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ON JORDAN IDEALS IN PRIME RINGS WITH GENERALIZED DERIVATIONS

Driss Bennis, Brahim Fahid, and Abdellah Mamouni

Abstract. Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Let F and G be two generalized derivations with associated derivations f and g, respectively. Our main result in this paper shows that if $F(x)x - xG(x) = 0$ for all $x \in J$, then R is commutative and $F = G$ or G is a left multiplier and $F = G + f$. This result with its consequences generalize some recent results due to El-Soufi and Aboubakr in which they assumed that the Jordan ideal J is also a subring of R.

1. Introduction

In what follows, unless stated otherwise, R will be an associative ring and $Z(R)$ the center of R. For any $x, y \in R$, the symbol $[x, y]$ and $x \circ y$ denote the Lie product $xy - yx$ and Jordan product $xy + yx$, respectively. Recall that a ring R is prime if for any $a, b \in R$, $aRb = \{0\}$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$.

In [2] Brešar introduced the definition of a generalized derivation: An additive mapping $F : R \longrightarrow R$ is called a generalized derivation if there exists a derivation $d : R \longrightarrow R$, called the associated derivation of F, such that $F(xy) = F(x)y + xd(y)$ for all $x, y \in R$. The notion of generalized derivations covers both the notions of a derivation and of a left multiplier (i.e., an additive mapping satisfying $f(xy) = f(x)y$ for all $x, y \in R$). A ring R is said to be n-torsion free, where $n \neq 0$ is a positive integer, if whenever $na = 0$, with $a \in R$, then $a = 0$. An additive subgroup J is said to be a Jordan ideal of R if $u \circ r \in J$ for all $u \in J$ and $r \in R$. Every ideal of R is a Jordan ideal of R but the converse need not be true. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. It is clear that if characteristic of R is 2, then Jordan ideals and Lie ideals of R are coincide.

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Several authors have proved commutativity theorems for prime and semi-prime rings admitting derivations or generalized derivations. It is worth mentioning that the investigation in this direction started with Posner in his famous paper [6] (see also the interesting work of Brešar [3]). Recently, in [4], El-Soufi and Aboubakr proved the following result:

Let R be a 2-torsion free prime ring, J be both a nonzero Jordan ideal and a subring of R, and F be a generalized derivation with associated derivation f. If one of the following properties holds: (i) $F(x)x = xf(x)$, (ii) $f(x^2) = 2F(x)x$, (iii) $F(x^2) = 2xF(x)$, (iv) $F(x^2) - 2xF(x) = f(x^2) - 2xf(x)$ for all $x \in J$, then $J \subseteq Z(R)$.

In [4, Example 3.9], they gave an example showing that the above result is not true in general if we assume that J is only a subring of R. In this paper we show that in fact, the condition of J being a subring is redundant. Indeed we prove this fact in a more general context. First, we focus on the generalization of the first assertion which is in fact our main result in this paper (see Theorem 3.2). As consequences we get generalizations of other assertions (Corollaries 3.5, 3.6 and 3.8).

2. Preliminary results

Let us begin with the following lemmas which will sometimes be used without explicit mention.

Lemma 2.1 ([7], Lemma 2.4)**.** *If J is a nonzero Jordan ideal of a ring R, then $2[R,R]J \subset J$ and $2J[R,R] \subset J$.*

Lemma 2.2 ([7], Lemma 2.6)**.** *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If, for two elements $a, b \in R$, $aJb = (0)$, then $a = 0$ or $b = 0$.*

Lemma 2.3 ([7], Lemma 2.7)**.** *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If $[J, J] = 0$, then R is commutative.*

Lemma 2.4 ([1], Proof of Lemma 3)**.** *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Then, $4j^2R \subset J$ and $4Rj^2 \subset J$ for all $j \in J$.*

Lemma 2.5 ([1], Proof of Theorem 2.12)**.** *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. Then, $4jRj \subset J$ for all $j \in J$.*

We will also make use of the following basic commutator identities:

$$[x, yz] = y[x, z] + [x, y]z \text{ and } [xy, z] = x[y, z] + [x, z]y.$$

3. Main results

For the sake of simplicity we prove at first the following particular case of our main theorem.

Lemma 3.1. *Let R be a 2-torsion free prime ring and two generalized derivations F and G associated with f and g, respectively. If $F(x)x - xG(x) = 0$ for all $x \in R$, then one of the following holds:*

(1) *R is commutative and $F = G$.*
(2) *G is a left multiplier and $F = G + f$.*

Proof. Assume that

$$F(x)x - xG(x) = 0 \quad \text{for all } x \in R. \tag{3.1}$$

The linearization of (3.1) gives

$$F(x)y + F(y)x - xG(y) - yG(x) = 0 \quad \text{for all } x, y \in R. \tag{3.2}$$

Replacing y by yx in (3.2) we find

$$yf(x)x - xyg(x) - yxG(x) + yG(x)x = 0 \quad \text{for all } x, y \in R. \tag{3.3}$$

Writing ry for y in (3.3) we obtain

$$ryf(x)x - xryg(x) - ryxG(x) + ryG(x)x = 0 \quad \text{for all } r, x, y \in R. \tag{3.4}$$

Left multiplying (3.3) by r we get

$$ryf(x)x - rxyg(x) - ryxG(x) + ryG(x)x = 0 \quad \text{for all } r, x, y \in R. \tag{3.5}$$

Subtracting (3.5) from (3.4), we conclude that

$$[x, r]Rg(x) = 0 \quad \text{for all } r, x \in R. \tag{3.6}$$

From the primeness of R, Equation (3.6) together with Brau's trick force that R is commutative or $g = 0$. So, for the case where R is commutative, Equation (3.1) becomes $(F(x) - G(x))x = 0$ for all $x \in R$, and so $F = G$. Otherwise, (3.4) becomes

$$ryf(x)x - ryxG(x) + ryG(x)x = 0 \quad \text{for all } r, x, y \in R. \tag{3.7}$$

That is

$$f(x)x - xG(x) + G(x)x = 0 \quad \text{for all } x \in R. \tag{3.8}$$

So that

$$f(x)x - F(x)x + G(x)x = 0 \quad \text{for all } x \in R. \tag{3.9}$$

The linearization of (3.9) gives

$$(f(x) - F(x) + G(x))y + (f(y) - F(y) + G(y))x = 0 \quad \text{for all } x, y \in R. \tag{3.10}$$

Replacing x by xt in the last equation we get

$$(f(xt) - F(xt) + G(xt))y + (f(y) - F(y) + G(y))xt = 0 \quad \text{for all } t, x, y \in R. \tag{3.11}$$

Right Multiplication of (3.10) by t gives

$$(f(x)-F(x)+G(x))yt+(f(y)-F(y)+G(y))xt = 0 \quad \text{for all } t, x, y \in R. \tag{3.12}$$

Subtracting (3.12) from (3.11), we result

$$(f(xt)-F(xt)+G(xt))y-(f(x)-F(x)+G(x))yt=0 \quad \text{for all } t,x,y\in R. \tag{3.13}$$

That is

$$(f(x)-F(x)+G(x))ty-(f(x)-F(x)+G(x))yt=0 \quad \text{for all } t,x,y\in R. \tag{3.14}$$

Replacing t by tr we get

$$(f(x)-F(x)+G(x))try-(f(x)-F(x)+G(x))ytr=0 \quad \text{for all } r,t,x,y\in R. \tag{3.15}$$

Right multiplying (3.14) by r we obtain

$$(f(x)-F(x)+G(x))tyr-(f(x)-F(x)+G(x))ytr=0 \quad \text{for all } r,t,x,y\in R. \tag{3.16}$$

Subtracting (3.16) from (3.15) we get

$$(f(x)-F(x)+G(x))t[y,r]=0 \quad \text{for all } r,t,x,y\in R. \tag{3.17}$$

Finally, the primeness of R together with (3.17) force that $f=F-G$. □

Now, we are in position to prove our main result.

Theorem 3.2. *Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R, and two generalized derivations F and G associated with f and g, respectively. If $F(x)x-xG(x)=0$ for all $x\in J$, then one of the following holds*:

(1) *R is commutative and $F=G$.*
(2) *G is a left multiplier and $F=G+f$.*

Proof. Assume that

$$F(x)x-xG(x)=0 \quad \text{for all } x\in J. \tag{3.18}$$

The linearization of (3.18) gives

$$F(x)y+F(y)x-xG(y)-yG(x)=0 \quad \text{for all } x,y\in J. \tag{3.19}$$

First case $Z(R)\cap J=\{0\}$.
Replacing x by $2x^2$ and y by $4yx^2$ in (3.19), we find

$$yf(x^2)x^2-x^2yg(x^2)-yx^2G(x^2)+yG(x^2)x^2=0 \quad \text{for all } x,y\in J. \tag{3.20}$$

Substituting $2[r,s]y$ in place of y in (3.20), where $r,s\in R$, we get

$$[[r,s],x^2]yg(x^2)=0.$$

Thus

$$[[r,s],x^2]Jg(x^2)=0 \quad \text{for all } x\in J \text{ and } r,s\in R. \tag{3.21}$$

By the primeness of R together with Lemma 2.2, we find $[[r,s],x^2]=0$ or $g(x^2)=0$. Clearly, in both cases, we arrive at $g(x^2)=0$ for all $x\in J$. This implies that $g = 0$ (by [5, Lemma 3]). Now, replacing y by $2[r, uv]x$ in (3.19), where $u, v \in J$ and $r \in R$, we get

$$[r, uv]f(x)x - [r, uv]xG(x) + [r, uv]G(x)x = 0 \quad \text{for all } u, v, x \in J \text{ and } r \in R. \tag{3.22}$$

That is

$$[r, uv](f(x)x - F(x)x + G(x)x) = 0 \quad \text{for all } u, v, x \in J \text{ and } r \in R. \tag{3.23}$$

The fact that R is a noncommutative prime ring forces that

$$f(x)x - F(x)x + G(x)x = 0 \quad \text{for all } x \in J. \tag{3.24}$$

The linearization of (3.24) yields

$$f(x)y - F(x)y + G(x)y + f(y)x - F(y)x + G(y)x = 0 \quad \text{for all } x, y \in J. \tag{3.25}$$

Replacing y by $2y[r, uv]$ in (3.25), we take, for all $u, v, x, y \in J$ and $r \in R$,

$$\begin{aligned} &f(x)y[r, uv] - F(x)y[r, uv] + G(x)y[r, uv] + f(y)[r, uv]x \\ &- F(y)[r, uv]x + G(y)[r, uv]x = 0. \end{aligned} \tag{3.26}$$

Right multiplying (3.25) by $[r, uv]$, we obtain, for all $u, v, x, y \in J$ and $r \in R$,

$$\begin{aligned} &f(x)y[r, uv] - F(x)y[r, uv] + G(x)y[r, uv] + f(y)x[r, uv] \\ &- F(y)x[r, uv] + G(y)x[r, uv] = 0. \end{aligned} \tag{3.27}$$

Subtracting (3.27) from (3.26), we conclude that

$$f(y)[[r, uv], x] - F(y)[[r, uv], x] + G(y)[[r, uv], x] = 0 \tag{3.28}$$

for all $u, v, x, y \in J$ and $r \in R$.

That is

$$(f(y) - F(y) + G(y))[[r, uv], x] = 0 \quad \text{for all } u, v, x, y \in J \text{ and } r \in R. \tag{3.29}$$

Replacing x by $2x[s, t]$ in (3.26), where $s, t \in R$, we obtain

$$(f(y) - F(y) + G(y))J[[r, uv], [s, t]] = 0 \quad \text{for all } u, v, y \in J \text{ and } r, s, t \in R. \tag{3.30}$$

Then, since R is a noncommutative prime ring, we get

$$f(y) - F(y) + G(y) = 0 \quad \text{for all } y \in J. \tag{3.31}$$

Replacing y by $4ry^2$ in (3.31), where $r \in R$, we get

$$(f(r) - F(r) + G(r))y^2 = 0 \quad \text{for all } y \in J \text{ and } r \in R. \tag{3.32}$$

Finally, we get

$$F = G + f.$$

Second case $Z(R) \cap J \neq \{0\}$.

Let $0 \neq z \in Z(R) \cap J$ and replacing y by $2yz = y \circ z$ in (3.18), we arrive at

$$yxf(z) = xyg(z) \quad \text{for all } x, y \in J. \tag{3.33}$$

Replacing y by $2[r, s]y$ in (3.33), where $r, s \in R$, we get

$$[r, s]yxf(z) = x[r, s]yg(z) \quad \text{for all } x, y \in J \text{ and } r, s \in R. \tag{3.34}$$

Left multiplication of (3.33) by $[r, s]$ gives

$$[r, s]yxf(z) = [r, s]xyg(z) \quad \text{for all } x, y \in J \text{ and } r, s \in R. \tag{3.35}$$

Subtracting (3.35) from (3.34), we arrive at $[[r, s], x]yg(z) = 0$, so

$$[[r, s], x]Jg(z) = 0 \quad \text{for all } x \in J \text{ and } r, s \in R. \tag{3.36}$$

Since R is a prime ring, Equation (3.36) forces that R is commutative or $g(z) = 0$. In this case where R is commutative we get, using simple calculation, $F = G$. Otherwise, (3.33) forces that $f(z) = 0$. So replacing in (3.18) x by $2rz$, where $r \in R$, we get

$$F(r)r = rG(r) \quad \text{for all } r \in R. \tag{3.37}$$

Therefore, using Lemma 3.1 together with (3.37), we get the desired result. □

As consequences of our main result we extend some results of [4] in more general context. To this end, we prefer at first giving the following general result.

It is clear that if F is a generalized derivation associated to a derivation f, then, for any homomorphism of right R-modules $h : R \to R$ and any nonzero integer α, $\alpha F + h$ is a generalized derivation associated to the derivation αf. Applying this remark to Theorem 3.2, we get the following result.

Corollary 3.3. *Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g, respectively. Then, for any homomorphism of right R-modules $h : R \to R$ and any nonzero integer α, if $F(x)x - \alpha xG(x) = xh(x)$ for all $x \in J$, then one of the following holds*:

(1) *R is commutative and $F = \alpha G + h$.*
(2) *αG is a left multiplier and $F = \alpha G + h + f$.*

For instance if we take (in Corollary 3.3) $h = \beta id_R$ (where id_R is the identity map on R and β is an integer), then we get the following result:

Corollary 3.4. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g. Then, for any two integers $\alpha \neq 0$ and β, if $F(x)x - \alpha xG(x) = \beta x^2$ for all $x \in J$, then one of the following holds*:

(1) *R is commutative and $F = \alpha G + \beta id_R$.*
(2) *αG is a left multiplier and $F = \alpha G + \beta id_R + f$.*

Now we give the first desired result which is a generalization of [4, Theorem 3.7].

Corollary 3.5. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If there are generalized derivations F and G of R associated with derivations $f \neq 0$ and g, respectively, such that $G(x^2) = 2xF(x)$ for all $x \in J$, then R is commutative and $2F = G + g$.*

Proof. By hypothesis,

$$G(x^2) + xg(x) = 2xF(x) \quad \text{for all } \ x \in J.$$

Then

$$G(x)x - 2xF(x) = -xg(x) \quad \text{for all } \ x \in J.$$

Therefore, the result follows using Corollary 3.3. □

The following result is a generalization of [4, Theorem 3.4].

Corollary 3.6. *Let R be a 2-torsion free prime ring and J a nonzero Jordan ideal. If R admits tow generalized derivations F and G associated to different derivations $f \neq 0$ and $g \neq 0$, respectively, such that $F(u^2) - 2uF(u) = G(u^2) - 2uG(u)$ for all $u \in J$, then R is commutative and $F - G = f - g$.*

Proof. By hypothesis,

$$F(u^2) - 2uF(u) = G(u^2) - 2uG(u) \ \text{ for all } \ u \in J. \tag{3.38}$$

Since F and G are additive maps, (3.38) can be rewritten as follows:

$$(F - G)(u^2) = 2u(F - G)(u) \ \text{ for all } \ u \in J. \tag{3.39}$$

If we set $K = F - G$, we get $K(u^2) = 2uK(u)$ for all $u \in J$. Then by Corollary 3.5, we obtain the result. □

Now we aim to give a generalization of [4, Theorem 3.6]. As done before we prefer at first giving the following general result.

Also, as before, if we consider a generalized derivation F associated to a derivation f, then, for any homomorphism of left R-modules $h : R \to R$ and any nonzero integer α, $\alpha F + h$ is a generalized derivation associated to the derivation αf. Applying this remark to Theorem 3.2, we get the following result.

Corollary 3.7. *Let R be a 2-torsion free prime ring, J be a nonzero Jordan ideal of R and two generalized derivations F and G associated with f and g, respectively. Then, for any homomorphism of left R-modules $h : R \to R$ and any nonzero integer α, if $F(x)x - \alpha xG(x) = h(x)x$ for all $x \in J$, then one of the following holds*:

(1) *R is commutative and $F - h = \alpha G$.*
(2) *αG is a left multiplier and $F - h = \alpha G + f$.*

As a consequence we get the following generalization of [4, Theorem 3.6].

Corollary 3.8. *Let R be a 2-torsion free prime ring and J be a nonzero Jordan ideal of R. If there are generalized derivations F and G of R associated with derivations f and $g \neq 0$, respectively, such that $G(x^2) = 2F(x)x$ for all $x \in J$, then R is commutative and $2F = G + g$.*

Proof. By hypothesis,

$$G(x)x + xg(x) = 2F(x)x \quad \text{for all} \quad x \in J.$$

Therefore, the result follows using Corollary 3.7. □

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Driss Bennis
Department of Mathematics
Faculty of Sciences
Mohammed V University in Rabat
Morocco
E-mail address: d.bennis@fsr.ac.ma; driss_bennis@hotmail.com

Brahim Fahid
Department of Mathematics
Faculty of Sciences
Mohammed V University in Rabat
Morocco
E-mail address: fahid.brahim@yahoo.fr

Abdellah Mamouni
Department of Mathematics
Faculty of Sciences and Techniques
Moulay Ismaïl University
Errachidia, Morocco
E-mail address: mamouni_1975@live.fr