Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1457–1470
https://doi.org/10.4134/BKMS.b160631
pISSN: 1015-8634 / eISSN: 2234-3016

STABILIZATION OF VISCOELASTIC WAVE EQUATION WITH VARIABLE COEFFICIENTS AND A DELAY TERM IN THE INTERNAL FEEDBACK

Fei Liang

Abstract. In this paper, we consider the stabilization of the viscoelastic wave equation with variable coefficients in a bounded domain with smooth boundary, subject to linear dissipative internal feedback with a delay. Our stabilization result is mainly based on the use of the Riemannian geometry methods and Lyapunov functional techniques.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary Γ. It is assumed that Γ consists of two parts Γ_1 and Γ_2 ($\Gamma = \Gamma_1 \cup \Gamma_2$) with $\Gamma_2 \neq \emptyset$, $\overline{\Gamma}_1 \cap \overline{\Gamma}_2 = \emptyset$. Let ν denote the outward normal vector field along the boundary and $div(X)$ denote the divergence of the vector field X in the Euclidean metric. Let $A(x) = (a_{ij}(x))$ be a matrix function, with $a_{ij} = a_{ji}$ of class C^1 satisfying

$$\lambda \sum_{i=1}^{n} \xi_i^2 \leq \sum_{i=1,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda \sum_{i=1}^{n} \xi_i,$$
$$x \in \Omega,\ 0 \neq \xi = (\xi_1, \xi_2, \ldots, \xi_n) \in \mathbb{R}^n$$

for positive constants λ and Λ. Define

$$\mathcal{A}u = -div\big(A(x)\nabla u\big) \quad \text{for } u \in H^1(\Omega).$$

Received July 30, 2016; Accepted December 2, 2016.
2010 *Mathematics Subject Classification.* 93D15, 35L05, 35L70.
Key words and phrases. viscoelastic wave equation, delay in internal feedback, the Riemannian geometry.
Supported in part by China NSF Grant No. 11501442, 11402194, Natural Science Basic Research Plan in Shaanxi Province of China No. 2016JM1025, the doctor scientific research start fund project of Xi An University of Science and Technology Grant No. 2016QDJ054 and Engagement Fund project of Xi An University of Science and Technology No. 6310216044.

We consider the viscoelastic wave equation with variable coefficients and a delay in the dissipative internal feedback

(1.1)
$$\begin{cases} u_{tt} + \mathcal{A}u - \displaystyle\int_0^t \beta(t-s)\mathcal{A}u(s)ds + [\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0, & x \in \Omega, t > 0, \\ u(x,t)\big|_{\Gamma_2} = 0, & t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}}\Big|_{\Gamma_1} = 0, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, t \in (0,\tau) \end{cases}$$

where $\frac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}}$ is the co-normal derivative

$$\frac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}} = \langle A(x)u, \nu\rangle,$$

where $\langle \cdot, \cdot \rangle$ denotes the standard metric of the Euclidean space $\mathbb{R}^n$. Moreover, $\tau > 0$ is a time delay, μ_1, μ_2 are real numbers with $\mu_1 > 0$, $\mu_2 \neq 0$, and the initial data u_0, u_1, f_0 are given functions belonging to suitable space. The purpose of this paper is to study the asymptotic stability for the solution of (1.1) with a delay term appearing in dissipative internal feedback.

Time delays arise in many applications because, in most instances, physical, chemical, biological, thermal, and economic phenomena naturally depend not only on the present state but also on some past occurrences. In recent years, the control of PDEs with time delay effects has become an active area of research, see for example [1, 12, 21] and references therein. In many cases, it was shown that delay is a source of instability and even an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay unless additional conditions or control terms have been used, see [7]. The stability issue of systems with delay is, therefore, of theoretical and practical importance.

If $A(x) = I$ is a constant matrix on $\overline{\Omega}$, that is to say the system (1.1) becomes the viscoelastic wave equation which has been considered by many authors during the past decades. For other related works, we refer the readers to [2, 3, 5, 6, 11, 13, 19, 20]. In [10], Kirane and Said-Houari considered the following linear viscoelastic wave equation

(1.2)
$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t \beta(t-s)\Delta u ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, & x \in \Omega, t > 0, \\ u(x,t) = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, t \in (0,\tau). \end{cases}$$

They proved the global existence of (1.2) by using the Faedo-Galerkin approximations together with some energy estimates and obtained the general decay results of energy via suitable Lyapunov functionals for $\mu_2 \leq \mu_1$. Comparing with wave equation [14, 22], the presence of the viscoelastic damping such that the solution of (1.2) is still asymptotically stable even if $\mu_1 = \mu_2$.

For a general $A(x)$, the main tools here to cope with the system (1.1) are the differential geometrical methods which were introduced by [23] and extended in [4, 9, 15, 18, 24, 25] and many others. For a survey on the differential geometric methods, see [8, 26]. Recently, Z. H. Ning et al. [17] studied the wave equation

$$\tag{1.3} \begin{cases} u_{tt} + \mathcal{A}u + a(x)[\mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau)] = 0, & x \in \Omega, t > 0, \\ u(x,t)\big|_{\Gamma_2} = 0, & t > 0, \\ \dfrac{\partial u(x,t)}{\partial \nu_{\mathcal{A}}}\Big|_{\Gamma_1} = 0, & t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \Omega, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & x \in \Omega, t \in (0,\tau) \end{cases}$$

and obtained the stabilization result based on the use of the Riemannian geometry methods, the energy-perturbed approach and the multiplier skills.

The main goal of the present paper is to obtain the stabilization of the viscoelastic wave equation with variable coefficients in a bounded domain with smooth boundary, subject to linear dissipative internal feedback with a delay. Our method of proof uses some ideas developed in [17] for the wave equation with Riemannian geometry methods and some estimates of the viscoelastic wave equation, enabling us to obtain suitable Lyapunov functionals, from which are derived the desired result.

This paper is organized as follows. In Section 2 we present some assumptions and state our main result. Section 3 is devoted to prove our main result.

2. Preliminaries and main result

First, let us introduce some notation used throughout this paper. Define

$$\tag{2.1} g = A^{-1}(x) \quad \text{for} \quad x \in \mathbb{R}^n$$

as a Riemannian metric on $\mathbb{R}^n$ and consider the couple $(\mathbb{R}^n, g)$ as a Riemannian manifold. For each $x \in \mathbb{R}^n$, the metric g introduces an inner product and the norm on the tangent space on $\mathbb{R}^n_x = \mathbb{R}^n$ by

$$\langle X, Y\rangle_g = \langle A^{-1}(x)X, Y\rangle, \quad |X|_g^2 = \langle X, X\rangle_g, \quad X,\ Y \in \mathbb{R}^n_x.$$

If $f \in C^1(\mathbb{R}^n)$, we define the gradient $\nabla_g f$ of f in the Riemannian metric g. It is easy to verify that $\nabla_g f = A(x)\nabla f$. On the other hand, we define the gradient $\nabla_g f$ of f in the Riemannian metric g, via the Riesz representation theorem, by

$$\tag{2.2} X(f) = \langle \nabla_g f, X\rangle_g,$$

where X is any vector field on $(\mathbb{R}^n, g)$.

For the relaxation function β, we assume

(G1) $\beta \in C^1[0,\infty)$ is a non-negative and non-increasing function satisfying

$$\beta(0) > 0, \quad 1 - \int_0^\infty \beta(s)ds = l > 0.$$

(G2) There exists a positive nonincreasing differentiable function $\zeta(t)$ such that

$$\beta'(t) \le -\zeta(t)\beta(t), \quad \forall t \ge 0,$$

and

$$\int_0^\infty \zeta(t)dt = +\infty.$$

As in [16], let us introduce the function

$$z(x,\rho,t) = u_t(x, t-\tau\rho), \quad x \in \Omega,\ \rho \in (0,1), t > 0. \tag{2.3}$$

Then, problem (1.1) is equivalent to

$$\begin{cases} u_{tt} + \mathcal{A}u - \displaystyle\int_0^t \beta(t-s)\mathcal{A}u(s)ds + \mu_1 u_t(x,t) + \mu_2 z(x,1,t) = 0,\ x \in \Omega, t > 0, \\ u(x,t)\big|_{\Gamma_2} = 0, t > 0, \\ \dfrac{u(x,t)}{\partial \nu_{\mathcal{A}}}\big|_{\Gamma_1} = 0, t > 0, \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \\ \tau z_t(x,\rho,t) + z_\rho(x,\rho,t) = 0, \quad x \in \Omega,\ \rho \in (0,1),\ t > 0, \\ z(x,0,t) = 0, \quad x \in \Gamma_2,\ t > 0, \\ z(x,0,t) = u_t(x,t), \quad x \in \Gamma_1,\ t > 0, \\ z(x,\rho,0) = f_0(x,-\rho\tau), \quad x \in \Omega, \rho \in (0,1). \end{cases} \tag{2.4}$$

Let

$$H^1_{\Gamma_2}(\Omega) = \{u \in H^1(\Omega) \,|\, u|_{\Gamma_2} = 0\},$$

and

$$L^2(\Omega \times (0,1)) = \left\{ u \,\Big|\, \int_0^1 \int_\Omega u^2(x,-\rho\tau)dxd\rho < \infty \right\}.$$

We now state, without a proof, a well-posedness result, which can be established by a similar proof with [10].

Theorem 2.1. *Assume that $|\mu_2| \le \mu_1$ and (G1) are satisfied. Then for given $u_0 \in H^0_{\Gamma_2}(\Omega)$, $u_1(x) \in L^2(\Omega)$ and $f_0 \in L^2(\Omega \times (0,1))$ and $T > 0$, there exists a unique weak solution (u,z) of the problem (2.4) on $(0,T)$ such that*

$$u \in C\big([0,T], H^1_{\Gamma_2}(\Omega)\big) \cap C^1\big([0,T], L^2(\Omega)\big),\ u_t \in L^2\big(0,T; H^1_0(\Omega)\big) \cap L^2\big((0,T)\times\Omega\big).$$

Define the new energy functional as

$$\begin{aligned} E(t) = E(t,u,z) &= \frac{1}{2}\int_\Omega u_t^2(t)dx + \frac{1}{2}\big(1 - \int_0^t \beta(s)ds\big)\int_\Omega |\nabla_g u|_g^2 dx \\ &+ \frac{1}{2}(\beta \circ \nabla_g u)(t) + \frac{\xi}{2}\int_\Omega \int_0^1 z^2(x,\rho,t)dxd\rho, \end{aligned} \tag{2.5}$$

where

$$(\beta \circ \nabla_g w)(t) = \int_0^t \beta(t-s) \int_\Omega |\nabla_g w(t) - \nabla_g w(s)|_g^2 dxds,$$

and ξ is a positive constant satisfying

$$\tau|\mu_2| < \xi < \tau(2\mu_1 - |\mu_2|) \quad \text{for} \quad |\mu_2| < \mu_1 \tag{2.6}$$

and

$$\xi = \tau\mu_1 \quad \text{for} \quad |\mu_2| = \mu_1 = \mu. \tag{2.7}$$

Our stability result is the following.

Theorem 2.2. *Let u be the solution of* (1.1). *Assume that $|\mu_2| \le \mu_1$ and β satisfies* $(G1)$ *and* $(G2)$. *Then there exist two positive constants K and k such that the energy of problem* (1.1) *satisfies*

$$E(t) \le Ke^{-k\int_{t_0}^t \zeta(s)ds}, \quad \forall t \ge 0. \tag{2.8}$$

Remark 2.1. Estimate (2.8) is also true for $t \in [0, t_0]$ by virtue of the continuity and boundedness of $E(t)$ and $\zeta(t)$.

3. Proof of Theorem 2.2

In this section, we show, using the Riemannian geometry methods and Lyapunov functionals that under the hypothesis $|\mu_2| \le \mu_1$, the energy of the solution of (1.1) decreases exponentially as t tends to infinity. We will discuss two case, the case where $|\mu_2| < \mu_1$ and the case $|\mu_2| = \mu_1$. We will separate the two cases since the proofs are slightly different.

3.1. Exponential stability for $|\mu_2| < \mu_1$

In this subsection, we will prove Theorem 2.2 for $|\mu_2| < \mu_1$. We have the following lemmas.

Lemma 3.1. *Let β satisfy* $(G1)$. *Then for all regular solutions of problem* (1.1)*, the energy functional defined by* (2.5) *is non-increasing and satisfies*

$$\begin{aligned} E'(t) \le\ & -C \int_\Omega \big(u_t^2(x,t) + u_t^2(x,t-\tau)\big)dx + \frac{1}{2}(\beta' \circ \nabla_g u)(t) \\ & - \frac{1}{2}\beta(t) \int_\Omega |\nabla_g u(t)|_g^2 dx \le 0 \end{aligned} \tag{3.1}$$

for some positive constant C.

Proof. Differentiating (2.5), applying Green's formula and by (2.4), we obtain

$$\begin{aligned} E'(t) =\ & \int_\Omega u_t u_{tt} dx + \big(1 - \int_0^t \beta(s)ds\big) \int_\Omega \nabla_g u \cdot \nabla u_t dx - \frac{1}{2}\beta(t) \int_\Omega |\nabla_g u|_g^2 dx \\ & + \int_0^t \beta(t-s) \int_\Omega [\nabla_g u(t) - \nabla_g u(s)] \cdot \nabla u_t(t) dxds + \frac{1}{2}(\beta' \circ \nabla_g u)(t) \end{aligned}$$

$$
\begin{aligned}
&+\xi\int_0^1\int_\Omega z_t z(x,\rho,t)\,dxd\rho\\
=&\int_\Omega\big[u_tu_{tt}+\nabla_g u\cdot\nabla u_t-\int_0^t\beta(t-s)\nabla_g u(s)\cdot\nabla u_t(t)ds\big]dx\\
&-\frac{1}{2}\beta(t)\int_\Omega|\nabla_g u|_g^2dx+\frac{1}{2}(\beta'\circ\nabla_g u)(t)+\xi\int_0^1\int_\Omega z_tz(x,\rho,t)dxd\rho\\
=&\int_\Omega\big[-\mu_1u_t^2(x,t)-\mu_2u_t(x,t)u_t(x,t-\tau)\big]dx-\frac{1}{2}\beta(t)\int_\Omega|\nabla_g u|_g^2dx\\
&+\frac{1}{2}(\beta'\circ\nabla_g u)(t)+\xi\int_0^1\int_\Omega z_tz(x,\rho,t)dxd\rho. \qquad (3.2)
\end{aligned}
$$

Noting the face that

$$
\begin{aligned}
z(x,\rho,t)&=u_t(x,t-\rho\tau)=-\frac{1}{\tau}u_\rho(x,\rho\tau),\\
z_t(x,\rho,t)&=u_{tt}(x,t-\rho\tau)=\frac{1}{\tau^2}u_{\rho\rho}(x,t-\rho\tau)
\end{aligned}
$$

and by integrating by parts, we arrive at

$$
\xi\int_0^1\int_\Omega z_tz(x,\rho,t)dxd\rho=\frac{\xi}{2\tau}\int_\Omega\big(u_t^2(x,t)-u_t^2(x,t-\tau)\big)dx. \qquad (3.3)
$$

From (3.2) and (3.3), and Young's inequality, we get

$$
\begin{aligned}
E'(t)\le\ &-(\mu_1-\frac{\xi}{2\tau}-\frac{|\mu_2|}{2})\int_\Omega u_t^2(x,t)dx-(\frac{\xi}{2\tau}-\frac{|\mu_2|}{2})\int_\Omega u_t^2(x,t-\tau)dx\\
&-\frac{1}{2}\beta(t)\int_\Omega|\nabla_g u|_g^2dx+\frac{1}{2}(\beta'\circ\nabla_g u)(t).
\end{aligned}
$$

Then, the inequality (3.1) follows directly from (2.8) and hypothesis $(G1)$. □

Now we are going to construct a Lyapunov functional $L(t)$ equivalent to $E(t)$. For this purpose, we first define the following functional

$$
I(t):=\int_\Omega u_tudx. \qquad (3.4)
$$

Then, we have the following estimate.

Lemma 3.2. *Under the assumption $(G1)$, the functional $I(t)$ satisfies, along the solution, the estimate*

$$
\begin{aligned}
I'(t)\le&\ \big(1+\frac{\mu_1}{4\delta_1}\big)||u_t||^2-\big(\frac{l}{2}-\delta_1C(\mu_1+|\mu_2|)\big)\int_\Omega|\nabla_g u(t)|_g^2dx\\
&+\frac{|\mu_2|}{4\delta_1}\int_\Omega u_t^2(x,t-\tau)dx+\frac{1-l}{2}(\beta\circ\nabla_g u)(t) \qquad (3.5)
\end{aligned}
$$

for any $\delta_1>0$.

Proof. Differentiating and applying Green's formula, we have

$$I'(t) = \int_\Omega u_t^2 dx - \int_\Omega |\nabla_g u(t)|_g^2 dx + \int_\Omega \Big\langle \nabla_g u(t), \int_0^t \beta(t-s)\nabla_g u(s) ds \Big\rangle_g dx$$
$$- \mu_1 \int_\Omega u_t(x,t) u dx - \mu_2 \int_\Omega u_t(x, t-\tau) u dx. \tag{3.6}$$

The third term in the right-hand side of (3.6) can be estimated as follows:

$$\begin{aligned}
&\int_\Omega \Big\langle \nabla_g u(t), \int_0^t \beta(t-s)\nabla_g u(s) ds \Big\rangle_g dx \\
&\le \frac{1}{2}\int_\Omega |\nabla_g u|_g^2 dx + \frac{1}{2}\int_\Omega \Big| \int_0^t \beta(t-s)\nabla_g u(s) ds\Big|_g^2 dx \\
&\le \frac{1}{2}\int_\Omega |\nabla_g u|_g^2 dx \\
&\quad + \frac{1}{2}\int_\Omega \Big(\int_0^t \beta(t-s)\big(|\nabla_g u(s) - \nabla_g u(t)|_g + |\nabla_g u(t)|_g\big) ds\Big)^2 dx.
\end{aligned} \tag{3.7}$$

Now, using Youngs inequality and $(G1)$, we obtain

$$\begin{aligned}
&\int_\Omega \Big(\int_0^t \beta(t-s)\big(|\nabla_g u(s) - \nabla_g u(t)|_g + |\nabla_g u(t)|_g\big) ds\Big)^2 dx \\
&\le \int_\Omega \Big[(1+\eta)\Big(\int_0^t \beta(t-s)|\nabla_g u(t)|_g ds\Big)^2 \\
&\quad + (1+\frac{1}{\eta})\Big(\int_0^t \beta(t-s)|\nabla_g u(s) - \nabla_g u(t)|_g ds\Big)^2\Big] dx \\
&\le (1+\eta)(1-l)^2 \int_\Omega |\nabla_g u(t)|_g^2 dx + (1+\frac{1}{\eta})(1-l)(\beta \circ \nabla_g u)(t).
\end{aligned} \tag{3.8}$$

Substituting (3.8) into (3.7), we have

$$\begin{aligned}
&\int_\Omega \Big\langle \nabla_g u(t), \int_0^t \beta(t-s)\nabla_g u(s) ds \Big\rangle_g dx \\
&\le \frac{1}{2}\big[1 + (1+\eta)(1-l)^2\big] \int_\Omega |\nabla_g u(t)|_g^2 dx + \frac{1}{2}(1+\frac{1}{\eta})(1-l)(\beta \circ \nabla_g u)(t).
\end{aligned} \tag{3.9}$$

Also, noting that $\int_\Omega |u|^2 dx \le c \int_\Omega |\nabla_g u|_g^2 dx$ for any $u \in H^1_{\Gamma_2}(\Omega)$, using Youngs and Poincaré inequalities gives

$$-\mu_1 \int_\Omega u_t u dx \le \mu_1 \delta_1 C \int_\Omega |\nabla_g u(t)|_g^2 dx + \frac{\mu_1}{4\delta_1}\int_\Omega u_t^2 dx, \tag{3.10}$$

and

$$-\mu_2 \int_\Omega u_t(x,t-\tau) u dx \le |\mu_2| \delta_1 C \int_\Omega |\nabla_g u(t)|_g^2 dx + \frac{|\mu_2|}{4\delta_1}\int_\Omega u_t^2(x,t-\tau) dx. \tag{3.11}$$

By inserting the estimates (3.9)-(3.10) into (3.6) and choosing $\eta = l/(1-l)$, then (3.5) holds. $\square$

Now, let us introduce the following functional

$$K(t) := \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx. \tag{3.12}$$

Differentiating (3.12) with respect t and using (2.4), we have

$$\begin{aligned} & \frac{d}{dt}\Big(\int_\Omega \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx \Big) dx \\ = \; & -\frac{2}{\tau} \int_\Omega \int_0^1 e^{-\tau\rho} z(x,\rho,t) z_\rho(x,\rho,t) dx d\rho \\ = \; & -\frac{1}{\tau} \int_\Omega \int_0^1 \frac{\partial}{\partial \rho}(z^2 e^{-\rho\tau}) d\rho dx - \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x,\rho,t) d\rho dx \\ = \; & \frac{1}{\tau} \int_\Omega u_t^2(x,t) dx - \frac{c}{\tau} \int_\Omega u_t^2(x,t-\tau) dx - K(t). \end{aligned} \tag{3.13}$$

Define the Lyapunov functional

$$\mathcal{L}(t) := E(t) + \varepsilon I(t) + \varepsilon K(t),$$

where ε is a positive real number which will be chosen later. It is straightforward to see that for $\varepsilon > 0$, $\mathcal{L}(t)$ and $E(t)$ are equivalent in the sense that there exist two positive constants α_1 and α_2 depending on ε such that for all $t \geq 0$

$$\alpha_1 E(t) \leq \mathcal{L}(t) \leq \alpha_2 E(t).$$

Now, we are ready to prove the general decay result for $|\mu_2| \leq \mu_1$.

Proof of Theorem 2.2 for $|\mu_2| < \mu_1$. Using the estimates (3.1), (3.5) and (3.13), we get

$$\begin{aligned} \mathcal{L}'(t) \leq \; & -\Big(C - \varepsilon(1 + \frac{\mu_1}{4\delta_1}) - \frac{\varepsilon}{\tau}\Big) \int_\Omega u_t^2(x,t) dx \\ & - \varepsilon\Big(\frac{l}{2} - \delta_1 C(\mu_1 + |\mu_2|)\Big) \int_\Omega |\nabla_g u(t)|_g^2 dx \\ & - \Big(C + \frac{\varepsilon c}{\tau} - \frac{\varepsilon|\mu_2|}{4\delta_1}\Big) \int_\Omega u_t^2(x,t-\tau) dx + \frac{\varepsilon(1-l)}{2}(\beta \circ \nabla_g u)(t) \\ & - \frac{1}{2}\beta(t) \int_\Omega |\nabla_g u|_g^2 dx + \frac{1}{2}(\beta' \circ \nabla_g u)(t) - \varepsilon K(t). \end{aligned}$$

By choosing δ_1 and ε small enough, we can find two positive constants γ_1 and γ_2 such that

$$\mathcal{L}'(t) \leq -\gamma_1 E(t) + \varepsilon\gamma_1 (\beta \circ \nabla_g u)(t), \quad \forall t \geq 0. \tag{3.14}$$

By multiplying (3.14) by $\zeta(t)$, we have

$$\zeta(t)\mathcal{L}'(t) \leq -\gamma_1 \zeta(t) E(t) + \gamma_2 \zeta(t)(\beta \circ \nabla_g u)(t), \quad \forall t \geq 0.$$

Recalling $(G2)$ and using (3.1), we have

$$\zeta(t)\mathcal{L}'(t) \leq -\gamma_1\zeta(t)E(t) - \gamma_2(\beta' \circ \nabla_g u)(t) \leq -\gamma_1\zeta(t)E(t) - 2\gamma_2 E'(t),\ \forall t \geq 0,$$

which implies

$$\big(\zeta(t)\mathcal{L}(t) + 2\gamma_2 E(t)\big)' - \zeta'(t)\mathcal{L}(t) \leq -\gamma_1\zeta(t)E(t),\ \forall t \geq 0.$$

Using the fact that $\zeta'(t) \leq 0$ for any $t \geq 0$ and letting

$$\mathcal{F}(t) = \zeta(t)\mathcal{L}(t) + 2\gamma_2 E(t) \sim E(t), \tag{3.15}$$

we obtain

$$\mathcal{F}'(t) \leq -\gamma_1\zeta(t)E(t) \leq -k\zeta(t)\mathcal{F}(t), \quad \forall t \geq 0. \tag{3.16}$$

A simple integration of (3.16) over $(0,t)$ leads to

$$\mathcal{F}(t) \leq \mathcal{F}(0)e^{-k\int_0^t \zeta(s)ds}, \quad \forall t \geq 0. \tag{3.17}$$

Consequently, (2.5) can be obtained by (3.15) and (3.17). The proof of Theorem 2.2 for $|\mu_2| < \mu_1$ is completed. □

3.2. Exponential stability for $|\boldsymbol{\mu_2}| = \boldsymbol{\mu_1}$

In this subsection, we assume that $\mu_1 = |\mu_2| = \mu$. As we will see, we cannot directly perform the same proof as for the case $|\mu_2| < \mu_1$.

Lemma 3.3. *Let β satisfy $(G1)$. Then for all regular solutions of problem* (1.1)*, the energy functional defined by* (2.5) *is non-increasing and satisfies*

$$E'(t) \leq \frac{1}{2}(\beta' \circ \nabla_g u)(t) - \frac{1}{2}\beta(t)\int_\Omega |\nabla_g u(t)|_g^2 dx \leq 0, \quad \forall t \geq 0. \tag{3.18}$$

Proof. The proof is an immediate consequence of Theorem 3.1 by choosing $\xi = \tau\mu$. □

Set

$$\chi(t) = -\int_\Omega u_t \int_0^t \beta(t-s)\big(u(t)-u(s)\big)dsdx. \tag{3.19}$$

Then we have:

Lemma 3.4. *Under the assumption $(G1)$, the functional $\chi(t)$ satisfies, along the solution, the estimate*

$$\begin{aligned}
\chi'(t) \leq\ & \big(\delta_2 + 2\delta_2(1-l)^2\big)\int_\Omega |\nabla_g u|_g^2 dx + \big(\delta_3(1+\mu) \\
& - \int_0^t \beta(s)ds\big)\int_\Omega u_t^2(x,t)dx \\
& + \big(\frac{1-l}{2\delta_2} + 2\delta_2(1-l) + \frac{\mu C^2}{4\delta_3} + \frac{\mu C^2}{4\delta_4}\big)(\beta \circ \nabla_g u)(t) \\
& + \mu\delta_4\int_\Omega u_t^2(x,t-\tau)dx - \frac{g(0)C^2}{4\delta_3}(\beta' \circ \nabla_g u)(t),
\end{aligned} \tag{3.20}$$

where ε_2, δ_2, δ_3 and δ_4 are arbitrary positive constants.

Proof. Differentiate (3.20) with respect to t, we have

$$\begin{aligned}\chi'(t) = &\int_\Omega \Big\langle \nabla_g u, \int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big\rangle_g dx\\ &-\int_\Omega u_t \int_0^t \beta'(t-s)\big(u(t)-u(s)\big)dsdx\\ &-\int_\Omega \Big\langle \int_0^t \beta(t-s)\nabla_g u(s)ds, \int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big\rangle_g dx\\ &-\Big(\int_0^t \beta(s)ds\Big)\int_\Omega u_t^2(x,t)dx\\ &-\mu_1\int_\Omega u_t \int_0^t \beta(t-s)\big(u(t)-u(s)\big)dsdx\\ &-\mu_2\int_\Omega u_t(x,t-\tau)\int_0^t \beta(t-s)\big(u(t)-u(s)\big)dsdx. \end{aligned} \tag{3.21}$$

In what follows we will estimate the right terms of (3.21) one by one. For the right-hand side first term, by Young's inequality and $(G1)$, we obtain for any $\delta_2 > 0$,

$$\begin{aligned}&\Big|\int_\Omega \Big\langle \nabla_g u, \int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big\rangle_g dx\Big|\\ &\le \delta_2\int_\Omega |\nabla_g u|_g^2 dx + \frac{1}{4\delta_2}\int_\Omega\Big(\int_0^t \beta(t-s)|\nabla_g u(s)-\nabla_g u(t)|_g ds\Big)^2 dx\\ &\le \delta_2\int_\Omega |\nabla_g u|_g^2 dx + \frac{1-l}{4\delta_2}(\beta\circ\nabla_g u)(t). \end{aligned} \tag{3.22}$$

For the right side second term, for any $\delta_3 > 0$, we have

$$\begin{aligned}&\int_\Omega u_t \int_0^t \beta'(t-s)\big(u(t)-u(s)\big)dsdx\\ &\le \delta_3\int_\Omega u_t^2(x,t)dx - \frac{\beta(0)C^2}{4\delta_3}(\beta'\circ\nabla_g u)(t), \end{aligned} \tag{3.23}$$

where we also use $\int_\Omega |v|^2dx \le c\int_\Omega |\nabla_g v|_g^2 dx$ for any $v \in H^1_{\Gamma_2}(\Omega)$. Similarly, the fifth term and the sixth can be estimated as follows

$$\begin{aligned}&\int_\Omega u_t \int_0^t \beta(t-s)\big(u(t)-u(s)\big)dsdx\\ &\le \delta_3\int_\Omega u_t^2(x,t)dx + \frac{C^2}{4\delta_3}(\beta\circ\nabla_g u)(t), \end{aligned} \tag{3.24}$$

and

$$\begin{aligned}
&\int_\Omega u_t(x,t-\tau)\int_0^t \beta(t-s)\big(u(t)-u(s)\big)dsdx \\
&\le \delta_4\int_\Omega u_t^2(x,t-\tau)dx+\frac{C^2}{4\delta_4}(\beta\circ\nabla_g u)(t).
\end{aligned}\tag{3.25}$$

For the third term, we have

$$\begin{aligned}
&\int_\Omega \Big\langle \int_0^t \beta(t-s)\nabla_g u(s)ds, \int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big\rangle_g dx \\
&\le \delta_2\int_\Omega \Big|\int_0^t \beta(t-s)\nabla_g u(s)ds\Big|_g^2 dx \\
&\quad+\frac{1}{4\delta_2}\int_\Omega \Big|\int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big|_g^2 dx \\
&\le \delta_2\int_\Omega \Big(\int_0^t \beta(t-s)\big(\big|\nabla_g u(t)-\nabla_g u(s)\big|_g+\big|\nabla_g u(t)\big|_g\big)ds\Big)^2 dx \\
&\quad+\frac{1}{4\delta_2}\int_\Omega \Big|\int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big|_g^2 dx \\
&\le \big(2\delta_2+\frac{1}{4\delta_2}\big)\int_\Omega \Big|\int_0^t \beta(t-s)\nabla_g\big(u(t)-u(s)\big)ds\Big|_g^2 dx \\
&\quad+2\delta_2(1-l)^2\int_\Omega |\nabla_g u(t)|_g^2dx \\
&\le \big(2\delta_2+\frac{1}{4\delta_2}\big)(1-l)(\beta\circ\nabla_g u)(t)+2\delta_2(1-l)^2\int_\Omega |\nabla_g u(t)|_g^2dx.
\end{aligned}\tag{3.26}$$

Inserting the above estimates (3.22)-(3.26) into (3.21), (3.20) is established. □

Proof of Theorem 2.2 for $|\mu_2|=\mu_1$. Define Lyapunov function $\mathscr{L}$ as

$$\mathscr{L}(t):=NE(t)+\epsilon_1 I(t)+\chi(t)+\epsilon_2 K(t),\tag{3.27}$$

where N, ϵ_1 and ϵ_2 are positive real numbers which will be chosen later. We claim that $\mathscr{L}(t)$ and $E(t)$ are equivalent for ϵ_1 and ϵ_2 small enough while N large enough, i.e., there exist two positive constants α_3 and α_4 depending on N, ϵ_1 and ϵ_2 such that

$$\alpha_3 E(t)\le \mathscr{L}(t)\le \alpha_4 E(t),\quad \forall t\ge 0.\tag{3.28}$$

Indeed, we consider the functional

$$H(t)=\epsilon_1 I(t)+\chi(t)+\epsilon_2 K(t)$$

and show that

$$|H(t)|\le CE(t),\quad C>0.\tag{3.29}$$

Using Young's inequality, Poincaré's inequality, we have

$$
\begin{aligned}
|\chi(t)| &= \Big| \int_\Omega u_t \int_0^t \beta(t-s)\big(u(t)-u(s)\big) ds dx \Big| \\
&\le \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{1}{2}\int_\Omega \Big(\int_0^t \beta(t-s)\big(u(t)-u(s)\big)ds\Big)^2 dx \\
&\le \frac{1}{2}\int_\Omega u_t^2(x,t)dx + \frac{1}{2}(1-l)C^2(\beta\circ\nabla_g u)(t). \qquad (3.30)
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&|\epsilon_1 I(t) + \epsilon_2 K(t)| = \Big|\epsilon_1 \int_\Omega u_t u dx\Big| + \Big|\epsilon_2 \int_\Omega \int_0^1 e^{-\tau\rho} z^2(x,\rho,t)d\rho dx\Big| \\
&\le \frac{\epsilon_1}{2}\int_\Omega u_t^2(x,t)dx + \frac{\epsilon_1}{2}\int_\Omega |\nabla_g u|_g^2 dx + \epsilon_2 c \int_\Omega\int_0^1 z^2(x,\rho,t)d\rho dx. \qquad (3.31)
\end{aligned}
$$

By the definition of $E(t)$, (3.30) and (3.31), we get (3.29) for some positive constant C. Now, it is obvious that (3.28) holds by choosing ϵ_1 and ϵ_2 small enough while N large enough.

Since the function β is positive, continuous and $\beta(0) > 0$, then for any $t \ge t_0 > 0$, we have

$$
\int_0^t \beta(s)ds \ge \int_0^{t_0} \beta(s)ds = \beta_0.
$$

Now, using (3.5), (3.13), (3.18) and (3.20), we have

$$
\begin{aligned}
\mathscr{L}'(t) \le{} & \Big\{\epsilon_1(1+\frac{\mu}{4\delta_1}) + \big(\delta_3(1+\mu) - \beta_0 + \frac{\epsilon_2}{\tau}\big)\Big\}\int_\Omega u_t^2(x,t)dx - \epsilon_2 K(t) \\
& + \Big\{\big(\delta_2 + 2\delta_2(1-l)^2\big) - \epsilon_1\big(\frac{l}{2} - 2\mu\delta_1 C\big)\Big\}\int_\Omega |\nabla_g u|_g^2 dx \\
& + \Big(\frac{N}{2} - \frac{\beta(0)C^2}{4\delta_3}\Big)(\beta'\circ\nabla_g u)(t) \\
& + \Big(\frac{\epsilon_1}{4\delta_1} + \mu\delta_4 - \frac{c\epsilon_2}{\tau}\Big)\int_\Omega u_t^2(x,t-\tau)dx \\
& + \Big\{\frac{\epsilon_1(1-l)}{2} + \Big(\frac{1-l}{2\delta_2} + 2\delta_2(1-l) + \frac{\mu C^2}{4\delta_3} + \frac{\mu C^2}{4\delta_4}\Big)\Big\}(\beta\circ\nabla_g u)(t). \qquad (3.32)
\end{aligned}
$$

Now we will choose the constants in (3.32) carefully. Firstly, let us take δ_1 small enough such that

$$
2\mu\delta_1 C \le \frac{l}{4}.
$$

Then, we select δ_3 small enough such that

$$
\delta_3(1+\mu) \le \frac{\beta_0}{2}.
$$

After that, we pick ϵ_2 so small that

$$\frac{\epsilon_2}{\tau} \leq \frac{\beta_0}{8}.$$

Once ϵ_2 is fixed, then we choose δ_4 small so that

$$\mu\delta_4 \leq \frac{\epsilon_2 c}{2\tau}.$$

Further, we take ϵ_1 small that

$$\epsilon_1 < \min\Big\{\frac{32\delta_1}{(4\delta_1+\mu)\beta_0}, \frac{\delta_1\epsilon_2 c}{\tau}\Big\}.$$

Also, let us take δ small so that

$$\delta_2\big(1+2(1-l)^2\big) \leq \frac{\epsilon_1 l}{8}.$$

Finally, we choose N large enough such that

$$N > \frac{\beta(0)C^2}{\delta_3}.$$

Consequently, there exist two positive constant γ_1 such that

$$\mathscr{L}'(t) \leq -\gamma_1 E(t) + \varepsilon\gamma_2(\beta \circ \nabla_g u)(t), \quad \forall t \geq t_0. \tag{3.33}$$

The remaining part of the proof of Theorem 2.2 for $|\mu_2| = \mu_1$ can be obtained following the same steps as in the proof Theorem 2.2 for $|\mu_2| < \mu_1$, so we omit the details. □

References

[1] C. Abdallah, P. Dorato, and R. Byrne, *Delayed positive feedback can stabilize oscillatory system*, Proceedings of the American Control Conference, San Francisco, pp. 3106–3107, 1993.

[2] F. Alabau-Boussouira, P. Cannarsa, and D. Sforza, *Decay estimates for second order evolution equations with memory*, J. Funct. Anal. **254** (2008), no. 5, 1342–1372.

[3] J. A. D. Appleby, M. Fabrizio, B. Lazzari, and D. W. Reynolds, *On exponential asymptotic stability in linear viscoelasticity*, Math. Models Methods Appl. Sci. **16** (2006), no. 10, 1677–1694.

[4] M. M. Cavalcanti, A. Khemmoudj, and M. Medjden, *Uniform stabilization of the damped Cauchy-Ventcel problem with variable coefficients and dynamic boundary conditions*, J. Math. Anal. Appl., **328** (2007), no. 2, 900–930.

[5] M. M. Cavalcanti and H. Oquendo, *Frictional versus viscoelastic damping in a semilinear wave equation*, SIMA J. Control Optim, **42** (2003), no. 4, 1310–1324.

[6] C. M. Dafermos, *An abstract Volterra equation with application to linear viscoelasticity*, J. Differential Equations **7** (1970), 554–589.

[7] R. Datko, J. Lagnese, and M. P. Polis, *An example on the of time delays in boundary feedback stabilization of wave equations*, SIAM J. Control Optim. **24** (1986), 152–156.

[8] R. Gulliver, I. Lasiecka, W. Littman, and R. Triggiani, *The case for differential geometry in the control of single and coupled PDEs: the structural acoustic chamber*, In: Geometric Methods in Inverse Problems and PDE Control. IMA Vol. Math. Appl., vol. 137, pp. 73–181. Springer, New York, 2004.

[9] B. Z. Guo and Z.C. Shao, *On exponential stability of a semilinear wave equation with variable coefficients under the nonlinear boundary feedback*, Nonlinear Anal. **71** (2009), no. 12, 5961–5978.

[10] M. Kirane and B. Said-Houari, *Existence and asymptotic stability of a viscoelastic wave equation with a delay*, Z. Angew. Math. Phys. **62** (2011), no. 6, 1065–1082.

[11] V. Komornik and E. Zuazua, *A direct method for the boundary stabilization of the wave equation*, J. Math. Pures Appl. **60** (1990), no. 1, 33–54.

[12] W. J. Liu, *Asymptotic behavior of solutions of time-delayed Burgers' equation*, Discrete Contin. Dyn. Syst. Ser. B **2** (2002), no. 1, 47–56.

[13] J. E. Mũnoz Rivera and J. Barbosa Sobrinho, *Existence and uniform rates of decay for contact problems in viscoelasticity*, Appl. Anal. **67** (1997), no. 3-4, 175–199.

[14] S. Nicaise and C. Pignotti, *Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks*, SIAM J. Control Optim. **45** (2006), no. 5, 1561–1585.

[15] ———, *Internal and boundary observability estimates for the heterogeneous Maxwells system*, Appl. Math. Optim. **54** (2006), no. 1, 47–70.

[16] ———, *Stabilization of the wave equation with boundary or internal distributed delay*, Differential Integral Equations **21** (2008), no. 9-10, 935–958.

[17] Z. H. Ning, C. X. Shen, and X. P. Zhao, *Stabilization of the wave equation with variable coefficients and a delay in dissipative internal feedback*, J. Math. Anal. Appl. **405** (2013), no. 1, 148–155.

[18] Z. H. Ning, C. X. Shen, X. P. Zhao, H. Li, C. S. Lin, and Y. M. Zhang, *Nonlinear boundary stabilization of the wave equations with variable coefficients and time dependent delay*, Appl. Math. Comput. **232** (2014), 511–520.

[19] J. E. M. Rivera, *Asymptotic behaviour in linear viscoelasticity*, Quart. Appl. Math. **52** (1994), 629–648.

[20] J. E. M. Rivera and A. P. Salvatierra, *Asymptotic behaviour of the energy in partially viscoelastic materials*, Quart. Appl. Math. **59** (2001), 557–578.

[21] I. H. Suh and Z. Bien, *Use of time delay action in the controller design*, IEEE Trans. Autom. Control **25** (1980), 600–503.

[22] C. Q. Xu, S.P. Yung, and L. K. Li, *Stabilization of wave systems with input delay in the boundary control*, ESAIM Control Optim. Calc. Var. **12** (2006), no. 4, 770–785.

[23] P. F. Yao, *On the observability inequalities for exact controllability of wave equations with variable coefficients*, SIAM J. Contr. Optim. **37** (1999), no. 5, 1568–1599.

[24] ———, *Global smooth solutions for the quasilinear wave equation with boundary dissipation*, J. Differential Equations **241** (2007), no. 1, 62–93.

[25] ———, *Boundary controllability for the quasilinear wave equation*, Appl. Math. Optim. **61** (2010), no. 2, 191–233.

[26] ———, *Modeling and Control in Vibrational and Structural Dynamics*, A Differential Geometric Approach, Chapman and Hall/CRC Applied Mathematics and Nonlinear Science Series. CRC Press, Boca Raton, FL, 2011.

Fei Liang
Department of Mathematics
Xi An University of Science and Technology
Xi'an 710054, P. R. China
E-mail address: `liangfei@xust.edu.cn`