Commun. Korean Math. Soc. **32** (2017), No. 3, pp. 503–509
https://doi.org/10.4134/CKMS.c160154
pISSN: 1225-1763 / eISSN: 2234-3024

ON WEAKLY LEFT QUASI-COMMUTATIVE RINGS

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Abstract. We in this note consider a generalized ring theoretic property of quasi-commutative rings in relation with powers. We will use the terminology of *weakly left quasi-commutative* for the class of rings satisfying such property. The properties and examples are basically investigated in the procedure of studying idempotents and nilpotent elements.

1. Introduction

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. We use $C(R)$ to denote the center of R, i.e., the set of all central elements in R. The n by n full (resp., upper triangular) matrix ring over R is written by $\text{Mat}_n(R)$ (resp., $U_n(R)$). $D_n(R)$ denotes the subring $\{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ of $U_n(R)$. Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. Let $J(R)$, $N_0(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the Jacobson radical, the Wedderburn radical, the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotent elements in a given ring R (possibly without identity), respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$. The polynomial ring with an indeterminate x over R is denoted by $R[x]$, and for any polynomial $f(x)$ in $R[x]$ $C_{f(x)}$ denotes the set of all coefficients of $f(x)$. $\mathbb{Z}_n$ denotes the ring of integers modulo n. $|S|$ denotes the cardinality of a given set S.

A ring (possibly without identity) is usually called *reduced* if it has no nonzero nilpotent elements. A ring is usually called *Abelian* if every idempotent is central.

Following Jung et al. [8], a ring R is said to be *quasi-commutative* if $ab \in C(R)$ for all $a \in C_{f(x)}$ and $b \in C_{g(x)}$ whenever two polynomials $f(x)$ and $g(x)$ over R satisfy $f(x)g(x) \in C(R)[x]$. In this note we consider a generalization of

Received July 25, 2016; Accepted October 5, 2016.
2010 *Mathematics Subject Classification.* 16D25, 16U70, 16S36.
Key words and phrases. weakly left quasi-commutative ring, polynomial ring, matrix ring, idempotent, nilpotent element.
This work was supported by 2-year Research Grant of Pusan National University.

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quasi-commutative rings. In the procedure, the following kind of matrix ring does a basic role. Next consider a generalization of quasi-commutative rings.

Definition 1.1. A ring R (possibly without identity) shall be said to be *weakly left quasi-commutative* provided that there exists positive integers $m = m(a)$, depending on a, such that $a^m b \in C(R)$ for each pair $(a, b) \in C_{f(x)} \times C_{g(x)}$ whenever two polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) \in C(R)[x]$. The weakly right quasi-commutative ring is defined similarly. A ring is called *weakly quasi-commutative* if it is both weakly left and weakly right quasi-commutative.

Every quasi-commutative ring is clearly weakly quasi-commutative. $C(R[x]) = C(R)[x]$ is easily shown, so we will use this fact freely.

Lemma 1.2. *Let R be a weakly left quasi-commutative ring. Then we have the following.*

(1) *Let $a \in R$. If $a^n \in C(R)$ for some $n \geq 1$, then $a \in C(R)$.*
(2) $N(R) \subseteq C(R)$.
(3) $N(R) = N_*(R) = N^*(R) = N_0(R)$.
(4) *Every weakly left or right quasi-commutative ring is Abelian.*

Proof. (1) The proof is almost same as one of [8, Lemma 1.5(1)]. But we write it here for the completeness. Let $a \in R$ and suppose that $a^n \in C(R)$ for some $n \geq 1$. Then $(1 - ax)(1 + ax + a^2x^2 + \cdots + a^{n-1}x^{n-1}) = 1 - a^n x^n \in C(R)[x]$. Since R is weakly left quasi-commutative, we have $a = 1a = 1^k a \in C(R)$ for some $k \geq 1$.

The proofs of (2) and (3) are equal to those of [8, Lemma 1.5(2, 3)].

(4) We apply the proof of [8, Proposition 1.8(1)]. Let R be a weakly left quasi-commutative ring. Assume on the contrary that there exist $e^2 = e, r \in R$ such that $er(1 - e) \neq 0$. Let $a = er(1 - e)$. Consider $f(x) = e + ax$ and $g(x) = (1 - e) - ax$ in $R[x]$. Then $f(x)g(x) = 0 \in C(R)[x]$. Since R is weakly left quasi-commutative, $a = ea = e^k a \in C(R)$ for some $k \geq 1$, and so $0 \neq a = ea = ae = 0$. This induces a contradiction. The proof of right case is similar, by the equality $a = a(1 - e) = a(1 - e)^l \in C(R)$ for some $l \geq 1$. □

By Lemma 1.2, we obtain [8, Lemma 1.5(1, 2, 3)] as corollaries.

Note. (1) Let R be a nil ring (possibly noncommutative) and $a, b \in R$. Then $a^n = 0$ and $a^n b = 0$ for some $n \geq 1$, so R is weakly left quasi-commutative. Similarly R is weakly right quasi-commutative. When R is a noncommutative ring, Lemma 1.2(3) is not valid here. Note that the proof of the results in Lemma 1.2 is done for the case of R having the identity.

(2) Let A be any ring and $R_0 = D_n(A)$ for $n \geq 5$. Next set $R = \{(a_{ij}) \in R_0 \mid a_{11} = \cdots = a_{nn} = 0\}$ be the nil subring of R_0. Then R is weakly quasi-commutative by (1). However R is not quasi-commutative as follows. Let

$$f(x) = E_{23} + E_{24}x \text{ and } g(x) = E_{45} - E_{35}x$$

in $R[x]$. Then $f(x)g(x) = 0 \in C(R)[x]$. But $E_{23}E_{35} = E_{25} \notin C(R)$ because $E_{25}E_{12} = 0 \neq E_{15} = E_{12}E_{25}$.

The following is an extension of weakly left quasi-commutative rings.

Proposition 1.3. (1) *Let R be a weakly left quasi-commutative ring. Then*

$$J(R[x]) = N_0(R[x]) = N_*(R[x]) = N^*(R[x]) = N_0(R)[x] = N(R)[x] = N(R[x]).$$

(2) *Let R be a weakly left quasi-commutative ring. Then $R[x]/J(R[x])$ is a reduced ring.*

(3) *Let R be a weakly left quasi-commutative ring. Then $N(R[x])$ is a commutative ring without identity.*

Proof. (1) The proof is much the same as [8, Proposition 1.6], by help of Lemma 1.2(3).

(2) is an immediate consequence of (1).

(3) is obtained by (1) and Lemma 1.2(2). □

By Proposition 1.3, we can obtain [8, Proposition 1.6] as a corollary.

Following [7], a ring is called *locally finite* if every finite subset generates a finite multiplicative semigroup. It is obvious that the class of locally finite rings contains finite rings and algebraic closures of finite fields. It is shown by [6, Theorem 2.2(1)] that a ring is locally finite if every finite subset generates a finite subring. In what follows, we extend [8, Corollary 1.10] to weakly left quasi-commutative rings.

Proposition 1.4. *Let R be a locally finite ring. Then the following conditions are equivalent*:

(1) *R is weakly left quasi-commutative*;

(2) *R is weakly right quasi-commutative*;

(3) *R is quasi-commutative*;

(4) *R is commutative.*

Proof. (4) $\Rightarrow$ (3), (3) $\Rightarrow$ (2), and (3) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (4) is shown by applying the proof of [8, Corollary 1.10]. Let R be quasi-commutative and $a \in R$. Since R is locally finite, a^k is an idempotent for some $k \geq 1$ by the proof of [7, Propostion 16]. But a^k is central for some $k \geq 1$ by Lemma 1.2. Consider $f(x) = 1 - ax$ and $g(x) = 1 + ax + \cdots + a^{k-1}x^{k-1}$ in $R[x]$. Then $f(x)g(x) = 1 - a^k x^k \in C(R)[x]$. So $a \in C(R)$ because R is weakly left quasi-commutative. Thus R is commutative.

The proof of (2) $\Rightarrow$ (4) is similar to the preceding one. □

Thus finite noncommutative rings cannot be weakly left quasi-commutative by Proposition 1.4.

Considering various situations above, one may conjecture that a ring R may be weakly left quasi-commutative when both R/I and I are weakly left quasi-commutative, where I is a proper ideal of R and is weakly left quasi-commutative as a ring without identity. However the following erases the possibility.

Example 1.5. Let K be a commutative ring, and $R = D_n(K)$ for $n \geq 3$. Then R is not weakly left quasi-commutative by the argument after Proposition 2.1 to follow, or by Lemma 1.2(2) because $N(R)$ is not contained in $C(R)$. Consider the ideal $I = \{(a_{ij}) \in R \mid a_{11} = \cdots = a_{nn} = 0\}$ of R. Then I is a nil ring, and so it is weakly quasi-commutative by Note after Lemma 1.2. Moreover R/I is isomorphic to K, so it is quasi-commutative.

2. Examples and counterexamples

In this section we are concerned with examples and counterexamples which are helpful to elaborate the structure of weakly left quasi-commutative rings.

Proposition 2.1. *For a ring R, the following conditions are all equivalent*:
(1) *R is a commutative ring*;
(2) *$D_2(R)$ is a commutative ring*;
(3) *$D_2(R)$ is a quasi-commutative ring*;
(4) *$D_2(R)$ is a weakly left quasi-commutative ring*;
(5) *$D_2(R)$ is a weakly right quasi-commutative ring.*

Proof. (1) ⇒ (2), (2) ⇒ (3), (3) ⇒ (4), and (3) ⇒ (5) are obvious.

(4) ⇒ (1): The proof is done by Lemma 1.2(2) and the proof of [8, Proposition 1.7]. But we write here another proof. Let $D_2(R)$ be a weakly left quasi-commutative ring, and assume on the contrary that R is not commutative. Say that $ab \neq ba$ for some $a, b \in R$. Consider $f(x) = 1 + \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) x$ and $g(x) = 1 + \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) x$ in $D_2(R)[x]$. Then $f(x)g(x) = 1 \in C(R)[x]$. But $1^k \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & a \\ 0 & 0 \end{smallmatrix}\right) \notin C(D_2(R))$ for all $k \geq 1$ because

$$\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} 0 & ab \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & ba \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}.$$

The proof of (5) ⇒ (1) is similar to the preceding one. □

Over any ring A, $D_n(A)$ cannot be weakly left quasi-commutative when $n \geq 3$ because $D_n(A)$ contains non-central nilpotent matrices E_{ij} for $i = 1, 2, \ldots$ and $j = i + 1$. Note $E_{ij}E_{j(j+1)} = E_{i(j+1)} \neq 0 = E_{j(j+1)}E_{ij}$.

Due to Bell [2], a ring R is said to be *IFP* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Reduced rings are easily shown to be IFP. It is also easily checked that IFP rings are Abelian. So, considering Lemma 1.2(4), one may ask whether IFP rings are weakly left quasi-commutative. However the following argument answers negatively. Let R be the Hamilton quaternions $\mathbb{H}$ over the real number field $\mathbb{R}$. Then R is clearly IFP. But it is not weakly left quasi-commutative. For, $(1 - ix)(1 + ix) = 1 + x^2 \in C(\mathbb{H})[x]$ and $1^k i = 1i = i \notin C(\mathbb{H}) = \mathbb{R}$ for all $k \geq 1$.

In what follows, we see an Abelian ring which is not weakly left quasi-commutative, showing that the converse of Lemma 1.2(4) need not hold.

Example 2.2. We use the ring in [7, Example 2]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra with noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$, and B be the set of polynomials of zero constant term in A.

Let I be the ideal of A generated by $a_0rb_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2)$, and $r_1r_2r_3r_4$, where $r \in A$ and $r_1, r_2, r_3, r_4 \in B$. Set $R = A/I$ here. Then R is an IFP ring by [7, Example 2]. We identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Note $B^4 = 0$.

However R is not weakly left quasi-commutative by Lemma 1.2(4) because there exists a nilpotent element a_0 which is not central (note $a_0b_0 \neq b_0a_0$). In fact, letting $f(x) = 1 + a_0^2x \in R[x]$, we have

$$f(x)^2 = (1 + a_0^2x)(1 + a_0^2x) = 1 + 2a_0^2x + a_0^4x^2 = 1 \in C(R)[x],$$

but $1^ka_0^2 = a_0^2 \notin C(R)$ for all $k \geq 1$ (note $a_0^2b_0 \neq b_0a_0^2$).

By help of Example 2.2, one can say that IFP rings need not be quasi-commutative.

Let R be a ring. Recall that an element u in R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. The *left regular* can be defined similarly. An element is *regular* if it is both left and right regular (i.e., not a zero divisor).

Proposition 2.3. *Let R be a ring and M be a multiplicatively closed subset of R consisting of central regular elements. Then R is weakly left quasi-commutative if and only if so is $M^{-1}R$.*

Proof. We apply the proof of [8, Proposition 2.3]. Write $E = M^{-1}R$ and note that $C(E) = M^{-1}C(R)$ by the argument in the proof of [8, Proposition 2.3].

Suppose that R is weakly left quasi-commutative. Let $F(x) = \sum_{i=0}^{m} \alpha_i x^i$ and $G(x) = \sum_{j=0}^{n} \beta_j x^j$ be in $E[x]$ such that $F(x)G(x) \in C(E)[x]$, where $\alpha_i = u^{-1}a_i$, $\beta_j = v^{-1}b_j$ with $a_i, b_j \in R$ for all i, j and regular $u, v \in R$. But $F(x)G(x) = u^{-1}(a_0 + a_1x + \cdots + a_mx^m)v^{-1}(b_0 + b_1x + \cdots + b_nx^n) = (uv)^{-1}(a_0 + a_1x + \cdots + a_mx^m)(b_0 + b_1x + \cdots + b_nx^n)$.

Here let $f(x) = a_0 + a_1x + \cdots + a_mx^m$ and $g(x) = b_0 + b_1x + \cdots + b_nx^n$. Then $f(x)$ and $g(x)$ are in $R[x]$. Moreover $f(x)g(x) \in C(R)[x]$ since

$$F(x)G(x) \in C(E)[x] \text{ and } C(E) = M^{-1}C(R).$$

Since R is weakly left quasi-commutative, there exists $k_i = k_i(a_i)$, depending on a_i, such that $a_i^{k_i}b_j \in C(R)$ for every tuple (i, j). This entails

$$\alpha_i^{k_i}\beta_j = u^{-k_i}a_i^{k_i}v^{-1}b_j = (u^{-k_i}v^{-1})a_i^{k_i}b_j \in M^{-1}C(R) = C(E).$$

Thus E is weakly left quasi-commutative.

Suppose that E is weakly left quasi-commutative. Let $f(x)g(x) \in C(R)[x]$ for $f(x), g(x) \in R[x]$. Then $f(x)g(x) \in (M^{-1}C(R))[x] = C(E)[x]$. Since E is weakly left quasi-commutative, $a^kb \in C(E)$ for some $k \geq 1$ for every tuple $(a, b) \in C_{f(x)} \times C_{g(x)}$. Thus $a^kb \in C(R)$ since $C(R) = R \cap C(E)$, and so R is weakly left quasi-commutative. □

Let R be a ring. Recall that the ring of *Laurent polynomials*, in an indeterminate x over R, consists of all formal sums $\sum_{i=k}^{n} a_i x^i$ with obvious addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers with $k \leq n$. We denote this ring by $R[x; x^{-1}]$.

Corollary 2.4. *Let R be a ring. $R[x]$ is weakly left quasi-commutative if and only if so is $R[x; x^{-1}]$.*

Proof. The proof is an immediate consequence of Proposition 2.3, noting that $R[x; x^{-1}] = M^{-1}R[x]$ if $M = \{1, x, x^2, \ldots\}$. □

We provide a weakly quasi-commutative ring which is not quasi-commutative in Note after Lemma 1.2. But this is the case of without identity. So we end this note by asking whether there exist weakly left quasi-commutative rings but not quasi-commutative when given rings have the identity.

Acknowledgment. The authors thank the referee for very careful reading of the manuscript and valuable suggestions that improved the paper by much.

References

[1] S. A. Amitsur, *Radicals of polynomial rings*, Canad. J. Math. **8** (1956), 355–361.

[2] H. E. Bell, *Near-rings in which each element is a power of itself*, Bull. Austral. Math. Soc. **2** (1970), 363–368.

[3] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee, *Completely prime ideals and associated radicals*, Proc. Biennial Ohio State-Denison Conference 1992, edited by S. K. Jain and S. T. Rizvi, World Scientific, 102–129, Singapore-New Jersey-London-Hong Kong, 1993.

[4] V. Camillo, C. Y. Hong, N. K. Kim, Y. Lee, and P. P. Nielsen, *Nilpotent ideals in polynomial and power series rings*, Proc. Amer. Math. Soc. **138** (2010), no. 5, 1607–1619.

[5] C. Huh, H. K. Kim, and Y. Lee, *p.p. rings and generalized p.p. rings*, J. Pure Appl. Algebra **167** (2002), no. 1, 37–52.

[6] C. Huh, N. K. Kim, and Y. Lee, *Examples of strongly π-regular rings*, J. Pure Appl. Algebra **189** (2004), no. 1-3, 195–210.

[7] C. Huh, Y. Lee, and A. Smoktunowicz, *Armendariz rings and semicommutative ring*, Comm. Algebra **30** (2002), no. 2, 751–761.

[8] D. W. Jung, B.-O. Kim, H. K. Kim, Y. Lee, S. B. Nam, S. J. Ryu, H. J. Sung, and S. J. Yun *On quasi-commutative rings*, J. Korean Math. Soc. **53** (2016), no. 2, 475–488.

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