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TOPOLOGICAL DIMENSION OF PSEUDO-PRIME SPECTRUM OF MODULES

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Abstract. Different topological dimensions related to the pseudo-prime spectrum of topological modules are studied. An example of topological modules is introduced. Also, we give a result about Noetherianness of the pseudo-prime spectrum of topological modules.

1. Introduction

Throughout the paper, all rings are commutative with identity and all modules are unital. For a submodule N of an R-module M, $(N :_R M)$ denotes the ideal $\{r \in R \mid rM \subseteq N\}$ and the *annihilator* of M, denoted by $\mathrm{Ann}_R(M)$, is the ideal $(\mathbf{0} :_R M)$. If there is no ambiguity we write $(N : M)$ (resp. $\mathrm{Ann}(M)$) instead of $(N :_R M)$ (resp. $\mathrm{Ann}_R(M)$). A proper submodule N of M is called *pseudo-prime* if $(N :_R M)$ is a prime ideal of R. We define the *pseudo-prime spectrum* of M to be the set of all pseudo-prime submodules of M and denote it by X^R_M. If there is no ambiguity we write only X_M instead of X^R_M (see [5]). For a submodule N of M we define $V^M(N) = \{L \in X_M \mid L \supseteq N\}$. If there is no ambiguity we write $V(N)$ instead of $V^M(N)$. For a submodule N of M, the *pseudo-prime radical* of N, denoted by $\mathbb{P}\mathrm{rad}(N)$, is the intersection of all pseudo-prime submodules of M containing N, that is

$$\mathbb{P}\mathrm{rad}(N) = \bigcap_{P \in V(N)} P.$$

If $V(N) = \emptyset$, then we set $\mathbb{P}\mathrm{rad}(N) = M$. A submodule N of M is said to be a *pseudo-prime radical submodule* if $N = \mathbb{P}\mathrm{rad}(N)$.

Let M be an R-module. Recall that a submodule N of M is said to be *pseudo-semiprime* if it is an intersection of pseudo-prime submodules. A pseudo-prime submodule H of M is called *extraordinary* if $N \cap L \subseteq H$, where N and L are pseudo-semiprime submodules of M, then either $L \subseteq H$ or $N \subseteq H$. M is said to be *topological* if $X_M = \emptyset$ or every pseudo-prime submodule of M

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is extraordinary (see [5]). If M is a topological R-module, then $\emptyset = V(M)$, $X_M = V(\mathbf{0})$ and for any family of submodules $\{N_i\}_{i\in I}$ of M,

$$\bigcap_{i\in I} V(N_i) = V(\sum_{i\in I} N_i).$$

Also for any submodules N and L of M there exists a submodule K of M such that $V(N) \cup V(L) = V(K)$. Thus, if $\zeta(M)$ denotes the collection of all subsets $V(N)$ of X_M, then $\zeta(M)$ satisfies the axioms of a topological space for the closed subsets. This topology is called the *Zariski topology*.

2. Main results

We begin this section by introducing some example of topological modules. Some examples of topological modules can be found in [5, Theorem 2.11]. Moreover, we present some conditions under which the pseudo-prime spectrum of a topological module is a Noetherian topological space.

Definition 2.1. We define *Zariski radical* of a submodule N of an R-module M, denoted by Z-$rad(N)$ as

$$\text{Z-}rad(N) = \bigcap \{P \in X_M \mid (P : M) \supseteq (N : M)\}.$$

We recall that an R-module M is called a *multiplication* module if every submodule N of M is of the form IM for some ideal I of R (see [1] and [4]).

Theorem 2.2. *Consider the following statements for an R-module M:*

(1) *M is a multiplication module*;
(2) *for every submodule N of M there exists an ideal I of R such that $V(N) = V(IM)$*;
(3) *M is a topological module.*

Then the implications (1) ⇒ (2) ⇒ (3) *hold. Moreover, if M is finitely generated then* (3) *implies* (1).

Proof. See [5, Theorem 2.10]. □

Theorem 2.3. *Let M be an R-module. If $\mathbb{P}\mathrm{rad}(N)$ = Z-$rad(N)$, for each submodule N of M, then M is topological. Moreover, if R is Noetherian, then X_M is a Noetherian topological space.*

Proof. Let N be a submodule of M. Then, we have

$$\begin{aligned} V(N) = V(\mathbb{P}\mathrm{rad}(N)) &= V(\text{Z-}rad(N)) \\ &= V(\text{Z-}rad((N : M)M)) \\ &= V(\mathbb{P}\mathrm{rad}((N : M)M)) \\ &= V((N : M)M). \end{aligned}$$

Therefore, by Theorem 2.2, M is a topological R-module.

Suppose that R is Noetherian. We show that X_M is a Noetherian topological space. For this aim, we will show that every open subset of X_M is quasi-compact (see [2, Proposition 9]). Let H be an open subset of X_M and let $\{E_\lambda\}_{\lambda\in\Lambda}$ be an open covering of H. Then there are submodules N and N_λ where $H = X_M \setminus V(N)$ and

$$E_\lambda = X_M \setminus V(N_\lambda)$$

for each $\lambda \in \Lambda$, such that

$$H \subseteq \bigcup_{\lambda\in\Lambda} E_\lambda = X_M \setminus \bigcap_{\lambda\in\Lambda} V(N_\lambda).$$

By the first part of proof, for each $\lambda \in \Lambda$, we have

$$V(N_\lambda) = V((N_\lambda : M)M).$$

Then

$$H \subseteq X_M \setminus V\left(\sum_{\lambda\in\Lambda}(N_\lambda : M)M\right) = X_M \setminus V\left((\sum_{\lambda\in\Lambda}(N_\lambda : M))M\right).$$

Since R is a Noetherian ring, there exists a finite subset Λ' of Λ such that

$$H \subseteq \bigcup_{\lambda\in\Lambda'} E_\lambda.$$

Hence, X_M is a Noetherian space. □

If $V(N)$ has at least one minimal member with respect to the inclusion, then such a minimal member is called a *minimal pseudo-prime* submodule of N or a pseudo-prime submodule minimal over N. A minimal pseudo-prime submodule of $(\mathbf{0})$ is called a minimal pseudo-prime submodule of M.

In the sequel, we are going to investigate the topological properties of X_M where M is a topological R-module. So, in the remainder of this paper we will assume that M is a topological R-module. First, we investigate the relationship between minimal pseudo-prime submodules of M and the irreducible closed subsets of X_M.

Corollary 2.4. *Let M be an R-module and N be a submodule of M.*

(1) *$V(L)$ is an irreducible closed subset of X_M for every pseudo-prime submodule L of an R-module M.*
(2) *$V(N)$ is an irreducible space if and only if $\mathbb{P}\mathrm{rad}(N)$ is a pseudo-prime submodule of M.*

Proof. See [5, Corollary 3.7]. □

Lemma 2.5. *Let M be an R-module and $N \le M$. Let Y be a nonempty subset of the closed set $V(N)$. Then Y is an irreducible closed subset of $V(N)$ if and only if $Y = V(P)$ for some $P \in V(N)$.*

Proof. There exists a submodule L of M, such that $Y = V(L)$. Since

$$Y = V(L) \subseteq V(N) \subseteq X_M$$

and $V(N)$ is closed in X_M, Y is an irreducible closed subset of X_M. By Corollary 2.4(2), $P := \mathbb{P}\mathrm{rad}(L)$ is a pseudo-prime submodule of M. But we have

$$Y = V(L) = V(\mathbb{P}\mathrm{rad}(L)) = V(P).$$

Since $L \subseteq \mathbb{P}\mathrm{rad}(L)$, $P \in V(P) \subseteq V(N)$, as desired. Other side follows from Corollary 2.4(1). □

Proposition 2.6. *Let M be an R-module and $N \leq M$.*

1. *The mapping $\varphi : P \mapsto V(P)$ is a bijection of $V(N)$ onto the set of irreducible closed subsets of $V(N)$.*
2. *The mapping $\theta : V(P) \mapsto P$ is a bijection of the set of irreducible components of $V(N)$ onto the set of minimal pseudo-prime submodule of N.*

Proof. (1) follows directly from Lemma 2.5. Moreover, the part (2) is a consequence of (1). □

Corollary 2.7. *Let M be an R-module. The correspondence $V(P) \mapsto P$ is a bijection of the set of irreducible components of X_M onto the set of minimal pseudo-prime submodules of M.*

Proof. Use Proposition 2.6. □

When $X_M \neq \emptyset$, the map $\psi : X_M \to \mathrm{Spec}(R/\mathrm{Ann}(M))$ defined by $\psi(L) = (L : M)/\mathrm{Ann}(M)$ for every $L \in X_M$, will be called the *natural map of X_M*. An R-module M is called *pseudo-primeful* if either $M = (\mathbf{0})$ or $M \neq (\mathbf{0})$ and the natural map of X_M is surjective. Also, M is called *pseudo-injective* if the natural map of X_M is injective (see [5, Definition 2.1]).

Lemma 2.8. *Let M be a nonzero pseudo-primeful R-module and let I be a radical ideal of R. Then $(IM : M) = I$ if and only if $\mathrm{Ann}(M) \subseteq I$. In particular, $\mathfrak{q}M$ is a pseudo-prime submodule of M for every $\mathfrak{q} \in V^R(\mathrm{Ann}(M))$.*

Proof. See [5, Lemma 3.10]. □

Corollary 2.9. *If M is pseudo-primeful, then the mapping*

$$\lambda : V(P) \mapsto (P : M)/\mathrm{Ann}(M) \in \mathrm{Spec}(R/\mathrm{Ann}(M))$$

is a bijection of the set of irreducible components of X_M onto the set of minimal prime ideal of $R/\mathrm{Ann}(M)$.

Proof. Let $V(P)$ be an irreducible component of X_M. Then P is a minimal pseudo-prime submodule of M by Corollary 2.7. We claim that $(P :$

$M)/\mathrm{Ann}(M)$ is a minimal prime ideal of $R/\mathrm{Ann}(M)$. Assume that $\mathfrak{q}/\mathrm{Ann}(M) \in \mathrm{Spec}(R/\mathrm{Ann}(M))$ with $\mathfrak{q}/\mathrm{Ann}(M) \subseteq (P : M)/\mathrm{Ann}(M)$. Then

$$\mathfrak{q}M \subseteq (P : M)M \subseteq P.$$

Since M is pseudo-primeful and $\mathfrak{q}M$ is a proper submodule of M, $\mathfrak{q}M$ is a pseudo-prime submodule of M with $(\mathfrak{q}M : M) = \mathfrak{q}$, by Lemma 2.8. By the minimality of P, $(P : M) = \mathfrak{q}$. Thus $(P : M)/\mathrm{Ann}(M)$ is a minimal prime ideal of $R/\mathrm{Ann}(M)$.

Now we show that λ is surjective. Suppose that $\mathfrak{p}/\mathrm{Ann}(M)$ is a minimal prime ideal of $R/\mathrm{Ann}(M)$. Again, by Lemma 2.8, $\mathfrak{p}M$ is a pseudo-prime submodule of M. We will show that $\mathfrak{p}M$ is a minimal pseudo-prime submodule of M. Let $Q \subseteq \mathfrak{p}M$ for some pseudo-prime submodule Q of M with $(Q : M) = \mathfrak{q}$. Then $\mathfrak{q}/\mathrm{Ann}(M) \subseteq \mathfrak{p}/\mathrm{Ann}(M)$. By the minimality of $\mathfrak{p}/\mathrm{Ann}(M)$, we have $\mathfrak{q} = \mathfrak{p}$. Since

$$\mathfrak{p}M = (Q : M)M \subseteq Q \subseteq \mathfrak{p}M,$$

we have $Q = \mathfrak{p}M$. Consequently the result follows from Corollary 2.7. □

Definition 2.10. A module M is said to have the Property (FC) (or simply is (FC)) if every closed subset of X_M has a finite number of irreducible components.

It is known that a ring R as an R-module has the property (FC) if and only if every ideal of R has a finite number of minimal prime divisors; furthermore if R has a Noetherian spectrum, then R has the property (FC) [9, p. 632].

Similar to the case that $M = R$, is it possible to give an algebraic condition which is equivalent to that an R-module M has the property (FC)? If M has a Noetherian spectrum, does it also have the property (FC)? In the following we answer both questions in the affirmative.

Theorem 2.11. *Let M be an R-module.*

(1) *M has the property (FC) if and only if every submodule N of M has a finite number of minimal pseudo-prime submodules.*
(2) *If M has a Noetherian spectrum, then M has the property (FC).*

Proof. (1) is a direct result of Proposition 2.6(2).

For (2), we apply the following well-known facts of Noetherian spaces: (i) every subspace of a Noetherian space is Noetherian [2, p. 79, Proposition 8(i)], and (ii) every Noetherian space has only finitely many irreducible components [2, p. 98, Proposition 10]. □

If a ring R has a Noetherian spectrum, then every radical ideal of R is the intersection of a finite number of prime ideals [6, p. 58, Theorem 87]. In next theorem we will show that this fact is true for pseudo-prime radical submodules of topological modules.

Theorem 2.12. *Let M be an R-module. If M has a Noetherian pseudo-prime spectrum, then every radical submodule of M is the intersection of a finite number of pseudo-prime submodules.*

Proof. By Theorem 2.11, every submodule N of M has a finite number of minimal pseudo-prime submodules. □

Let T be a topological space. We consider strictly decreasing (or strictly increasing) chain $Z_0, Z_1, \ldots, Z_r$ of length r of irreducible closed subsets Z_i of T. The supremum of the lengths, taken over all such chains, is called the *combinatorial dimension* of T and denoted by $C.\dim(T)$. For the empty set $\emptyset$, the combinatorial dimension is defined to be -1.

The Krull dimension of a ring R, $\dim(R)$, equals the combinatorial dimension of $\mathrm{Spec}(R)$ equipped with the Zariski topology (see [8] and [9]).

The main purpose of this paper is the investigation of combinatorial dimension of a pseudo-prime spectrum X_M of a topological R-module M. Ohm and Pendleton have shown in [9] that if a ring R has a Noetherian spectrum, then every closed subset of $\mathrm{Spec}(R)$ has a finite number of irreducible components. We are going to find a similar result for a topological R-module M with a Noetherian pseudo-prime spectrum.

Remark 2.13. Let M be an R-module.

(1) The classical Krull dimension of M is denoted by $\dim(M)$ and is defined by $\dim(M) = \dim(R/\mathrm{Ann}(M))$.
(2) If M is finitely generated, then $\dim(M)$ is the combinatorial dimension of the closed subspace $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$ of $\mathrm{Spec}(R)$ (see [8, p. 31]).

Definition 2.14. The *pseudo-prime submodule dimension of* M, $P.\dim(M)$, is defined as

$$P.\dim(M) = \sup_n \{P_0 \subset P_1 \subset \cdots \subset P_n \mid P_i \text{ is a pseudo-prime submodule of } M\}.$$

Proposition 2.15. *Let M be a pseudo-primeful and pseudo-injective R-module. Then X_M has a chain of irreducible closed subsets of X_M of length r if and only if R has a chain of prime ideals of length r.*

Proof. Let $Z_0 \subset Z_1 \subset \cdots \subset Z_r$ be a strictly increasing chain of irreducible closed subsets Z_i of X_M of length r. By Lemma 2.5, $Z_i = V(P_i)$ for some $P_i \in X_M$. Hence, we have

$$V(P_0) \subset V(P_1) \subset \cdots \subset V(P_r).$$

Thus we obtain $P_0 \supset P_1 \supset \cdots \supset P_r$, a strictly decreasing chain of pseudo-prime submodules of M of length r. Since M is pseudo-injective, we can infer that

$$(P_0 : M) \supset (P_1 : M) \supset \cdots \supset (P_r : M)$$

is a strictly decreasing chain of prime ideals of R of length r.

For the converse, let $\mathfrak{q}_0 \supset \mathfrak{q}_1 \supset \cdots \supset \mathfrak{q}_r$ be a strictly decreasing chain of prime ideals in R of length r. Since M is a pseudo-primeful R-module, we have

$$\mathfrak{q}_0 M \supset \mathfrak{q}_1 M \supset \cdots \supset \mathfrak{q}_r M$$

a strictly decreasing chain of pseudo-prime submodules of M of length r, by Lemma 2.8. Consequently, by Lemma 2.5, we get

$$V(\mathfrak{q}_0 M) \subset V(\mathfrak{q}_1 M) \subset \cdots \subset V(\mathfrak{q}_r M)$$

a strictly increasing chain of length r of irreducible closed subsets of X_M. □

We remark that pseudo-primeful modules and primeful modules which are introduced in [7] are not the same. More precisely, every primeful module is a pseudo-primeful module. However, the converse is not true in general (see [5, Example 2.4]).

Theorem 2.16. *Let M be an R-module.*

(1) $P.\dim(M) = C.\dim(X_M)$.

(2) *If M is primeful and pseudo-injective, then*

$$C.\dim(X_M) = \dim(R/\mathrm{Ann}(M)) = C.\dim(\mathrm{Supp}(M)) = \dim(M).$$

Proof. (1) It is clear by Lemma 2.5.

(2) By Proposition 2.15, $C.\dim(X_M) = C.\dim(\mathrm{Spec}(R/\mathrm{Ann}(M)))$. Since M is primeful by [7, Proposition 3.4], we have $\mathrm{Supp}(M) = V(\mathrm{Ann}(M))$. So, we have that

$$\mathrm{Supp}(M) = V(\mathrm{Ann}(M)) \cong \mathrm{Spec}(R/\mathrm{Ann}(M)).$$

Thus

$$C.\dim(\mathrm{Supp}(M)) = C.\dim(\mathrm{Spec}(R/\mathrm{Ann}(M))) = \dim(R/\mathrm{Ann}(M)).$$

On the other hand we have $\dim(M) = \dim(R/\mathrm{Ann}(M))$. Hence

$$C.\dim(\mathrm{Supp}(M)) = \dim(M) = \dim(R/\mathrm{Ann}(M)).$$

This completes the proof. □

Corollary 2.17. *Let M be a primeful and pseudo-injective R-module such that X_M has the combinatorial dimension zero. If M has a Noetherian spectrum, then the set of irreducible components of X_M is*

$$\{V(\mathfrak{m}_1 M), V(\mathfrak{m}_2 M), \ldots, V(\mathfrak{m}_k M)\}$$

for some $k \in \mathbb{N}$, where the $\mathfrak{m}_i$ for $i = 1, 2, \ldots k$ are all the minimal prime ideal of R.

Proof. Since X_M is a Noetherian topological space, X_M has only finitely many irreducible components, say $Z_0, Z_1, \ldots, Z_r$. By Lemma 2.5 and Proposition 2.6, $Z_i = V(P_i)$ for some minimal pseudo-prime submodule P_i of M such that $\mathfrak{p}_i := (P_i : M)$ is a minimal prime ideal of R. By assumption $C.\dim(X_M) = 0$, hence by Theorem 2.16, $\dim(R/\mathrm{Ann}(M)) = 0$. Thus every prime ideal of R is maximal and so,

$$\mathfrak{p}_i = (\mathfrak{p}_i M : M) = (P_i : M).$$

Therefore $P_i = \mathfrak{p}_i M$, since M is pseudo-injective. This completes the proof. □

Definition 2.18. Let T be a Noetherian topological space. The *connectedness dimension* $Cd(T)$ of T is defined to be the minimum of dimension of those closed subsets Z of T for which $T \setminus Z$ is disconnected.

Notation 2.19. For $r \in \mathbb{N}$, denote by $\Omega(r)$ the set of all pairs (A, B) of non-empty subsets of $\{1, \ldots, r\}$ for which $A \cup B = \{1, \ldots, r\}$.

Theorem 2.20. *Let M be a faithfully flat and pseudo-injective R-module with a Noetherian pseudo-prime spectrum. Then*

$$Cd(X_M) = \min\left\{\dim\left(R/(\bigcap_{\mathfrak{p}\in C}\mathfrak{p} + \bigcap_{\mathfrak{p}\in D}\mathfrak{p})\right) : C \cup D = Min(R)\right\}.$$

Proof. By [3, Lemma 19.1.15], for the Noetherian topological space X_M, we have

$$Cd(X_M) = \min\left\{C.\dim\left((\bigcup_{i\in A} T_i)\bigcap(\bigcup_{j\in B} T_j)\right) : (A, B) \in \Omega(n)\right\},$$

where, $T_1, T_2, \ldots, T_n$ are irreducible components of X_M. Hence by Lemma 2.5, for each $i = 1, \ldots, n$, $T_i = V(P_i)$ for some pseudo-prime submodule P_i of M, with $(P_i : M) = \mathfrak{p}_i$. So, we have

$$Cd(X_M) = \min\left\{C.\dim\left((\bigcup_{i\in A} V(P_i))\bigcap(\bigcup_{j\in B} V(P_j))\right) : (A, B) \in \Omega(n)\right\}.$$

Recall that a faithfully flat module is always flat and faithful. Thus, $\mathrm{Ann}(M) = 0$. Since X_M is a Noetherian topological space, R has finitely many minimal prime ideals by Corollary 2.9, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$. Since M is faithfully flat, so it is pseudo-primeful (see [5]). Thus,

$$(\mathfrak{p}_i M : M) = (P_i : M) = \mathfrak{p}_i,$$

whence $P_i = \mathfrak{p}_i M$, because M is pseudo-injective. Thus

$$Cd(X_M) = \min\left\{C.\dim\left((\bigcup_{\mathfrak{p}\in C} V(\mathfrak{p}M))\bigcap(\bigcup_{\mathfrak{p}\in D} V(\mathfrak{p}M))\right) : C \cup D = Min(R)\right\}.$$

Since M is topological and $Min(R)$ is finite,

$$Cd(X_M) = \min\left\{C.\dim\left(V(\bigcap_{\mathfrak{p}\in C}(\mathfrak{p}M))\bigcap V(\bigcap_{\mathfrak{p}\in D}(\mathfrak{p}M))\right) : C \cup D = Min(R)\right\}.$$

Thus

$$Cd(X_M)=\min\left\{C.\dim\left(V(\bigcap_{\mathfrak{p}\in C}(\mathfrak{p}M)+\bigcap_{\mathfrak{p}\in D}(\mathfrak{p}M))\right):C\cup D=Min(R)\right\}.$$

Since M is flat and $Min(R)$ is a finite set,

$$\bigcap_{\mathfrak{p}\in C}(\mathfrak{p}M)+\bigcap_{\mathfrak{p}\in D}(\mathfrak{p}M)=(\bigcap_{\mathfrak{p}\in C}\mathfrak{p})M+(\bigcap_{\mathfrak{p}\in D}\mathfrak{p})M=(\bigcap_{\mathfrak{p}\in C}\mathfrak{p}+\bigcap_{\mathfrak{p}\in D}\mathfrak{p})M.$$

Now set

$$I:=(\bigcap_{\mathfrak{p}\in C}\mathfrak{p}+\bigcap_{\mathfrak{p}\in D}\mathfrak{p}).$$

Therefore

$$Cd(X_M)=\min\{C.\dim(V(IM)):C\cup D=Min(R)\}. \tag{2.1}$$

We claim that

$$C.\dim(V(IM))=\dim(R/I). \tag{2.2}$$

Let $Y_0\subset Y_1\subset\cdots\subset Y_r$ be a strictly increasing chain of irreducible closed subsets Y_i of $V(IM)$ of length r. By Lemma 2.5, $Z_i=V(P_i)$ for some $P_i\in V(IM)$. Hence, we have

$$V(P_0)\subset V(P_1)\subset\cdots\subset V(P_r).$$

Thus, we obtain $P_0\supset P_1\supset\cdots\supset P_r$, a strictly decreasing chain of pseudo-prime submodules of M containing IM of length r. Since M is pseudo-injective, we can infer that

$$(P_0:M)\supset(P_1:M)\supset\cdots\supset(P_r:M),$$

is a strictly decreasing chain of prime ideals of R containing I of length r. Therefore,

$$(P_0:M)/I\supset(P_1:M)/I\supset\cdots\supset(P_r:M)/I$$

is a strictly decreasing chain of prime ideals of R/I of length r. On the other hand, let $\mathfrak{q}_0/I\supset\mathfrak{q}_1/I\supset\cdots\supset\mathfrak{q}_r/I$ be a strictly decreasing chain of prime ideals in R/I of length r. Since M is a pseudo-primeful R-module, we have

$$\mathfrak{q}_0M\supset\mathfrak{q}_1M\supset\cdots\supset\mathfrak{q}_rM$$

a strictly decreasing chain of pseudo-prime submodules of M containing IM of length r, by Lemma 2.8. Consequently, by Lemma 2.5, we get

$$V(\mathfrak{q}_0M)\subset V(\mathfrak{q}_1M)\subset\cdots\subset V(\mathfrak{q}_rM)$$

a strictly increasing chain of length r of irreducible closed subsets of $V(IM)$. Consequently, $C.\dim(V(IM))=\dim(R/I)$.

Now, it follows from (2.1) and (2.2) that

$$Cd(X_M)=\min\left\{\dim\left(R/(\bigcap_{\mathfrak{p}\in C}\mathfrak{p}+\bigcap_{\mathfrak{p}\in D}\mathfrak{p})\right):C\cup D=Min(R)\right\}.$$

This completes the proof. □

Definition 2.21. The *subdimension* of a topological space T, denoted by $sdim(T)$ (see [3, Definition 19.2.1]), is defined as

$$sdim(T) = \min\{C.\dim(H) \mid H \text{ is an irreducible component of } T\}.$$

Theorem 2.22. *Let M be a faithfully flat and pseudo-injective R-module with a Noetherian pseudo-prime spectrum. Then*

$$sdim(X_M) = \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in Min(R)\}.$$

Proof. By definition we have

$$sdim(X_M) = \min\{C.\dim(H) \mid H \text{ is an irreducible component of } X_M\}.$$

By Corollary 2.9,

$$sdim(X_M) = \min\{C.\dim(V(\mathfrak{p}M)) : \mathfrak{p} \in Min(R)\}.$$

As we mentioned in the proof of Theorem 2.20, we have

$$C.\dim(V(\mathfrak{p}M)) = \dim(R/\mathfrak{p}).$$

Hence,

$$sdim(X_M) = \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in Min(R)\}.$$ □

Corollary 2.23. *Let R be a Noetherian ring and let N be a pseudo-injective R-module such that $\mathbb{P}\mathrm{rad}(K) = Z\text{-}rad(K)$ for each submodule K of N. If N is faithfully flat, then we have*

(1) $Cd(X_N) = \min\left\{\dim\left(R/(\bigcap_{\mathfrak{p}\in C}\mathfrak{p} + \bigcap_{\mathfrak{p}\in D}\mathfrak{p})\right) : C \cup D = Min(R)\right\}$,

(2) $sdim(X_N) = \min\{\dim(R/\mathfrak{p}) : \mathfrak{p} \in Min(R)\}$.

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