Commun. Korean Math. Soc. **32** (2017), No. 3, pp. 543–552
https://doi.org/10.4134/CKMS.c160211
pISSN: 1225-1763 / eISSN: 2234-3024

A GENERALIZATION OF THE PRIME RADICAL OF IDEALS IN COMMUTATIVE RINGS

Javad Bagheri Harehdashti and Hosein Fazaeli Moghimi

Abstract. Let R be a commutative ring with identity, and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be a function where $\mathscr{I}(R)$ is the set of all ideals of R. Following [2], a proper ideal P of R is called a ϕ-prime ideal if $x, y \in R$ with $xy \in P - \phi(P)$ implies $x \in P$ or $y \in P$. For an ideal I of R, we define the ϕ-radical $\sqrt[\phi]{I}$ to be the intersection of all ϕ-prime ideals of R containing I, and show that this notion inherits most of the essential properties of the usual notion of radical of an ideal. We also investigate when the set of all ϕ-prime ideals of R, denoted $\mathrm{Spec}_\phi(R)$, has a Zariski topology analogous to that of the prime spectrum $\mathrm{Spec}(R)$, and show that this topological space is Noetherian if and only if ϕ-radical ideals of R satisfy the ascending chain condition.

1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Let R be a ring. We denote the set of all ideals (resp. proper ideals) of R by $\mathscr{I}(R)$ (resp. $\mathscr{I}^*(R)$). Anderson and Smith [3], defined a weakly prime ideal, i.e., a proper ideal P of R with the property that for $a, b \in R$, $0 \neq ab \in P$ implies $a \in P$ or $b \in P$. Weakly prime elements were introduced by Galovich in [7], and used by Agargün et al. [1], to study the unique factorization in rings with zero-divisors. In studying unique factorization domains, Bhatwadekar and Sharma [5] defined the notion of almost prime ideal, i.e., an ideal $P \in \mathscr{I}^*(R)$ with the property that if $a, b \in R$, $ab \in P - P^2$, then either $a \in P$ or $b \in P$. Thus a weakly prime ideal is almost prime and any proper idempotent ideal is also almost prime. Moreover, an ideal P of R is almost prime if and only if P/P^2 is a weakly prime ideal of R/P^2. Anderson and Bataineh in [2], extended these concepts to ϕ-prime ideals. Let $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be a function. A proper ideal P of R is called ϕ-*prime* if for $x, y \in R$, $xy \in P - \phi(P)$ implies $x \in P$ or $y \in P$. In fact, P is a ϕ-prime ideal of R if and only if $P/\phi(P)$ is a weakly prime ideal of $R/\phi(P)$.

Received October 9, 2016; Accepted February 16, 2017.
2010 *Mathematics Subject Classification.* 13A15, 13A99.
Key words and phrases. ϕ-prime ideal, ϕ-radical of an ideal, ϕ-prime spectrum, ϕ-top ring.

©2017 Korean Mathematical Society

Since $P - \phi(P) = P - (P \cap \phi(P))$, there is no loss of generality in assuming that $\phi(P) \subseteq P$. We henceforth make this assumption. Given two functions $\psi_1, \psi_2 : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$, we define $\psi_1 \leq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathscr{I}(R)$.

For a ring R, we consider the following functions $\phi_\alpha : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ and the corresponding ϕ_α-prime ideals which were introduced in [2] and we will refer to these frequently:

$\phi_\varnothing$	$\phi(P) = \varnothing$	prime ideal
ϕ_0	$\phi(P) = 0$	weakly prime ideal
ϕ_2	$\phi(P) = P^2$	almost prime ideal
ϕ_n $(n \geq 2)$	$\phi(P) = P^n$	n-almost prime ideal
ϕ_ω	$\phi(P) = \bigcap_{n=1}^{\infty} P^n$	ω-prime ideal
ϕ_1	$\phi(P) = P$	any ideal.

Observe that

$$(*) \qquad \phi_\varnothing \leq \phi_0 \leq \phi_\omega \leq \cdots \leq \phi_n \leq \cdots \leq \phi_2 \leq \phi_1.$$

In the rest of this paper, the set of these functions is denoted by $\mathcal{A}$, that is $\mathcal{A} = \{\phi_\varnothing, \phi_0, \phi_\omega, \ldots, \phi_n, \ldots, \phi_2, \phi_1\}$, and $\mathcal{A}^* = \mathcal{A} - \{\phi_1\}$. Let $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be a function. We define the *ϕ-radical* of an ideal I, denoted by $\sqrt[\phi]{I}$, to be the intersection of all ϕ-prime ideals of R containing I. When $\phi = \phi_\varnothing$, we use $\sqrt{I}$ instead of $\sqrt[\phi_\varnothing]{I}$. It follows from $(*)$ that:

$$\sqrt{I} \supseteq \sqrt[\phi_0]{I} \supseteq \sqrt[\phi_\omega]{I} \supseteq \cdots \supseteq \sqrt[\phi_{n+1}]{I} \supseteq \sqrt[\phi_n]{I} \supseteq \cdots \supseteq \sqrt[\phi_2]{I} \supseteq \sqrt[\phi_1]{I} = I.$$

If $\phi, \psi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ are two functions such that $\phi \leq \psi$, then $\sqrt[\psi]{\sqrt[\phi]{I}} = \sqrt[\phi]{\sqrt[\psi]{I}} = \sqrt[\phi]{I}$ (Theorem 2.5). It is shown that if R is a Noetherian integral domain and $I \in \mathscr{I}^*(R)$, then $\sqrt[\phi]{I} = \sqrt{I}$ for all $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ with $\phi \leq \phi_3$ (Proposition 2.12). In particular, if R is a PID, then $\sqrt[\phi]{I} = \sqrt{I}$ for all $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ with $\phi \leq \phi_2$ (Proposition 2.14).

The set of all ϕ-prime ideals of R is called the *ϕ-prime spectrum* of R and denoted by $\mathrm{Spec}_\phi(\mathrm{R})$ or simply X_ϕ. Now, by $(*)$, we have:

$$X_{\phi_\varnothing} \subseteq X_{\phi_0} \subseteq X_{\phi_\omega} \subseteq \cdots \subseteq X_{\phi_{n+1}} \subseteq X_{\phi_n} \subseteq \cdots \subseteq X_{\phi_2} \subseteq X_{\phi_1} = \mathscr{I}^*(R).$$

In particular, if $\phi = \phi_\varnothing$, then $\mathrm{Spec}_\phi(\mathrm{R}) = \mathrm{Spec}(R)$ and if $\phi = \phi_1$, then $\mathrm{Spec}_\phi(\mathrm{R}) = \mathscr{I}^*(R)$. For any ideal I of R we define $V_\phi(I)$ to be the set of all ϕ-prime ideals of R containing I. Of course, $V_\phi(R)$ is just the empty set and $V_\phi(0)$ is X_ϕ. Note that for any family of ideals $\{I_\gamma |\ \gamma \in \Gamma\}$ of R, $\bigcap_{\gamma \in \Gamma} V_\phi(I_\gamma) = V_\phi(\sum_{\gamma \in \Gamma} I_\gamma)$. Thus if $\zeta_\phi(R)$ denotes the collections of all subsets $V_\phi(I)$ of X_ϕ, then $\zeta_\phi(R)$ contains the empty set and X_ϕ, and $\zeta_\phi(R)$ is closed under arbitrary intersections. We shall say R is a ring with a ϕ-Zariski topology, or a ϕ-top ring for short, if $\zeta_\phi(R)$ is closed under finite union, for in this case $\zeta_\phi(R)$ satisfies the axioms for the closed sets of a topological space. In this paper, we investigate the behaviour of ϕ-top rings under the idealization of a module and finite direct products. In particular, we study this topological space from the point of view of Noetherian spaces. It is shown that, for a ϕ-top ring R, X_ϕ with ϕ-Zariski topology is a Noetherian space if and only if ascending chain condition holds for ϕ-radical ideals of R (Theorem 3.10).

2. ϕ-radical of ideals

Let R be a ring and I be an ideal of R. It is well-known that the radical of I is the intersection of all prime ideals of R containing I and characterized as the set of all $a \in R$ for which $a^n \in I$ for some positive integer n. A natural generalization of this notion is the following.

Definition. Let R be a ring, I an ideal of R and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be a function. The *ϕ-radical* of I, denoted by $\sqrt[\phi]{I}$, is defined to be the intersection of all ϕ-prime ideals of R containing I. In other words,

$$\sqrt[\phi]{I} = \cap\{P \in \mathrm{Spec}_\phi(\mathrm{R}) : P \supseteq I\}.$$

Moreover, I is called a *ϕ-radical* ideal if $\sqrt[\phi]{I} = I$.

We note that, by definition, an ideal I of R is a $\phi_\varnothing$-radical ideal if and only if I is a radical ideal.

Example 2.1. In this example we compute and compare the ϕ-radical of some ideals for some $\phi \in \mathcal{A}$.

(1) Let $R = \mathbb{Z}_4$, $0 = (\bar{0})$, and $P = (\bar{2})$. Then

$$\sqrt{0} = \sqrt[\phi_\varnothing]{0} = P \supsetneq 0 = \sqrt[\phi_0]{0}.$$

This also shows that 0 is a ϕ_0-radical ideal of R which is not radical.

(2) Let $R = \mathbb{Z}_{12}$, $0 = (\bar{0})$, $P_1 = (\bar{2})$, $P_2 = (\bar{3})$, $I = (\bar{4})$, and $J = (\bar{6})$. Then $\mathscr{I}^*(R) = \{0, I, J, P_1, P_2\}$, $X = X_{\phi_\varnothing} = \{P_1, P_2\}$, $X_{\phi_0} = \{0, P_1, P_2\}$ and $X_{\phi_2} = \{0, P_1, P_2, I\}$. Hence we have

$$\sqrt[\phi_2]{0} = \sqrt[\phi_0]{0} = 0 \subsetneq \sqrt{0} = P_1 \cap P_2 = J \ , \quad \sqrt[\phi_2]{I} = I \subsetneq \sqrt[\phi_0]{I} = \sqrt{I} = P_1$$

and

$$\sqrt[\phi_2]{J} = \sqrt[\phi_0]{J} = \sqrt{J} = P_1 \cap P_2 = J.$$

(3) Let $(R, \mathcal{M})$ be a quasilocal ring with $\mathcal{M}^2 = 0$. Let $I \subset \mathcal{M}$. Then by [3, Example 12], the $\mathcal{M}[X]$-primary ideal $I[X]$ of $R[X]$ is weakly prime. Hence we have

$$\mathcal{M}[X] = \sqrt{I[X]} \supset I[X] = \sqrt[\phi_0]{I[X]}.$$

Thus $I[X]$ is a ϕ-radical ideal of $R[X]$ for all $\phi \in \mathcal{A} - \{\phi_\varnothing\}$.

(4) Let S be a ring such that $\sqrt{0} \neq 0$, T a ring and $R = S \times T$. Let $I = 0 \times T$. Then $\sqrt[\phi_\varnothing]{I} \subsetneq \sqrt[\phi_0]{I}$. In fact, the weakly prime ideals of R containing I are exactly the prime ideals of R containing I [3, Theorem 7]. Hence $\sqrt[\phi_0]{I} = \sqrt{I} = \sqrt{0} \times T \supsetneq 0 \times T = I$. On the other hand, since

0 is a weakly prime ideal, $I = 0 \times T$ is a ϕ_ω-prime ideal of R, by [2, Theorem 8] and hence $\sqrt[\phi_\omega]{I} = I$.

Lemma 2.2. *Let R be a ring, $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function and $I, J \in \mathscr{I}(R)$. Then*

(1) *If $I \subseteq J$, then $\sqrt[\phi]{I} \subseteq \sqrt[\phi]{J}$.*
(2) $\sqrt[\phi]{\bigcap_{\gamma\in\Gamma} I_\gamma} \subseteq \bigcap_{\gamma\in\Gamma} \sqrt[\phi]{I_\gamma}$ $I_\gamma(\gamma \in \Gamma)$ *of ideals of R.*
(3) $\sqrt[\phi]{I_1 I_2 \cdots I_n} \subseteq \sqrt[\phi]{I_1 \cap I_2 \cap \cdots \cap I_n} \subseteq \sqrt[\phi]{I_1} \cap \sqrt[\phi]{I_2} \cap \cdots \cap \sqrt[\phi]{I_n}$ *for each finite set $\{I_1, \ldots, I_n\}$ of ideals of R, and the equality holds if $I_1 I_2 \cdots I_n \nsubseteq \phi(P)$ for all ϕ-prime ideals P of R containing $I_1 I_2 \cdots I_n$.*
(4) $\sqrt[\phi]{\sqrt[\phi]{I}} = \sqrt[\phi]{I}$.

Proof. (1) It is clear, since every ϕ-prime ideal P of R containing J contains also I. (2) It is a direct result of (1). (3) Given inclusions are clear by (1). Let P be a ϕ-prime ideal of R containing $I_1 I_2 \cdots I_n$. By assumption, $I_1 I_2 \cdots I_n \nsubseteq \phi(P)$ and hence $I_{i_1} I_{i_2} \cdots I_{i_j} \nsubseteq \phi(P)$ for all $1 \le j \le n$. Now [2, Theorem 13] gives the result. (4) Since $I \subseteq \sqrt[\phi]{I}$, by (1), $\sqrt[\phi]{I} \subseteq \sqrt[\phi]{\sqrt[\phi]{I}}$. The reverse containment follows from the fact that every ϕ-prime ideal of R containing I contains also $\sqrt[\phi]{I}$. □

Corollary 2.3. *Let R be a ring and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be a function. Then I is a ϕ-radical ideal of R if and only if I is an intersection of ϕ-prime ideals of R.*

Proof. (⇒) It follows from definition.
(⇐) Use Lemma 2.2(2). □

Corollary 2.4. *Let R be a ring, n a positive integer and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. If I is an ideal of R such that for every ϕ-prime ideal P of R containing I, $I^n \nsubseteq \phi(P)$, then $\sqrt[\phi]{I^n} = \sqrt[\phi]{I}$. In particular, $\sqrt[\phi_0]{I^n} = 0$ or $\sqrt[\phi_0]{I^n} = \sqrt[\phi_0]{I}$.*

Proof. The first part follows from Lemma 2.2. For the "in particular" part, it is clear that if $I^n = 0$, then $\sqrt[\phi_0]{I^n} = 0$, and if $I^n \neq 0$, then, by the first part, $\sqrt[\phi_0]{I^n} = \sqrt[\phi_0]{I}$. □

Theorem 2.5. *Let R be a ring, $\phi, \psi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ two functions such that $\phi \le \psi$ and $I \in \mathscr{I}(R)$. Then*

(1) $\sqrt[\psi]{I} \subseteq \sqrt[\phi]{I}$.
(2) $\sqrt[\psi]{\sqrt[\phi]{I}} = \sqrt[\phi]{\sqrt[\psi]{I}} = \sqrt[\phi]{I}$. *In particular,* $\sqrt{\sqrt[\psi]{I}} = \sqrt[\psi]{\sqrt{I}} = \sqrt{I}$.
(3) $\sqrt[\lambda_n]{\cdots \sqrt[\lambda_2]{\sqrt[\lambda_1]{I}}} = \sqrt[\phi]{I}$ *for $\lambda_1, \lambda_2, \ldots, \lambda_n \in \{\phi, \psi\}$ and for $n \ge 2$.*
(4) *If I is a ϕ-radical ideal of R, then it is ψ-radical.*

Proof. (1) Since $\phi \leq \psi$, every ϕ-prime ideal is a ψ-prime ideal. Thus the desired result is clear. (2) By (1), $\sqrt[\phi]{I} \subseteq \sqrt[\phi]{\sqrt[\psi]{I}} \subseteq \sqrt[\phi]{\sqrt[\phi]{I}}$. Hence, by Lemma 2.2(4), we have $\sqrt[\phi]{\sqrt[\psi]{I}} = \sqrt[\phi]{I}$. On the other hand, by (1) and Lemma 2.2(4), $\sqrt[\psi]{\sqrt[\phi]{I}} \subseteq \sqrt[\phi]{\sqrt[\phi]{I}} = \sqrt[\phi]{I} \subseteq \sqrt[\psi]{\sqrt[\phi]{I}}$. Thus we have the asserted equality. For "in particular" part, put $\phi = \phi_{\varnothing}$ in the first part. (3) It follows inductively by (2) and Lemma 2.2. (4) It follows directly from (2). □

Corollary 2.6. *Let R be a ring, I and J ideals of R, and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. Then*

(1) $\sqrt[\phi]{I} = R$ *if and only if* $I = R$.

(2) $\sqrt[\phi]{I + J} = \sqrt[\phi]{\sqrt[\phi]{I} + \sqrt[\phi]{J}}$. *In particular,* $I + J = R$ *if and only if* $\sqrt[\phi]{I} + \sqrt[\phi]{J} = R$.

Proof. (1) ($\Rightarrow$) Since $\sqrt[\phi]{I} \subseteq \sqrt{I}$, $\sqrt[\phi]{I} = R$ implies that $\sqrt{I} = R$ and this gives that $I = R$. ($\Leftarrow$) Clear.

(2) Clearly $I + J \subseteq \sqrt[\phi]{I} + \sqrt[\phi]{J} \subseteq \sqrt[\phi]{I + J}$. Now taking ϕ-radical and using Lemma 2.2 give the result. □

Let R be a ring and M be an R-module. Then $R_{(+)}M$ with coordinate-wise addition, and multiplication $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ is a commutative ring with identity called the idealization of M. The prime ideals of $R_{(+)}M$ have the form $P_{(+)}M$ where P is a prime ideal of R [4, Theorem 3.2]. The homogeneous ideals of (the graded ring) $R_{(+)}M$ have the form $I_{(+)}N$, where I is an ideal of R, N is a submodule of M, and $IM \subseteq N$ [4, Theorem 3.3]. A ring $R_{(+)}M$ is called a homogeneous ring if every ideal of $R_{(+)}M$ is homogeneous.

Proposition 2.7. *Let R be a ring, M an R-module, and $R_{(+)}M$ a homogeneous ring. Let $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ and $\psi : \mathscr{I}(R_{(+)}M) \to \mathscr{I}(R_{(+)}M) \cup \{\varnothing\}$ be two functions such that $\psi(I_{(+)}N) = \phi(I)_{(+)}N$. Then*

(1) *If $Q = P_{(+)}N$ is a ψ-prime ideal of $R_{(+)}M$, then P is a ϕ-prime ideal of R.*

(2) *P is a ϕ-prime ideal of R if and only if $P_{(+)}M$ is a ψ-prime ideal of $R_{(+)}M$.*

(3) $\sqrt[\psi]{I_{(+)}M} = \sqrt[\phi]{I}_{(+)}M$.

Proof. (1) Let $r_1r_2 \in P - \phi(P)$ for $r_1, r_2 \in R$. Then $(r_1, 0)(r_2, 0) \in Q - \psi(Q)$. Since Q is ψ-prime, $(r_1, 0) \in Q$ or $(r_2, 0) \in Q$ which implies that $r_1 \in P$ or $r_2 \in P$. Thus P is a ϕ prime ideal of R. (2) Let P be a ϕ-prime ideal of R, $Q = P_{(+)}M$ and let $(r_1, m_1)(r_2, m_2) \in Q - \psi(Q)$ for $r_1, r_2 \in R$ and $m_1, m_2 \in M$. Then $r_1r_2 \in P - \phi(P)$. Since P is ϕ-prime, $r_1 \in P$ or $r_2 \in P$. Thus $(r_1, m_1) \in Q$ or $(r_2, m_2) \in Q$. Hence Q is a ψ-prime ideal of $R_{(+)}M$. The converse follows from (1). (3) Let Q be a ψ-prime ideal of $R_{(+)}M$ containing $I_{(+)}M$. Since Q contains $0_{(+)}M$, $Q = P_{(+)}M$ where P is a ϕ-prime ideal of R containing I by (2). Hence $\sqrt[\phi]{I}_{(+)}M \subseteq \sqrt[\psi]{I_{(+)}M}$. Also, if P is a ϕ-prime ideal of R containing I, then $P_{(+)}M$ is a ψ-prime ideal containing $I_{(+)}M$. This follows $\sqrt[\psi]{I_{(+)}M} \subseteq \sqrt[\phi]{I}_{(+)}M$. □

Proposition 2.8. *Let R be a ring and $I \in \mathscr{I}^*(R)$. Then either $\sqrt[\phi]{I} = \sqrt{I}$ or $(\sqrt[\phi]{I})^2 \subseteq \phi(P)$ for some ϕ-prime ideal P of R containing I.*

Proof. If every ϕ-prime ideal of R containing I is prime, then clearly $\sqrt[\phi]{I} = \sqrt{I}$. Now let P be a ϕ-prime ideal of R containing I which is not prime and let $x, y \in \sqrt[\phi]{I}$. Then $x, y \in P$ and hence $xy \in P^2 \subseteq \phi(P)$, by [2, Theorem 5]. Thus $(\sqrt[\phi]{I})^2 \subseteq \phi(P)$. □

Corollary 2.9. *Let R be a ring, $I \in \mathscr{I}^*(R)$ and $\phi \in \mathcal{A}^*$. Then either $\sqrt[\phi]{I} = \sqrt{I}$ or $(\sqrt[\phi]{I})^2 \subseteq P^2$ for some ϕ_2-prime ideal P of R containing I.*

Proof. Let $\sqrt[\phi]{I} \neq \sqrt{I}$. Then there exists a ϕ-prime ideal P of R containing I which is not prime. Since $\phi \in \mathcal{A}^*$, P is a ϕ_2-prime ideal, and hence by Proposition 2.8, we have

$$(\sqrt[\phi]{I})^2 \subseteq \phi(P) \subseteq \phi_2(P) = P^2.$$ □

Corollary 2.10. *Let R be a ring, and $I \in \mathscr{I}^*(R)$. Then either $\sqrt[\phi_0]{I} = \sqrt{I}$ or $(\sqrt[\phi_0]{I})^2 = 0$.*

Proof. It is obtained by considering $\phi = \phi_0$ in Proposition 2.8. □

Proposition 2.11. *Let R be a ring, $I \in \mathscr{I}^*(R)$ and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function such that $\phi_\omega \leq \phi \leq \phi_3$. Then $\sqrt[\phi]{I} = \sqrt[\phi_\omega]{I}$. In particular, if I is a ϕ-radical ideal, then I is ϕ_ω-radical.*

Proof. Since $\phi_\omega \leq \phi$, by Theorem 2.5, $\sqrt[\phi]{I} \subseteq \sqrt[\phi_\omega]{I}$. Let P be a ϕ-prime ideal of R containing I. Since $\phi \leq \phi_3$, by [2, Corollary 6], P is a ϕ_ω-prime ideal and so $\sqrt[\phi_\omega]{I} \subseteq \sqrt[\phi]{I}$. The "In particular" part follows immediately from the first part. □

The next three propositions, which are easily obtained from some assertions of [2], provide a good supply of examples of ϕ-radical ideals for some $\phi \in \mathcal{A}$.

Proposition 2.12. *Let R be a Noetherian integral domain, $I \in \mathscr{I}^*(R)$ and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function such that $\phi \leq \phi_3$. Then $\sqrt[\phi]{I} = \sqrt{I}$. In particular, I is a radical ideal of R if and only if I is a ϕ-radical ideal of R.*

Proof. By [2, Corollary 10], P is a prime ideal of R if and only if P is a ϕ_3-prime ideal of R. Hence we have $\sqrt[\phi_3]{I} = \sqrt{I}$. Thus $\sqrt[\phi]{I} = \sqrt{I}$ for all functions $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ with $\phi \leq \phi_3$, which also yields the "in particular" part. □

Example 2.13. Let K be a field, $R = K[[X^3, X^4, X^5]]$, and $\mathcal{M} = (X^3, X^4, X^5)$. By [5, Example, p. 47], $I = (X^3, X^4)$ is an $\mathcal{M}$-primary ideal of R which is ϕ_2-prime. On the other hand, as we show in Proposition 2.12, $\sqrt[\phi]{I} = \sqrt{I}$ for all $\phi \in \mathcal{A} - \{\phi_1, \phi_2\}$. It follows that $\sqrt[\phi_3]{I} = \mathcal{M} \supsetneq I = \sqrt[\phi_2]{I}$.

Proposition 2.14. *Let R be a PID, $I \in \mathscr{I}^*(R)$ and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function such that $\phi \leq \phi_2$. Then $\sqrt[\phi]{I} = \sqrt{I}$. In particular, I is a radical ideal of R if and only if I is a ϕ-radical ideal of R.*

Proof. By [2, Theorem 12], P is a prime ideal of R if and only if P is a ϕ-prime ideal of R, where $\phi \leq \phi_2$. Hence the result follows clearly. □

Proposition 2.15. *Let R be a ring, $I \in \mathscr{I}^*(R)$ and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. If R is von Neumann regular or $(R, \mathcal{M})$ is quasilocal with $\mathcal{M}^2 = 0$, then I is a ϕ-radical ideal for each $\phi_\omega \leq \phi \leq \phi_2$.*

Proof. By [2, Corollary 18]. □

3. ϕ-top rings

Let R be a ring and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. Recall that the ϕ-prime spectrum of R, denoted by $\mathrm{Spec}_\phi(\mathrm{R})$ or simply X_ϕ, is the set of all ϕ-prime ideals of R. For an ideal I of R, let $V_\phi(I)$ denotes the set of all ϕ-prime ideals P of R such that $P \supseteq I$, i.e., $V_\phi(I) = \{P \in X_\phi : P \supseteq I\}$. When $\phi = \phi_\varnothing$, we use X and $V(I)$ instead of $X_{\phi_\varnothing}$ and $V_{\phi_\varnothing}(I)$, respectively. The following lemma collects some elementary facts about the sets $V_\phi(I)$.

Lemma 3.1. *With the above notations we have,*

(1) $V_\phi(\varnothing) = X_\phi$ *and* $V_\phi(R) = \varnothing$.
(2) $\bigcap_{\gamma \in \Gamma} V_\phi(I_\gamma) = V_\phi(\sum_{\gamma \in \Gamma} I_\gamma)$ *for every family* $I_\gamma (\gamma \in \Gamma)$ *of ideals of* R.
(3) $V_\phi(I) \cup V_\phi(J) \subseteq V_\phi(I \cap J) \subseteq V_\phi(IJ)$ *for any ideals* I, J *of* R.
(4) $V_\phi(I) = V_\phi(\sqrt[\phi]{I})$ *for any ideal* I *of* R.

Proof. Clear. □

From the above lemma, we can easily see that there exists a topology, τ_ϕ say, on X_ϕ having the set $\zeta_\phi(R) = \{V_\phi(I_\gamma) \mid I_\gamma \text{ is an ideal of } R\}$ as the collection of all closed sets if and only if $\zeta_\phi(R)$ is closed under finite union. When this is the case, we call the ring R a *ϕ-top ring*. It is well-known that any ring R is a $\phi_\varnothing$-top ring.

Example 3.2. (1) Let R be a ring which is an injective R-module, and $0 \neq \mathcal{M}$ be a maximal ideal of R with $\mathcal{M}^2 = 0$. By [8, Lemma 2.25], $0 \subsetneq \mathcal{M} \subsetneq R$ is the only ideal of R. Hence we have $X = V(\mathcal{M}) = V(0) = \{\mathcal{M}\}$. Moreover, for any $\phi \geq \phi_0$, we have $X_\phi = \{0, \mathcal{M}\} = \mathscr{I}^*(R)$, $V_\phi(0) = \mathscr{I}^*(R)$ and $V_\phi(\mathcal{M}) = \{\mathcal{M}\}$. Now, it is easily seen that R is a ϕ-top ring for all $\phi \geq \phi_0$ and hence R is a ϕ-top ring for each $\phi \in \mathcal{A}$.

(2) Let R be a valuation ring and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. It is well-known that the ideals of R are totally ordered by inclusion. Hence, for any ideals I and J of R, $V_\phi(I) \subseteq V_\phi(J)$ or $V_\phi(J) \subseteq V_\phi(I)$. Therefore, every valuation ring is a ϕ-top ring.

(3) Let $R = K[X,Y]/(X,Y)^2$, where K is a field. Then R is not a ϕ-top ring, for each $\phi \geq \phi_0$. To see this, assume the contrary. Let $\mathcal{M} = (X,Y)/(X,Y)^2$, $I = (\bar{X})$ and $J = (\bar{Y})$. Since $\mathcal{M}^2 = 0$, every proper ideal of R is ϕ_0-prime and hence ϕ-prime. Since R is assumed to be ϕ-top, there exists $K \in \mathscr{I}^*(R)$ such that $V_\phi(I) \cup V_\phi(J) = V_\phi(K)$. This implies that $I, J \in V_\phi(K)$ and hence $I \supseteq K$ and $J \supseteq K$. On the other hand $K \in V_\phi(I)$ or $K \in V_\phi(J)$. Let $K \in V_\phi(I)$. Then $I \subseteq K$ which shows that $I \subseteq K \subseteq J$, a contradiction.

(4) Let R be an integral domain. It is clear that R is a ϕ_0-top ring. Now, let I and J be two ideals of R such that $I \nsubseteq J$, $J \nsubseteq I$ and $I \cap J$ be an almost prime ideal of R. Then $I \cap J \in V_{\phi_2}(I \cap J)$ but $I \cap J \notin V_{\phi_2}(I)$ and $I \cap J \notin V_{\phi_2}(J)$ and hence $I \cap J \notin V_{\phi_2}(I) \cup V_{\phi_2}(J)$, that is, R is not a ϕ_2-top ring.

Proposition 3.3. *Let R be a ring, $I, J \in \mathscr{I}(R)$ and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. If for every ϕ-prime ideal P of R containing IJ, $IJ \nsubseteq \phi(P)$, then R is a ϕ-top ring.*

Proof. Let $P \in V_\phi(IJ)$. Then, by [2, Theorem 13], $P \in V_\phi(I)$ or $P \in V_\phi(J)$. Thus, by the Lemma 3.1(3), $V_\phi(I) \cup V_\phi(J) = V_\phi(I \cap J) = V_\phi(IJ)$, that is, R is a ϕ-top ring. □

Proposition 3.4. *Let R be a ring, $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be a function and $P \in X_\phi - X$. Then $V(\phi(P)) = V(P)$.*

Proof. Since $\phi(P) \subseteq P$, we have $V(P) \subseteq V(\phi(P))$. On the other hand, since P is not a prime ideal of R, by [2, Theorem 5], $P^2 \subseteq \phi(P)$. Now, if $Q \in V(\phi(P))$, then $P^2 \subseteq Q$; hence $Q \supseteq P$. So $V(\phi(P)) \subseteq V(P)$. □

Corollary 3.5. *Let R be a ring and $P \in X_{\phi_0} - X$. Then $V(P) = V(0)$.*

Proposition 3.6. *Let R be a ring and $\phi, \psi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be two functions such that $\psi \leq \phi$. If R is a ϕ-top ring, then R is a ψ-top ring.*

Proof. For any ideal I of R we have $V_\psi(I) = V_\phi(I) \cap X_\psi$. Now let I_1 and I_2 be two ideals of R. Then $V_\psi(I_1) \cup V_\psi(I_2) = (V_\phi(I_1) \cap X_\psi) \cup (V_\phi(I_2) \cap X_\psi) = (V_\phi(I_1) \cup V_\phi(I_2)) \cap X_\psi$. Since R is a ϕ-top ring, there exists an ideal K of R such that $V_\phi(I_1) \cup V_\phi(I_2) = V_\phi(K)$. Thus $V_\psi(I_1) \cup V_\psi(I_2) = V_\phi(K) \cap X_\psi = V_\psi(K)$. Therefore R is a ψ-top ring. □

Theorem 3.7. *Let R be a ring and $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ a function. Then R is a ϕ-top ring if and only if $V_\phi(I) \cup V_\phi(J) = V_\phi(I \cap J)$ for any ϕ-radical ideals I and J of R.*

Proof. ($\Rightarrow$) Let P be any ϕ-prime ideal of R and let I and J be ϕ-radical ideals of R such that $I \cap J \subseteq P$. Since R is a ϕ-top ring, there exists an ideal K of R such that $V_\phi(I) \cup V_\phi(J) = V_\phi(K)$. Now, since I is assumed to be ϕ-radical, we have $I = \bigcap_{\gamma \in \Gamma} P_\gamma$ for some ϕ-prime ideals $P_\gamma (\gamma \in \Gamma)$ of R. So $P_\gamma \in V_\phi(I) \subseteq V_\phi(K)$ (for each $\gamma \in \Gamma$) and hence $K \subseteq P_\gamma$(for each $\gamma \in \Gamma$) which shows that $K \subseteq \bigcap_{\gamma \in \Gamma} P_\gamma = I$. Similarly, $K \subseteq J$. Thus $K \subseteq I \cap J$ and therefore $V_\phi(I) \cup V_\phi(J) \subseteq V_\phi(I \cap J) \subseteq V_\phi(K) = V_\phi(I) \cup V_\phi(J)$ which implies that $V_\phi(I) \cup V_\phi(J) = V_\phi(I \cap J)$. ($\Leftarrow$) Let I and J be ideals of R. Then, by Lemma 3.1 and hypothesis, we have $V_\phi(I) \cup V_\phi(J) = V_\phi(\sqrt[\phi]{I}) \cup V_\phi(\sqrt[\phi]{J}) = V_\phi(\sqrt[\phi]{I} \cap \sqrt[\phi]{J})$. Thus, R is a ϕ-top ring. $\square$

Corollary 3.8. *Let R be a ring, and M an R-module. Let $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$, and $\psi : \mathscr{I}(R_{(+)}M) \to \mathscr{I}(R_{(+)}M) \cup \{\varnothing\}$ be two functions such that $\psi(I_{(+)}M) = \phi(I)_{(+)}M$. If $R_{(+)}M$ is a ψ-top ring, then R is a ϕ-top ring.*

Proof. Let I_1 and I_2 be two ϕ-radical ideals of R and $P \in V_\phi(I_1 \cap I_2)$. By Proposition 2.7, $J_1 = I_{1(+)}M$ and $J_2 = I_{2(+)}M$ are ψ-radical ideals of $R_{(+)}M$, and $P_{(+)}M$ is a ψ-prime ideal of $R_{(+)}M$. So $P_{(+)}M \in V_\psi(J_1 \cap J_2)$. Thus, by Theorem 3.7, $P_{(+)}M \in V_\psi(J_1) \cup V_\psi(J_2)$. It follows that $P \supseteq I_1$ or $P \supseteq I_2$. Hence $V_\phi(I_1) \cup V_\phi(I_2) = V_\phi(I_1 \cap I_2)$, i.e., R is ϕ-top. $\square$

Theorem 3.9. *Let R_1 and R_2 be rings, $\psi_i : \mathscr{I}(R_i) \to \mathscr{I}(R_i) \cup \{\varnothing\}$ (for $i = 1, 2$) be functions, and let $\phi = \psi_1 \times \psi_2$. If R_i is a ψ_i-top ring such that for any non-trivial ideal I_i of R_i, $\psi_i(I_i) \neq I_i$ (for $i = 1, 2$), then $R_1 \times R_2$ is a ϕ-top ring.*

Proof. Let $I_1 \times I_2$ and $J_1 \times J_2$ be ideals of $R_1 \times R_2$. Since R_i is ψ_i-top, $V_{\psi_i}(I_i) \cup V_{\psi_i}(J_i) = V_{\psi_i}(K_i)$ for some ideal K_i of R_i ($i = 1, 2$). Now, since by [2, Theorem 16], every ϕ-prime ideal of $R_1 \times R_2$ has the form $P_1 \times R_2$ or $R_1 \times P_2$, where P_i is a ψ_i-prime ideal of R_i ($i = 1, 2$), we have $V_\phi(I_1 \times I_2) \cup V_\phi(J_1 \times J_2) = V_\phi(K_1 \times K_2)$. $\square$

A topological space T is Noetherian provided that the open (respectively, closed) subsets of T satisfy the ascending (respectively, descending) chain condition, or the maximal (respectively, minimal) condition [6, §4.2]. Recall that a ring R has Noetherian spectrum (i.e., Spec(R) is a Noetherian space with the Zariski topology) if and only if the ascending chain condition (ACC) for radical ideals holds [9, p. 631]. We next generalize this result.

Theorem 3.10. *Let R be a ϕ-top ring, where $\phi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ is a function. Then (X_ϕ, τ_ϕ) is a Noetherian space if and only if ACC holds for ϕ-radical ideals of R. In particular, if R is a Noetherian ϕ-top ring, then (X_ϕ, τ_ϕ) is a Noetherian space.*

Proof. ($\Rightarrow$) Let $I_1 \subseteq I_2 \subseteq \cdots$ be an ascending chain of ϕ-radical ideals of R. Then we have the descending chain $V_\phi(I_1) \supseteq V_\phi(I_2) \supseteq \cdots$ of closed subsets of (X_ϕ, τ_ϕ). Now, by hypothesis, there is a positive integer n such that $V_\phi(I_n) = V_\phi(I_{n+1}) = \cdots$. It follows that $\sqrt[\phi]{I_n} = \sqrt[\phi]{I_{n+1}} = \cdots$; hence $I_n = I_{n+1} = \cdots$.

($\Leftarrow$) Let $V_\phi(I_1) \supseteq V_\phi(I_2) \supseteq \cdots$ be a descending chain of closed subsets of X_ϕ. Then $\sqrt[\phi]{I_1} \subseteq \sqrt[\phi]{I_2} \subseteq \cdots$. Since ACC holds for ϕ-radical ideals, there is a positive integer n such that $\sqrt[\phi]{I_n} = \sqrt[\phi]{I_{n+1}} = \cdots$, and hence $V_\phi(\sqrt[\phi]{I_n}) = V_\phi(\sqrt[\phi]{I_{n+1}}) = \cdots$. Now, by Lemma 3.1(4), we have $V_\phi(I_n) = V_\phi(I_{n+1}) = \cdots$.

The "In particular" statement follows immediately from the first part. $\square$

Corollary 3.11. *Let R be a ring, and let $\phi, \psi : \mathscr{I}(R) \to \mathscr{I}(R) \cup \{\varnothing\}$ be two functions such that $\phi \leq \psi$. If (X_ψ, τ_ψ) is a Noetherian space, then (X_ϕ, τ_ϕ) is a Noetherian space, and in particular, R has Noetherian spectrum.*

Proof. Apply Theorem 2.5(4) and Theorem 3.10. □

Corollary 3.12. *Let R be a Noetherian domain. Then (X_ϕ, τ_ϕ) is a Noetherian space for all $\phi \in \mathcal{A} - \{\phi_1, \phi_2\}$. In particular, if R is a PID, then (X_ϕ, τ_ϕ) is a Noetherian space for all $\phi \in \mathcal{A} - \{\phi_1\}$.*

Proof. If R is a Noetherian domain, by [2, Corollary 10], $V_\phi(I) = V(I)$ for all $\phi \in \mathcal{A} - \{\phi_1, \phi_2\}$, and if R is a PID, by [2, Theorem 12], $V_\phi(I) = V(I)$ for all $\phi \in \mathcal{A} - \{\phi_1\}$. Thus R is a ϕ-top ring for these functions ϕ, and hence (X_ϕ, τ_ϕ) is a Noetherian space by Theorem 3.10. □

References

[1] A. G. Agargün, D. D. Anderson, and S. Valdes-Leon, *Unique factorization rings with zero divisors*, Comm. Algebra, **27** (1999), no. 4, 1967–1974.

[2] D. D. Anderson and M. Bataineh, *Generalization of prime ideals*, Comm. Algebra **36** (2008), no. 2, 686–696.

[3] D. D. Anderson and E. Smith, *Weakly prime ideals*, Houston J. Math. **29** (2003), no. 4, 831–840.

[4] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1** (2009), no. 1, 3–56.

[5] S. M. Bhatwadekar and P. K. Sharma, *Unique factorization and birth of almost primes*, Comm. Algebra **33** (2005), no. 1, 43–49.

[6] N. Bourbaki, *Commutative Algebra*, Chap. 1, 2. Paris: Hermann, 1961.

[7] S. Galovich, *Unique factorization rings with zero divisors*, Math. Mag. **51** (1978), no. 5, 276–283.

[8] S. K Jain, A. K. Srivastava, and A. A. Tuganbaev, *Cyclic modules and the structure of rings*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2012.

[9] J. Ohm and R. L. Pendleton, *Rings with Noetherian spectrum*, Duke Math. J. **35** (1968), 631–639.

Javad Bagheri Harehdashti
Department of Mathematics
University of Birjand
Birjand, Iran
E-mail address: j_bagheri@birjand.ac.ir

Hosein Fazaeli Moghimi
Department of Mathematics
University of Birjand
Birjand, Iran
E-mail address: hfazaeli@birjand.ac.ir