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REMARKS ON GENERALIZED (α, β)-DERIVATIONS IN SEMIPRIME RINGS

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Abstract. Let R be an associative ring and $\alpha, \beta : R \to R$ ring homomorphisms. An additive mapping $d : R \to R$ is called an (α, β)-derivation of R if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for any $x, y \in R$, and an additive mapping $D : R \to R$ is called a generalized (α, β)-derivation of R associated with an (α, β)-derivation d if $D(xy) = D(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for all $x, y \in R$. In this note, we intend to generalize a theorem of Vukman [5], and a theorem of Daif and El-Sayiad [2].

1. Introduction

Throughout this paper, R will represent an associative ring with center $Z(R)$ and $\alpha, \beta : R \to R$ ring homomorphisms. Given an integer $n \geq 2$, a ring R is said to be n-torsion free, if for $x \in R$, $nx = 0$ implies $x = 0$. An additive mapping $d : R \to R$ is called an (α, β)-derivation of R if $d(xy) = d(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for any $x, y \in R$, and an additive mapping $D : R \to R$ is called a generalized (α, β)-derivation of R associated with an (α, β)-derivation d if $D(xy) = D(x)\alpha(y) + \beta(x)d(y)$ is fulfilled for all $x, y \in R$, we denote this generalized (α, β)-derivation as (D, d). Now we call an additive mapping $F : R \to R$ an (α, β)-G-mapping of R if $F(xy) = F(x)\alpha(y) + \beta(x)D(y)$ is fulfilled for all $x, y \in R$ and for some generalized (α, β)-derivation (D, d). If $\alpha = \beta$ is an identity map of R, then we call a $(1, 1)$-derivation d a derivation, we call a generalized $(1, 1)$-derivation D a generalized derivation associated a derivation d, and we call a $(1, 1)$-G-mapping F a G-mapping.

Example 1.1. Let S be a semiprime ring, and let $R = \left\{ \left(\begin{smallmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{smallmatrix} \right) | x, y, z \in S \right\}$. Now, we define maps $F, D, d : R \to R$ by

$$F \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}, D \begin{pmatrix} 0 & x & y \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix} = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z \end{pmatrix}$$

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and

$$d\begin{pmatrix}0 & x & y\\ 0 & 0 & 0\\ 0 & 0 & z\end{pmatrix}=\begin{pmatrix}0 & x & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Then, it can be verified that R is a ring which is not semiprime, d is a derivation of R, (D,d) is a generalized derivation and (F,D) is a G-mapping which is not a generalized derivation.

An additive mapping $D : R \to R$ is called a Jordan (α,β)-derivation if $D(x^2) = D(x)\alpha(x) + \beta(x)D(x)$ is fulfilled for all $x \in R$. An additive mapping $D : R \to R$ is called a generalized (α,β)-Jordan derivation if $D(x^2) = D(x)\alpha(x) + \beta(x)d(x)$ for all $x \in R$ and for some (α,β)-derivation d. We call a generalized $(1,1)$-Jordan derivation a generalized Jordan derivation.

In [5], J. Vukman introduced additive mappings $F : R \to R$ such that $F(xyx) = F(xy)x + xyF(x)$ for all $x,y \in R$, and $G : R \to R$ such that $G(xyx) = G(x)yx + xG(yx)$ for all $x,y \in R$. We call this additive mappings F (resp. G) a left (resp. right) V-derivation. In [5], Vukman obtained the following result:

Theorem A. *Let R be a 2-torsion free semiprime ring and let $D : R \to R$ be an additive mapping. Suppose that either $D(xyx) = D(xy)x + xyD(x)$ or $D(xyx) = D(x)yx + xD(xy)$ holds for all pairs $x,y \in R$. In both cases D is a derivation.*

Further, in [2], M. N. Daif and M. N. Tammam El-Sayiad introduced an additive mapping $G : R \to R$ such that $G(xyx) = G(x)yx + xD(yx)$ is fulfilled for all $x,y \in R$ and for some derivation D, and we call this additive mapping G a DS-derivation. And, Daif and Tammam El-Sayiad [2] proved the following result.

Theorem B. *Let R be a 2-torsion free semiprime ring and let $G : R \to R$ be an additive mapping. If $G(xyx) = F(x)yx + xD(yx)$ for all $x,y \in R$ for some derivation D of R, then G is a generalized Jordan derivation.*

We call an additive mapping $F : R \to R$ a left (resp. right) Vukman-(α,β)-derivation if $F(xyx) = F(xy)\alpha(x) + \beta(xy)F(x)$ (resp. $F(x)\alpha(yx) + \beta(x)F(yx)$) for all $x,y \in R$ (abbreviated as V-(α,β)-derivation). And we call an additive mapping F a generalized left (resp. right) Vukman-(α,β)-derivation (abbreviated as GV-(α,β)-derivation) if $F(xyx) = F(xy)\alpha(x) + \beta(xy)D(x)$ (resp. $F(x)\alpha(yx) + \beta(x)D(yx)$) for all $x,y \in R$ and for some left (resp. right) Vukman-(α,β)-derivation D.

Now, we denote the relationships of above various derivations as follows:

$$\begin{array}{ccccc}
\text{V-}(\alpha,\beta)\text{-derivations} & \Longleftarrow & (\alpha,\beta)\text{-derivations} & \Longrightarrow & \text{Jordan }(\alpha,\beta)\text{-derivation}\\
\Downarrow & & \Downarrow & & \Downarrow\\
\text{GV-}(\alpha,\beta)\text{-derivations} & \Longrightarrow & \text{generalized }(\alpha,\beta)\text{-derivations} & \Longrightarrow & \text{generalized Jordan }(\alpha,\beta)\text{-derivations}\\
\Downarrow & & & & \\
(\alpha,\beta)\ G\text{-mappings} & & & &
\end{array}$$

In this note, we intend to generalize above theorem of Vukman [5], and a theorem of Daif and El-Sayiad [2].

2. Results

We will prepare a few lemmas which are essential for developing the proof of our main result.

Lemma 2.1 ([3] Corollary 2.1(1))**.** *Let R be a 2-torsion free semiprime ring, L be a square-closed Lie ideal of R such that $L \nsubseteq Z(R)$ and let $a \in L$. If $aLa = 0$, then $a = 0$.*

Lemma 2.2 ([4] Theorem 2)**.** *Let R be a 2-torsion-free semiprime ring and D a Jordan (α,β)-derivation of R with α or β an automorphism of R. Then D is an (α,β)-derivation of R.*

Lemma 2.3 ([1] Theorem 3.1)**.** *Let R be a 2-torsion free semiprime ring, α an automorphism of R and β an endmorphism of R. If F is a generalized Jordan (α,β)-derivation with some Jordan (α,β)-derivation D, then F is a generalized (α,β)-derivation associated with D.*

We shall start our investigations with the following proposition concerning (α,β)-G-mappings.

Proposition 2.1. *Let R be a semiprime ring, and β an epimorphism. If F is an (α,β)-G-mapping of R associated with a generalized (α,β)-derivation (D,d), then $D = d$, and so F is a generalized (α,β)-derivation of R associated with an (α,β)-derivation d.*

Proof. By our hypothesis on F,

$$F(xyx) = F(x)\alpha(yx) + \beta(x)D(yx) = F(x)\alpha(yx) + \beta(x)D(y)\alpha(x) + \beta(xy)d(x)$$

for all $x, y \in R$. While, we have

$$F(xyx) = F(xy)\alpha(x) + \beta(xy)D(x) = F(x)\alpha(yx) + \beta(x)D(y)\alpha(x) + \beta(xy)D(x)$$

for all $x, y \in R$. Comparing above two equations, we get

$$\beta(xy)(D-d)(x) = 0 \text{ for all } x, y \in R. \tag{2.1}$$

Hence, we obtain that $(D-d)(x)\beta(x)\beta(y)(D-d)(x)\beta(x) = 0$ for all $y \in R$. Since β is an epimorphism and R is semiprime, we have $(D-d)(x)\beta(x) = 0$. While,

$$\beta(x)(D-d)(x)\beta(y)\beta(x)(D-d)(x) = 0 \text{ for all } y \in R$$

by (2.1). So, we have $\beta(x)(D-d)(x) = 0$. By linearizing,

$$\beta(x)(D-d)(z) + \beta(z)(D-d)(x) = 0 \text{ for all } x, z \in R.$$

Multiplying $(D-d)(x)$ from the left, we have

$$\begin{aligned} 0 &= (D-x)(x)\beta(x)(D-d)(z) + (D-d)(x)\beta(z)(D-d)(x) \\ &= (D-d)(x)\beta(z)(D-d)(x) \end{aligned}$$

for all $z \in R$. By semiprimeness of R, we have $(D - d)(x) = 0$ for all $x \in R$. And so, $D = d$, that is, F is a generalized derivation associated with d. □

Now, we prove our main theorem.

Theorem 2.1. *Let R be a 2-torsion free semiprime ring and $L \nsubseteq Z(R)$ be a square-closed Lie ideal of R. Let $F, D : R \to R$ be additive mappings such that $F(L) \subseteq L$ and $D(L) \subseteq L$, and let α, β be ring homomorphisms of R such that $\alpha(L) \subseteq L$ and $\beta(L) \subseteq L$.*

(i) *If $F(xyx) = F(xy)\alpha(x) + \beta(xy)D(x)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a Jordan (α, β)-derivation on L.*

(ii) *If $F(xyx) = F(x)\alpha(yx) + \beta(x)D(yx)$ and $D(xyx) = D(x)\alpha(yx) + \beta(x)D(yx)$ hold for all $x, y \in L$ and $\alpha(L) = L$, then F is a generalized Jordan (α, β)-derivation associated with a Jordan (α, β)-derivation D on L.*

(iii) *If $F(xyx) = \alpha(x)F(yx) + D(x)\beta(yx)$ holds for all $x, y \in L$ and $\beta(L) = L$, then D is a Jordan (β, α)-derivation on L.*

(iv) *If $F(xyx) = \alpha(xy)F(x) + D(xy)\beta(x)$ and $D(xyx) = \alpha(xy)D(x) + D(xy)\beta(x)$ hold for all $x, y \in L$ and $\alpha(L) = L$, then F is a generalized Jordan (β, α)-derivation associated with a Jordan (β, α)-derivation D on L.*

Proof. (i) We have

$$F(xyx) = F(xy)\alpha(x) + \beta(xy)D(x) \tag{2.2}$$

for all $x, y \in L$. Linearizing above relation, we have

$$F(xyz + zyx) = F(xy)\alpha(z) + F(zy)\alpha(x) + \beta(xy)D(z) + \beta(zy)D(x) \tag{2.3}$$

for all $x, y \in L$. Replacing z by x^2 in (2.3), we get

$$F(xyx^2+x^2yx) = F(xy)\alpha(x^2)+F(x^2y)\alpha(x)+\beta(xy)D(x^2)+\beta(x^2y)D(x). \tag{2.4}$$

On the other hand, in (2.2), substituting $xy + yx$ for y, we obtain that

$$\begin{aligned} F(x^2yx + xyx^2) &= F(x^2y + xyx)\alpha(x) + \beta(x^2y + xyx)D(x) \\ &= F(x^2y)\alpha(x) + F(xy)\alpha(x^2) + \beta(xy)D(x)\alpha(x) \\ &\quad + \beta(x^2y + xyx)D(x). \end{aligned} \tag{2.5}$$

Comparing (2.4) with (2.5), we have

$$\beta(x)\beta(y)\{D(x^2) - D(x)\alpha(x) - \beta(x)D(x)\} = 0$$

for all $x, y \in L$. Since β is a ring homomorphism and $\beta(L) = L$, we find that

$$\beta(x)z\{D(x^2) - D(x)\alpha(x) - \beta(x)D(x)\} = 0$$

for all $x, z \in L$. Now, we set $A(x) = D(x^2) - D(x)\alpha(x) - \beta(x)D(x)$. Since $D(L) \subseteq L$, $\alpha(L) \subseteq L$ and $\beta(L) \subseteq L$, we find that

$$\beta(x)A(x)z\beta(x)A(x) = 0$$

and

$$A(x)\beta(x)zA(x)\beta(x) = 0.$$

Since R is semiprime, we have

$$A(x)\beta(x) = 0 \tag{2.6}$$

and

$$\beta(x)A(x) = 0 \tag{2.7}$$

by Lemma 2.1. In (2.6), substituting $x + z$ for x, we have

$$A(x)\beta(z) + A(z)\beta(x) + B(x, z)\beta(x) + B(x, z)\beta(z) = 0, \tag{2.8}$$

where

$$B(x, z) = D(xz + zx) - D(x)\alpha(z) - D(z)\alpha(x) - \beta(x)D(z) - \beta(z)D(x).$$

In (2.8), substituting $-x$ for x, we get

$$A(x)\beta(z) - A(z)\beta(x) + B(x, z)\beta(x) - B(x, z)\beta(z) = 0. \tag{2.9}$$

By comparing (2.8) and (2.9), we get

$$2\{A(x)\beta(z) + B(x, z)\beta(x)\} = 0.$$

Since R is 2-torsion free, we have

$$A(x)\beta(z) + B(x, z)\beta(x) = 0.$$

And so we have

$$0 = A(x)\beta(z)A(x) + B(x, z)\beta(x)A(x) = A(x)\beta(z)A(x)$$

by (2.7). Since $\beta(L) = L$, we get

$$A(x)yA(x) = 0 \text{ for all } x, y \in L.$$

By semiprimeness of R, we obtain that $A(x) = 0$ for all $x \in L$ by Lemma 2.1 and hence, D is a Jordan (α, β)-derivation on L.

(ii) Now, assume that

$$F(xyx) = F(x)\alpha(yx) + \beta(x)D(yx) \tag{2.10}$$

and

$$D(xyx) = D(x)\alpha(yx) + \beta(x)D(yx) \tag{2.11}$$

for all $x, y \in L$. In (2.10), by linearizing, we have

$$F(xyz + zyx) = F(x)\alpha(yz) + F(z)\alpha(yx) + \beta(x)D(yz) + \beta(z)D(yx).$$

Now, substituting x^2 for z, we have

$$F(xyx^2 + x^2yx) = F(x)\alpha(yx^2) + F(x^2)\alpha(yx) + \beta(x)D(yx^2) + \beta(x^2)D(yx). \tag{2.12}$$

In (2.10), substituting $xy + yx$ for y, we have

$$\begin{aligned} F(x^2yx + xyx^2) &= F(x)\alpha(xyx + yx^2) + \beta(x)D(xyx + yx^2) \\ &= F(x)\alpha(xyx) + F(x)\alpha(yx^2) \\ &\quad + \beta(x)\{D(x)\alpha(yx) + \beta(x)D(yx) + D(yx^2)\}. \end{aligned} \tag{2.13}$$

By comparing (2.12) with (2.13), we get

$$\{F(x^2) - F(x)\alpha(x) - \beta(x)D(x)\}\alpha(y)\alpha(x) = 0$$

for all $x \in L$. Now, we set

$$E(x) = F(x^2) - F(x)\alpha(x) - \beta(x)D(x).$$

Since $\alpha(L) = L$, we have $E(x)z\alpha(x) = 0$ for all $x, z \in L$. As a similar way to the proof of (i), we obtain $E(x) = 0$, that is,

$$F(x^2) = F(x)\alpha(x) + \beta(x)D(x) \text{ for all } x \in L.$$

In the case of $D(xyx) = D(x)\alpha(yx) + \beta(x)D(yx)$, D is a Jordan (α, β)-derivation on L by the similar arguments to the above arguments, and so F is a generalized Jordan (α, β)-derivation on L associated with a Jordan (α, β)-derivation D on L.

(iii) The proof is similar to that of (i).

(iv) The proof is similar to that of (ii). □

In the following there are some immediate consequences of the above theorem.

Corollary 2.1. *Let R be a 2-torsion free semiprime ring, α, β endomorphisms of R, and let $F, D : R \to R$ be additive mappings.*

(i) *If $F(xyx) = F(xy)\alpha(x) + \beta(xy)D(x)$ holds for all $x, y \in R$, and β is an automorphism of R, then D is an (α, β)-derivation.*

(ii) *If $F(xyx) = F(x)\alpha(yx) + \beta(x)D(yx)$ and $D(xyx) = D(x)\alpha(yx) + \beta(x)D(yx)$ hold for all $x, y \in R$ and α is an automorphism of R, then F is a generalized (α, β)-derivation associated with an (α, β)-derivation D.*

(iii) *If $F(xyx) = \alpha(x)F(yx) + D(x)\beta(yx)$ holds for all $x, y \in R$, and β is an automorphism of R, then D is a (β, α)-derivation.*

(iv) *If $F(xyx) = \alpha(xy)F(x) + D(xy)\beta(x)$ and $D(xyx) = \alpha(xy)D(x) + D(xy)\beta(x)$ hold for all $x, y \in R$, and α is an automorphism of R, then F is a generalized (β, α)-derivation associated with a (β, α)-derivation D.*

Corollary 2.2. *Let R be a 2-torsion free semiprime ring, $D : R \to R$ an additive mapping. Then the followings are equivalent:*

(1) *$D(xyx) = D(xy)x + xy(D(x)$ for all $x, y \in R$.*

(2) *$D(xyx) = D(x)yx + xD(yx)$ for all $x, y \in R$.*

(3) *$D(xyx) = D(xy)x + xyD(x)$ or $D(xyx) = D(x)yx + xyD(x)$ for all $x, y \in R$.*
(4) *D is a derivation.*

Proof. (1) ⇒ (4). In Corollary 1, by putting $F = D$, D is a derivation.
Similarly, (2) ⇒ (4) is proved.
(3) ⇒ (4). We put $R_x = \{y \in R \,|\, D(xyx) = D(xy)x + xD(x)$ for all $x \in R\}$ and $R_x^* = \{y \in R \,|\, D(xyx) = D(x)yx + xD(yx)$ for all $x \in R\}$. Then we have $R = R_x \cup R_x^*$. Since R_x and R_x^* are additive groups, $R = R_x$ or $R = R_x^*$ by Brauer's Trick. By the same method, we have $R = \{x \in R \,|\, R = R_x\}$ or $R = \{R = R_x^*\}$. Therefore, by (1) and (2), D is a derivation.
(4) ⇒ (1), (4) ⇒ (2) and (4) ⇒ (3) are clear. □

Corollary 2.3. *Let R be a 2-torsion free semiprime ring, and let $F, D : R \to R$ be additive mappings.*

(i) *If one of the following conditions is fulfilled, then F is a generalized derivation associated with a derivation D.*
  (1) *$F(xyx) = F(x)yx + xD(yx)$ and $D(xyx) = D(x)yx + xD(yx)$ for all $x, y \in R$.*
  (2) *$F(xyx) = xyF(x) + D(xy)x$ and $D(xyx) = xyD(x) + D(xy)x$ for all $x, y \in R$.*
  (3) *$F(xyx) = F(x)yx + xD(yx)$ and $D(xyx) = D(x)yx + xD(yx)$, or $F(xyx) = xyF(x) + D(xy)x$ and $D(xyx) = xyD(x) + D(xy)x$ for all $x, y \in R$.*

(ii) *If one of the following conditions is fulfilled, then D is a derivation.*
  (4) *$F(xyx) = F(xy)x + xyD(x)$ for all $x, y \in R$.*
  (5) *$F(xyx) = xF(yx) + D(x)yx$ for all $x, y \in R$.*
  (6) *$F(xyx) = F(xy)x + xyD(x)$ or $F(xyx) = xF(yx) + D(x)yx$ for all $x, y \in R$.*

Proof. By the similar method of Corollary 2.2, this corollary is proved. □

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