z|)^{1-\alpha}|^2 K(\mathrm{g}(z,a)) dA(z) \le \|T_g f\|_{Q_K} < \infty.$$

It follows

$$M = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} (1-|z|^2)^{2-2\alpha} |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \infty.$$

3) $\Rightarrow$ 1) Suppose 3) holds. By Lemma 2.1, for $\forall f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned}
&\sup_{z\in\mathbb{D}}\int_{\mathbb{D}} |(T_g f)'(z)|^2 K(\mathrm{g}(z,a))dA(z)\\
\le\ &\sup_{z\in\mathbb{D}}\int_{\mathbb{D}} |f(z)|^2|g'(z)|^2 K(\mathrm{g}(z,a))dA(z)\\
\le\ &C\|f\|^2_{\mathcal{B}^\alpha}\sup_{a\in\mathbb{D}}\int_{\mathbb{D}}(1-|z|^2)^{2-2\alpha}|g'(z)|^2K(\mathrm{g}(z,a))dA(z)<\infty.
\end{aligned}$$

It follows that $T_g : \mathcal{B}^\alpha \to Q_K$ is bounded. $\square$

Theorem 2.7. *Let $g \in H(\mathbb{D})$ and $\alpha > 0$. Suppose K satisfies* (2.1)*, then $T_g : Q_K \to \mathcal{B}^\alpha$ is bounded if and only if*

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha \log\frac{1}{1-|z|}|g'(z)|<\infty. \tag{2.2}$$

Proof. Suppose $T_g : Q_K \to \mathcal{B}^\alpha$ is bounded. Then for all $f \in Q_K$ and $\alpha > 0$, we have

$$\|T_g(f)\|_{\mathcal{B}^\alpha} \le C\|f\|_{Q_K}.$$

Taking test function $f(z) = \log\frac{1}{1-e^{-i\theta}z}$, where $\theta \in [0, 2\pi)$. By Lemma 2.3 we have $f \in Q_K$. Then

$$\begin{aligned}
\infty > \|T_g\|_{Q_K\to\mathcal{B}^\alpha} &\ge \sup_{z\in\mathbb{D}}\left|\log\frac{1}{1-e^{-i\theta}z}g'(z)\right|(1-|z|^2)^\alpha\\
&= \sup_{z\in\mathbb{D}}|(T_g f)'(z)|(1-|z|^2)^\alpha.
\end{aligned}$$

Let $z = re^{i\theta}(r = |z|, \theta \in [0, 2\pi))$, then we get

$$\sup_{z\in\mathbb{D}}\left|\log\frac{1}{1-|z|}g'(z)\right|(1-|z|^2)^\alpha \le \|T_g f\|_{\mathcal{B}^\alpha}<\infty.$$

It follows that (2.2) holds

On the other hand, assume that (2.2) holds. By Lemma 2.2, for $f \in Q_K$, we have

$$\begin{aligned}
\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|(T_g f)'(z)| &\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)||g'(z)|\\
&\le \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha\log\frac{1}{1-|z|}|g'(z)|\|f\|_{Q_K}<\infty.
\end{aligned}$$

It follows that $T_g : Q_K \to \mathcal{B}^\alpha$ is bounded. $\square$

3. The compactness

Lemma 3.1. *Let $\alpha > 0$ and $g \in H(\mathbb{D})$, then $T_g : \mathcal{B}^\alpha(\mathcal{B}^\alpha_0) \to Q_K$(or $T_g : Q_K \to \mathcal{B}^\alpha$) is compact if and only if $T_g : \mathcal{B}^\alpha(\mathcal{B}^\alpha_0) \to Q_K$(or $T_g : Q_K \to \mathcal{B}^\alpha$) is bounded and for every bounded sequence $\{f_n\}_{n\in N}$ in $\mathcal{B}^\alpha(\mathcal{B}^\alpha_0$or $Q_K)$ which converges to 0 uniformly on compact subsets of $\mathbb{D}$, $\|T_g(f_n)\|_{Q_K}$(or $\|T_g(f_n)\|_{\mathcal{B}^\alpha}) \to 0(n \to \infty)$.*

Proof. Using Montel theorem and the definition of compact operator can prove this theorem. □

Theorem 3.2. *Suppose K satisfies* (1.1) *and $0 < \alpha < +\infty$, then $T_g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to Q_K$ is compact if and only if $T_g : \mathcal{B}^\alpha(\mathcal{B}_0^\alpha) \to Q_K$ is bounded and*

$$\lim_{t\to 1} \sup_{a\in\mathbb{D}, f\in\mathbb{B}_{\mathcal{B}^\alpha}(f\in\mathbb{B}_{\mathcal{B}_0^\alpha})} \int_{|z|>t} |f(z)|^2|g'(z)|^2K(\mathrm{g}(z,a))dA(z) = 0. \tag{3.1}$$

Proof. We only consider the case $T_g : \mathcal{B}^\alpha \to Q_K$ is compact and the case of $\mathcal{B}_0^\alpha$ can be proved similar. Suppose (3.1) holds. Without loss of generality, we choose a sequence $\{f_n\}_{n=1}^\infty \subset \mathbb{B}_{\mathcal{B}^\alpha}$, which converges to 0 uniformly on the compact subsets of $\mathbb{D}$, then by the definition of Cesàro operator,

$$\|T_g(f_n)\|_K^2 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z).$$

By (3.1), for all $\epsilon > 0$, there exists $t \in (0,1)$ such that for all $f_n \in \mathbb{B}_{\mathcal{B}^\alpha}$ and for all $a \in \mathbb{D}$,

$$\sup_{a\in\mathbb{D}} \int_{|z|>t} |f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z) < \epsilon. \tag{3.2}$$

Let $\mathbb{D}_t = \{z \in \mathbb{D} : |z| \le t\}$. So $\mathbb{D}_t$ is a compact subsets of $\mathbb{D}$ and f_n converges to 0 uniformly on $\mathbb{D}_t$. By $1 \in \mathcal{B}^\alpha$, we see that $g \in Q_K$. Then for given $\epsilon > 0$, there exists a N such that

$$\sup_{a\in\mathbb{D}} \int_{|z|\le t} |f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z) < \epsilon\|g\|_{Q_K}^2, \tag{3.3}$$

where $n > N$. By (3.2), (3.3) and Lemma 3.1, we see that $T_g : \mathcal{B}_\alpha \to Q_K$ is compact.

Suppose $T_g : \mathcal{B}^\alpha \to Q_K$ is compact. To verify (3.1) consider $\forall f \in \mathbb{B}_{\mathcal{B}^\alpha}$ and let $f_s(t) = f(st)$ for $\forall s \in (0,1)$ and $t \in \mathbb{D}$. Note that f_s converges to f uniformly on compact subsets of $\mathbb{D}$ as $s \to 1$. By [6] we know $\{f_s, 0 < s < 1\}$ is bounded in $\mathcal{B}^\alpha$. Since T_g is compact,

$$\|T_gf_s - T_gf\|_{Q_K} \to 0(s \to 1).$$

That is for given $\epsilon > 0$, there exists $s_0 \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_s(z) - f(z)|^2|g'(z)|^2K(\mathrm{g}(z,a))dA(z) < \epsilon,$$

where $s > s_0$. For $t \in (0,1)$ and the above s_0, the triangle inequality gives

$$\begin{aligned} &\sup_{a\in\mathbb{D}} \int_{|z|>t} |f(z)|^2|g'(z)|^2K(\mathrm{g}(z,a))dA(z) \\ &\le \|f_s\|_\infty^2 \sup_{a\in\mathbb{D}} \int_{|z|>t} |g'(z)|^2K(\mathrm{g}(z,a))dA(z) + \epsilon. \end{aligned} \tag{3.4}$$

Let $h_n(z) = 2z^n$. Then $h_n \in \mathcal{B}^\alpha$. Note that h_n converges to 0 uniformly on compact subsets of $\mathbb{D}$. Since T_g is compact,

$$\lim_{n\to\infty} \|T_g(h_n)\|_{Q_K} = 0.$$

That is for given $\epsilon > 0$, there exists a N such that for $\forall a \in \mathbb{D}$

$$4 \sup_{a\in\mathbb{D}} \int_{|z|>t} |z|^{2n} |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \epsilon,$$

where $n > N$. Further we imply

$$4t^{2N} \sup_{a\in\mathbb{D}} \int_{|z|>t} |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \epsilon.$$

Given $t = 4^{-\frac{1}{2N}}$, then

$$\|f_s\|_\infty^2 \sup_{a\in\mathbb{D}} \int_{|z|>t} |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \epsilon. \tag{3.5}$$

Hence by (3.4) and (3.5), we have already proved that for $\forall \epsilon > 0$ and $f \in \mathbb{B}_{\mathcal{B}^\alpha}$, there exists a $\delta = \delta(\epsilon, f) \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{|z|>t} |f(z)|^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \epsilon,$$

where $\delta < t < 1$. Next we finish our proof by showing that the above δ is independent of $f \in \mathbb{B}_{\mathcal{B}^\alpha}$.

Since T_g is compact, $T_g(\mathbb{B}_{\mathcal{B}^\alpha})$ is a relative compact subset of Q_K. It means that for all $\epsilon > 0$ and $f \in \mathbb{B}_{\mathcal{B}^\alpha}$, there exists a finite collection of functions $f_1, f_2, \ldots, f_n$ in $\mathbb{B}_{\mathcal{B}^\alpha}$ such that for all $\epsilon > 0$ and $f \in \mathbb{B}_{\mathcal{B}^\alpha}$, there is a $k, 1 \le k \le n$ satisfying

$$\sup_{a\in D} \int_{\mathbb{D}} |f(z) - f_k(z)|^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \epsilon. \tag{3.6}$$

Let $\delta_0 = \max_{1\le k\le n} \delta(\epsilon, f_k) < t < 1$, we have proved for all $k = 1, 2, \ldots, n$,

$$\sup_{a\in\mathbb{D}} \int_{|z|>t} |f_k(z)|^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < \epsilon. \tag{3.7}$$

The triangle inequality, together with (3.6) and (3.7), gives

$$\sup_{a\in\mathbb{D}} \int_{|z|>t} |f(z)|^2 |g'(z)|^2 K(\mathrm{g}(z,a)) dA(z) < 2\epsilon$$

where $\delta_0 < t < 1$. The proof is complete. □

Theorem 3.3. *Suppose $g \in H(\mathbb{D})$ and $0 < \alpha < 1$, then the following statements are equivalent.*

1) $T_g : \mathcal{B}^\alpha \to Q_K$ *is compact*;
2) $T_g : \mathcal{B}_0^\alpha \to Q_K$ *is compact*;

3)

$$\lim_{t\to 1}\sup_{a\in\mathbb{D}}\int_{|z|>t}|g'(z)|^2K(\mathrm{g}(z,a))dA(z)=0. \tag{3.8}$$

Proof. (1) $\Rightarrow$ (2): It is obvious by $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(2) $\Rightarrow$ (3): Assume $T_g : \mathcal{B}_0^\alpha \to Q_K$ is compact. Given $f(z) = 1 \in \mathcal{B}_0^\alpha$, by theorem 3.2, for all $a \in \mathbb{D}$,

$$\lim_{t\to 1}\sup_{a\in\mathbb{D}}\int_{|z|>t}|g'(z)|^2K(\mathrm{g}(z,a))dA(z)=0.$$

(3) $\Rightarrow$ (1): Suppose (3.8) holds. Without loss of generality, given a sequence $\{f_n\}_{n=1}^\infty$ in $\mathbb{B}_{\mathcal{B}^\alpha}$ and f_n converges to 0 on compact subsets of $\mathbb{D}$ as $n \to \infty$. By Lemma 2.1,

$$\begin{aligned}
\|T_gf_n\|_K^2 &= \sup_{a\in\mathbb{D}}\int_{\mathbb{D}}|f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z)\\
&= \sup_{a\in\mathbb{D}}\int_{|z|\le t}|f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z)\\
&\quad+\sup_{a\in\mathbb{D}}\int_{|z|>t}|f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z)\\
&\le \sup\{|f_n(z)|^2 : |z|\le t\}\|g\|_K^2\\
&\quad+C\cdot\|f_n\|_{\mathcal{B}^\alpha}^2\sup_{a\in\mathbb{D}}\int_{|z|>t}|g'(z)|^2K(\mathrm{g}(z,a))dA(z)\\
&= I_1+I_2.
\end{aligned}$$

For I_1, since f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ and $g \in Q_K$, $I_1 \to 0(n \to \infty)$. For I_2, (3.8) gives $I_2 \to 0(t \to 1)$. Hence by Lemma 3.1 we follow that $T_g : \mathcal{B}^\alpha \to Q_K$ is compact. $\square$

Theorem 3.4. *Suppose $g \in H(\mathbb{D})$ and $\alpha = 1$, then the following statements are equivalent.*

1) $T_g : \mathcal{B}^\alpha \to Q_K$ *is compact*;

2) $T_g : \mathcal{B}_0^\alpha \to Q_K$ *is compact*;

3)

$$\lim_{t\to 1}\sup_{a\in\mathbb{D}}\int_{|z|>t}(\log\frac{2}{1-|z|^2})^2|g'(z)|^2K(\mathrm{g}(z,a))dA(z)=0. \tag{3.9}$$

Proof. (1) $\Rightarrow$ (2): It is obvious by $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(2) $\Rightarrow$ (3): Assume $T_g : \mathcal{B}_0^\alpha \to Q_K$ is compact. Given $f(z) = \log\frac{2}{1-e^{-i\theta}z} \in \mathcal{B}_0^\alpha(\forall\theta \in [0, 2\pi))$, by Theorem 3.2, for all $a \in \mathbb{D}$,

$$\lim_{t\to 1}\sup_{a\in\mathbb{D}}\int_{|z|>t}|\log\frac{2}{1-e^{-i\theta}z}|^2|g'(z)|^2K(\mathrm{g}(z,a))dA(z)=0.$$

We obtain (3.9) by integrating with respect to θ, the Fubini theorem and the Poisson formula.

(3) $\Rightarrow$ (1): Suppose (3.9) holds. Without loss of generality, given a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathbb{B}_{\mathcal{B}^\alpha}$ and f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$. By Lemma 2.1,

$$\begin{aligned}
\|T_g f_n\|_K^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)g'(z)|^2 K(\mathrm{g}(z,a)))dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{|z|\leq t} |f_n(z)g'(z)|^2 K(\mathrm{g}(z,a))dA(z) \\
&\quad + \sup_{a\in\mathbb{D}} \int_{|z|>t} |f_n(z)g'(z)|^2 K(\mathrm{g}(z,a))dA(z) \\
&\leq \sup\{|f_n(z)|^2 : |z| \leq t\}\|g\|_K^2 \\
&\quad + C \cdot \|f_n\|_{\mathcal{B}^\alpha}^2 \sup_{a\in\mathbb{D}} \int_{|z|>t} (\log \frac{2}{1-|z|^2})^2 |g'(z)|^2 K(\mathrm{g}(z,a))dA(z) \\
&= I_1 + I_2.
\end{aligned}$$

For I_1, since f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n \to \infty$ and $g \in Q_K$, $I_1 \to 0(n \to \infty)$. For I_2, (3.9) gives $I_2 \to 0(t \to 1)$. Hence by Lemma 3.1 we follow that $T_g : \mathcal{B}^\alpha \to Q_K$ is compact. □

Theorem 3.5. *Suppose $g \in H(\mathbb{D})$ and $\alpha > 1$, then the following statements are equivalent.*

1) $T_g : \mathcal{B}^\alpha \to Q_K$ *is compact*;

2) $T_g : \mathcal{B}_0^\alpha \to Q_K$ *is compact*;

3)

$$\lim_{t\to 1} \sup_{a\in\mathbb{D}} \int_{|z|>t} (1-|z|^2)^{2-2\alpha} |g'(z)|^2 K(\mathrm{g}(z,a))dA(z) = 0. \tag{3.10}$$

Proof. (1) $\Rightarrow$ (2): It is obvious by $\mathcal{B}_0^\alpha \subset \mathcal{B}^\alpha$.

(2) $\Rightarrow$ (3): Assume $T_g : \mathcal{B}_0^\alpha \to Q_K$ is compact. Given $f(z) = (1 - e^{-i\theta}z)^{1-\alpha} \in \mathcal{B}_0^\alpha(\forall\theta \in [0, 2\pi))$ by Theorem 3.2, for all $a \in \mathbb{D}$,

$$\lim_{t\to 1} \sup_{a\in\mathbb{D}} \int_{|z|>t} (1-e^{-i\theta}z)^{2-2\alpha} |g'(z)|^2 K(\mathrm{g}(z,a))dA(z) = 0.$$

We obtain (3.10) by integrating with respect to θ, the Fubini theorem and the Poisson formula.

(3) $\Rightarrow$ (1): Suppose (3.10) holds. Without loss of generality, given a sequence $\{f_n\}_{n=1}^{\infty} \subset \mathbb{B}_{\mathcal{B}^\alpha}$ and f_n converges to 0 on compact subsets of $\mathbb{D}$ as $n \to \infty$. By Lemma 2.1,

$$\begin{aligned}
\|T_g f_n\|_K^2 &= \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_n(z)g'(z)|^2 K(\mathrm{g}(z,a))dA(z) \\
&= \sup_{a\in\mathbb{D}} \int_{|z|\leq t} |f_n(z)g'(z)|^2 K(\mathrm{g}(z,a))dA(z)
\end{aligned}$$

$$
\begin{aligned}
&+\sup_{a\in\mathbb{D}}\int_{|z|>t}|f_n(z)g'(z)|^2K(\mathrm{g}(z,a))dA(z)\\
\leq\;& \sup\{|f_n(z)|^2:|z|\leq t\}\|g\|_K^2\\
&+C\cdot\|f_n\|^2_{\mathcal{B}^\alpha}\sup_{a\in\mathbb{D}}\int_{|z|>t}(1-|z|^2)^{2-2\alpha}|g'(z)|^2K(\mathrm{g}(z,a))dA(z)\\
=\;& I_1+I_2.
\end{aligned}
$$

For I_1since f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$ and $g\in Q_K$, $I_1\to 0(n\to\infty)$. For I_2, (3.10) gives $I_2\to 0(t\to 1)$. Hence by Lemma 3.1 we follow that $T_g:\mathcal{B}^\alpha\to Q_K$ is compact. □

Theorem 3.6. *Suppose $g\in H(\mathbb{D})$ and $\alpha>0$. Let K satisfy* (2.1)*, then $T_g:Q_K\to\mathcal{B}^\alpha$ is compact if and only if*

$$\lim_{|z|\to 1^-}(1-|z|^2)^\alpha\log\frac{1}{1-|z|}|g'(z)|=0. \tag{3.11}$$

Proof. Let $T_g:Q_K\to\mathcal{B}^\alpha$ be compact. Now suppose that the condition (3.11) fails. Then there exists a number $\epsilon_0>0$ and a sequence $\{z_n\}_{n=1}^\infty\subset\mathbb{D}$, such that

$$(1-|z_n|^2)^\alpha\log\frac{1}{1-|z_n|}|g'(z)|>\epsilon_0, \tag{3.12}$$

whenever $n>N_0$, where N_0 is a fixed positive integer.
Taking test function

$$f_n(z)=\left(\log\frac{2}{1-|z_n|^2}\right)^{-1}\left(\log\frac{2}{1-\overline{z_n}z}\right)^2,$$

from easy calculation, we have $f_n(z)\in Q_K$. It is obvious f_n converges to 0 uniformly on compact subsets of $\mathbb{D}$. From Lemma 3.1, we obtain $\|T_g(f_n)\|_{\mathcal{B}^\alpha}=0$ as $n\to\infty$. However

$$
\begin{aligned}
\|T_g(f_n)\|_{\mathcal{B}^\alpha}&\geq(1-|z_n|^2)^\alpha|(T_gf_n)'(z_n)\\
&=(1-|z_n|^2)^\alpha\log\frac{2}{1-|z_n|^2}|g'(z)|\\
&\geq(1-|z_n|^2)^\alpha\log\frac{1}{1-|z_n|}|g'(z)|>\epsilon_0>0.
\end{aligned}
$$

There is a contradiction. So $T_g:Q_K\to\mathcal{B}^\alpha$ is compact.

Conversely, suppose (3.11) holds. By Theorem 2.7, It is obvious $T_g:Q_K\to\mathcal{B}^\alpha$ is bounded. Without loss of generality, we choose a sequence $\{f_n\}_{n=1}^\infty\subset\mathbb{B}_{Q_K}$, which converges to 0 uniformly on subsets of $\mathbb{D}$, then by the definition of extend Cesàro operator,

$$\|T_g(f_n)\|_{\mathcal{B}^\alpha}=\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f_n(z)g'(z)|.$$

By (3.11), for all $\epsilon > 0$, there exists a $t(0 < t < 1)$ such that

$$(1 - |z|^2)^\alpha \log \frac{1}{1 - |z|} |g'(z)| < \epsilon, \tag{3.13}$$

whenever $t < |z| < 1$. Let $\mathbb{D}_t = \{z \in \mathbb{D} : |z| \le t\}$. So $\mathbb{D}_t$ is a compact subset of $\mathbb{D}$ and f_n converges to 0 uniformly on $\mathbb{D}_t$. By $1 \in Q_K$, we can see $g \in \mathcal{B}^\alpha$. Then for given $\epsilon > 0$, there exists a N such that

$$\sup_{z \in \mathbb{D}_t} (1 - |z|^2)^\alpha |f_n(z) g'(z)| \le \epsilon \|g\|_{\mathcal{B}^\alpha}, \tag{3.14}$$

whenever $n > N$. By Lemma 3.1, combining (3.13) and (3.14), we obtain $T_g : Q_K \to \mathcal{B}^\alpha$ is compact. □

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