Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1443–1455
https://doi.org/10.4134/BKMS.b160618
pISSN: 1015-8634 / eISSN: 2234-3016

REDUCING SUBSPACES OF A CLASS OF MULTIPLICATION OPERATORS

Bin Liu and Yanyue Shi

Abstract. Let $M_{z^N}(N \in \mathbb{Z}_+^d)$ be a bounded multiplication operator on a class of Hilbert spaces with orthogonal basis $\{z^n : n \in \mathbb{Z}_+^d\}$. In this paper, we prove that each reducing subspace of M_{z^N} is the direct sum of some minimal reducing subspaces. For the case that $d = 2$, we find all the minimal reducing subspaces of $M_{z^N}(N = (N_1, N_2), N_1 \neq N_2)$ on weighted Bergman space $A_\alpha^2(\mathbb{B}_2)(\alpha > -1)$ and Hardy space $H^2(\mathbb{B}_2)$, and characterize the structure of $\mathcal{V}^*(z^N)$, the commutant algebra of the von Neumann algebra generated by M_{z^N}.

1. Introduction

Let T be a bounded linear operator on a Hilbert space. If $\mathcal{M}$ is a closed subspace satisfying $T\mathcal{M} \subseteq \mathcal{M}$, then $\mathcal{M}$ is called an *invariant subspace* of T. In addition, if $\mathcal{M}$ also is invariant subspace of T^*, then $\mathcal{M}$ is called a *reducing subspace* of T. Combining the methods in analysis, algebra and geometry, the reducing subspaces of multiplication operators with Blaschke products are characterized. The details can be found in the book [3] and its references.

On the polydisk, the research begins with some special functions. The reducing subspaces of $M_{z_1^n z_2^m}$ are described in [4, 5, 6]. For $p(z_1, z_2) = \alpha z_1^n + \beta z_2^m$ or $z_1^n \overline{z}_2^m$, the reducing subspaces of Toeplitz operator T_p are described in [1, 2, 7]. A reducing subspace $\mathcal{M}$ is called *minimal* if there is no nonzero reducing subspace $\mathcal{N}$ such that $\mathcal{N}$ is a proper subspace of $\mathcal{M}$. For $N_1 \neq N_2$, the results in [6] shows that $M_{z_1^{N_1} z_2^{N_2}}$ has more minimal reducing subspaces on unweighted Bergman space than on the weighted Bergman space in several cases. It is prove that all $L_{n,m} = \overline{\text{span}}\{z_1^{n+lN_1} z_2^{m+lN_2} : l \in \mathbb{Z}_+\}$ are the only minimal reducing subspaces of $M_{z_1^{N_1} z_2^{N_2}}$ on $A_\alpha^2(\mathbb{D}^2)$ with $\alpha > -1$ and $\alpha \neq 0$. While on the unweighted Bergman space $A^2(\mathbb{D}^2)$, $L_{n,m}^* = \overline{\text{span}}\{a z_1^{n+hN_1} z_2^{m+hN_2} + b z_1^{\rho_1(m+hN_2)} z_2^{\rho_2(n+hN_1)}; h = 0, 1, 2, \ldots\}$ $(a, b \in \mathbb{C})$ are all the minimal reducing

Received July 25, 2016; Revised October 12, 2016; Accepted November 29, 2016.
2010 *Mathematics Subject Classification.* Primary 47B35; Secondary 47C15.
Key words and phrases. multiplication operator, reducing subspace, commutant algebra, unit ball.
This research is partially supported by NSFC No. 11201438, 11171315.

subspaces of $M_{z_1^{N_1} z_2^{N_2}}$, if $\rho_1(m) = \frac{(m+1)N_1}{N_2} - 1$ and $\rho_2(n) = \frac{(n+1)N_2}{N_1} - 1$ are nonnegative integers.

Denote by $\mathbb{Z}_+$ and $\mathbb{N}$ the set of all the nonnegative integers and all the positive integers, respectively. For $d \in \mathbb{N}$, write $m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}_+^d$, $z^m = {z_1}^{m_1} {z_2}^{m_2} \cdots {z_d}^{m_d}$, $m! = m_1! m_2! \cdots m_d!$ and $|m| = m_1 + m_2 + \cdots + m_d$.

Let $\mathcal{H}$ be a Hilbert space with the orthogonal basis $\{z^m\}_{m \in \mathbb{Z}_+^d}$, and satisfy that the multiplication operator M_q is bounded for each polynomial q. This kind of space contains a lot of classical spaces, such as weighted Bergman spaces over polydisk $A^2_\alpha(\mathbb{D}^d)$, weighted Bergman spaces over unit ball $A^2_\alpha(\mathbb{B}_d)$, Hardy space over unit ball $H^2(\mathbb{B}_d)$, and so on. Recall that $\mathbb{B}_d = \{z \in \mathbb{C}^d : \sum_{i=1}^d |z_i|^2 < 1\}$ and $S = \{z \in \mathbb{C}^d : \sum_{i=1}^d |z_i|^2 = 1\}$. Denote by $d\sigma$ the Haar measure on S, and by $H(\mathbb{B}_d)$ all the analytic functions on $\mathbb{B}_d$. The Hardy space $H^2(\mathbb{B}_d)$ is defined by

$$H^2(\mathbb{B}_d) = \{f \in H(\mathbb{B}_d) : \lim_{r \to 1^-} \int_S |f(rz)|^2 d\sigma < +\infty\}.$$

Let $dA(z)$ denote the normalized area measure over $\mathbb{B}_d$, and let $dA_\alpha(z) = C_\alpha (1 - |z|^2)^\alpha dA(z)$, where C_α is a constant such that dA_α is normalized. The weighted Bergman space $A^2_\alpha(\mathbb{B}_d)$ is the Hilbert space of all holomorphic functions over $\mathbb{B}_d$, which are square integrable with respect to $dA_\alpha(z)$.

Guo and Huang [3] point that $\mathcal{M}$ is a nonzero reducing subspace for $M_{z^N} = M_{{z_1}^{N_1} {z_2}^{N_2} \cdots {z_d}^{N_d}}$ on the Hilbert space $\mathcal{H}$ if and only if

$$\mathcal{M} = \bigoplus_n [\mathcal{M}_n],$$

where $[\mathcal{M}_n]$ is the closure of the linear span of $\{z^{kN} \mathcal{M}_n\} (k \geq 0)$ and $\mathcal{M}_n$ is a closed linear subspace of $E_n = \overline{\text{span}}\{z^m : M^{*h}_{z^N} M^h_{z^N} z^m = M^{*h}_{z^N} M^h_{z^N} z^n, \forall h \in \mathbb{Z}_+\}$, where $n \in \{m = (m_1, m_2, \ldots, m_d) \in \mathbb{Z}_+^d : 0 \leq m_i < N_i \text{ for some } i\}$.

In this paper, we continue to consider the reducing subspaces of M_{z^N} on $\mathcal{H}$, and prove that every $[\mathcal{M}_n]$ is the direct sum of some minimal reducing subspaces. In particular, on $A^2_\alpha(\mathbb{B}_2)$ and $H^2(\mathbb{B}_2)$, we describe all the minimal reducing subspaces of $M_{z_1^{N_1} z_2^{N_2}}$ with $N_1 \neq N_2$, and characterize the commutant algebra $\mathcal{V}^*(z_1^{N_1} z_2^{N_2})$.

2. The results in general Hilbert space

Let $\mathcal{H}$ be the Hilbert space defined in above section, and

$$\Omega = \{n = (n_1, n_2, \ldots, n_d) \in \mathbb{Z}_+^d : 0 \leq n_i < N_i \text{ for some } i\}.$$

Define an equivalence on Ω by

$$q \sim n \Leftrightarrow \frac{\gamma_{q+hN}}{\gamma_q} = \frac{\gamma_{n+hN}}{\gamma_n}, \forall h \geq 1,$$

where $\gamma_m = \| z^m \|^2$. For $n \in \Omega$, set $\Im_n := \{q \in \Omega : q \sim n\}$ and $\mathcal{H}_n := \overline{\text{span}}\{z^J : J \in \Im_n\}$. Then $\cup_{n \in F} \Im_n = \Omega$ and $\oplus_{n \in F} \mathcal{H}_n = \overline{\text{span}}\{z^J : J \in \Omega\}$, where F is the partition of Ω by the equivalence $\sim$. Let P_m be the orthogonal projection from $\mathcal{H}$ onto $\mathcal{H}_m$. Denote by M the multiplication operator M_{z^N}. It is easy to check that

$$M^*(z^{m+hN}) = \frac{\gamma_{m+hN}}{\gamma_{m+(h-1)N}} z^{m+(h-1)N}$$
$$M^{*h}M^h z^m = \frac{\gamma_{m+hN}}{\gamma_m} z^m$$

for any $m \in \Im_n$ and $h \in \mathbb{N}$. For $n \in \Omega$, denote by $\widetilde{P}_n$ the orthogonal projection from $\mathcal{H}$ onto $\overline{\text{span}}\{z^J : \frac{\gamma_{J+hN}}{\gamma_J} = \frac{\gamma_{n+hN}}{\gamma_n}, J \in \mathbb{Z}_+^d, \forall h \in \mathbb{Z}_+\}$. By the spectrum decomposition, we see that $\widetilde{P}_n$ is in the von Neumann algebra generated by M_{z^N}. For every reducing subspace $\mathcal{M}$ of M, denote by $P_{\mathcal{M}}$ the orthogonal projection from $\mathcal{H}$ onto $\mathcal{M}$. Therefore, $\widetilde{P}_n P_{\mathcal{M}} = P_{\mathcal{M}} \widetilde{P}_n$. Since

$$\langle P_{\mathcal{M}} z^m, z^l \rangle = \langle P_{\mathcal{M}} z^m, M z^{l-N} \rangle = \langle P_{\mathcal{M}} M^* z^m, z^{l-N} \rangle = 0$$

for $l \notin \Omega$ and $m \in \Omega$, we have $P_{\mathcal{M}} z^m \in \overline{\text{span}}\{z^J : J \in \Omega\}$ and $P_{\mathcal{M}} z^l \perp \{z^J : J \in \Omega\}$. Therefore, $P_n P_{\mathcal{M}} = P_{\mathcal{M}} P_n$.

In the following, we prove that each nonzero reducing subspace for M_{z^N} always contains a minimal reducing subspace, and every reducing subspace is the direct sum of several minimal reducing subspaces.

Theorem 2.1. *Suppose $\mathcal{M}$ be a nonzero reducing subspace of M on $\mathcal{H}$. Then*

$$\mathcal{M} = \bigoplus_{n \in F} [\mathcal{M}_n] = \bigoplus_{n \in F} \bigoplus_{j=1}^{q_n} [e_{nj}],$$

where $\{e_{nj}\}_{j=1}^{q_n}$ $(1 \le q_n \le +\infty)$ is the orthogonal basis of $\mathcal{M}_n \ne \{0\}$.

Proof. (1) Choose a nonzero function g in $\mathcal{M}$. Let h_0 be the minimal nonnegative integer such that

$$P_\Omega M^{*h_0}(g) \ne 0,$$

where P_Ω is the orthogonal projection from $\mathcal{H}$ onto $\overline{\text{span}}\{z^J : J \in \Omega\}$. Clearly, there exists $n \in \Omega$, such that $f = P_n P_\Omega M^{*h_0} g \ne 0$. In this case, $f = P_n P_{\mathcal{M}} M^{*h_0} g = P_{\mathcal{M}} P_n M^{*h_0} g = \Sigma_{J \in \Im_n} b_J z^J$. Then $f \in \mathcal{M} \cap \mathcal{H}_n$. By $f \in \mathcal{H}_n$, we obtain that

$$M^{*q}(f z^{hN}) = \begin{cases} \frac{\gamma_{n+hN}}{\gamma_{n+(h-q)N}} f z^{(h-q)N} & \text{if } h \ge q \ge 0 \\ 0 & \text{if } q > h \ge 0. \end{cases}$$

Moreover, $M^q(f z^{hN}) = f z^{(h+q)N}$ for $h, q \ge 0$; $f z^{h_1 N} \perp f z^{h_2 N}$ with $h_1 \ne h_2$, since

$$\begin{aligned} \langle f z^{h_1 N}, f z^{h_2 N} \rangle &= \langle M^{h_1} f, M^{h_2} f \rangle \\ &= \begin{cases} \frac{\gamma_{n+(h_1-1)N}}{\gamma_{n+(h_1-h_2-1)N}} \langle f z^{(h_1-h_2-1)N}, M^* f \rangle, & \text{if } h_1 > h_2 \ge 0 \\ \frac{\gamma_{n+(h_2-1)N}}{\gamma_{n+(h_2-h_1-1)N}} \langle M^* f, f z^{(h_2-h_1-1)N} \rangle, & \text{if } h_2 > h_1 \ge 0 \end{cases} \end{aligned}$$

$$= 0.$$

Thus, we conclude that $[f] = \overline{\mathrm{span}}\{fz^{hN} : h \in \mathbb{Z}_+\} = \bigoplus_{h=0}^{+\infty} \mathrm{span}\{fz^{hN}\} \subset \mathcal{M}$ is a reducing subspace of M. It is easy to see that $[f_1] = [f]$ for each $f_1 \in [f]$. Thus $[f]$ is minimal.

(2) Denote by $\mathcal{M}_n = P_n\mathcal{M}$. Notice that $P_n\mathcal{M} \perp P_m\mathcal{M}$ for $m \notin \Im_n$. If $P_n\mathcal{M} \neq \{0\}$, choose an orthogonal basis $\{e_{nj}\}_{j=1}^{q_n} (1 \leq q_n \leq +\infty)$ of $P_n\mathcal{M}$. Notice that $[e_{nj}] \perp [e_{mi}]$ for $(n, j) \neq (m, i)$, since

$$\begin{aligned}
&\langle e_{nj} z^{h_1 N}, e_{mi} z^{h_2 N}\rangle \\
&= \langle M^{h_1} e_{nj}, M^{h_2} e_{mi}\rangle \\
&= \begin{cases} \frac{\gamma_{n+(h_1-1)N}}{\gamma_{n+(h_1-h_2-1)N}} \langle e_{nj} z^{(h_1-h_2-1)N}, M^* e_{mi}\rangle, & \text{if } h_1 > h_2 \geq 0 \\ \frac{\gamma_{m+(h_2-1)N}}{\gamma_{m+(h_2-h_1-1)N}} \langle M^* e_{nj}, e_{mi} z^{(h_2-h_1-1)N}\rangle, & \text{if } h_2 > h_1 \geq 0 \\ \frac{\gamma_{m+hN}}{\gamma_m} \langle e_{nj}, e_{mi}\rangle, & \text{if } h_2 = h_1 = h \geq 0 \end{cases} \\
&= 0.
\end{aligned}$$

By the result in (1), we know that $[e_{nj}] = \bigoplus\limits_{h=0}^{+\infty} \mathbb{C} e_{nj} z^{hN}$ is a minimal reducing subspace of M. Thus $[P_n\mathcal{M}] = \bigoplus\limits_{h=0}^{+\infty} z^{hN} P_n\mathcal{M} = \bigoplus\limits_{h=0}^{+\infty} \bigoplus\limits_{j=1}^{q_n} \mathbb{C} e_{nj} z^{hN} = \bigoplus\limits_{j=1}^{q_n} [e_{nj}]$. So we finish the proof. □

Put $\mathcal{V}^*(z^N)$ the commutant algebra of the von Neumann algebra generated by M_{z^N}. Then $\mathcal{V}^*(z^N)$ is a von Neumann algebra and is the norm closed linear span of its projections. Recall that two reducing subspaces M_1 and M_2 of M_{z^N} are called *unitarily equivalent* if there exists a unitary operator U from M_1 onto M_2 and U commutes with M_{z^N}. One can show that M_1 is unitarily equivalent to M_2 if and only if P_{M_1} and P_{M_2} are equivalent in $\mathcal{V}^*(z^N)$, that is, there is a partial isometry V in $\mathcal{V}^*(z^N)$ such that

$$V^*V = P_{M_1},\ VV^* = P_{M_2}.$$

Proposition 2.2. *Let $n, m \in \Omega$ and e_{nj}, e_{mi} be defined as in Theorem 2.1. Then the following statements hold.*

(i) *$L_n = [z^n]$ and $L_m = [z^m]$ are unitarily equivalent if and only if $n \sim m$;*
(ii) *$[e_{nj}]$ and $[e_{mi}]$ are unitarily equivalent if and only if $n \sim m$.*

Proof. (i) On the one hand, assume that L_n and L_m are unitatily equivalent, then there is a partial isometry $U \in \mathcal{V}^*(z^N)$ such that $U|_{L_n}$ is a unitary operator from L_n onto L_m. Obviously, $UM^*M(z^{n+hN}) = M^*MU(z^{n+hN})$. It follows that

$$\frac{\gamma_{n+(h+1)N}}{\gamma_n} U(z^{n+hN}) = \frac{\gamma_{m+(h+1)N}}{\gamma_m} U(z^{n+hN}).$$

Since $U(z^{n+hN}) \neq 0$, we have $\frac{\gamma_{n+(h+1)N}}{\gamma_{n+hN}} = \frac{\gamma_{m+(h+1)N}}{\gamma_{m+hN}}$ for $h \geq 0$, i.e., $n \sim m$.

On the other hand, if $n \sim m$, let

$$U(z^J) = \begin{cases} \sqrt{\frac{\gamma_n}{\gamma_m}} z^{m+hN}, & \text{if } J = n + hN \\ 0, & \text{if } J \neq n + hN \end{cases}$$

for $h = 0, 1, 2, \ldots$. Then U is a partial isometry on $\mathcal{H}$ and $U|_{L_n}$ is a unitary operator from L_n onto L_m. It is easy to check that $U \in \mathcal{V}^*(z^N)$ by direct calculation.

(ii) Let P_{nj} be the orthogonal projection from $\mathcal{H}$ onto $[e_{nj}]$. Obviously, there is $n_0 \sim n$ such that $\langle e_{nj}, z^{n_0} \rangle \neq 0$, that is, $P_{n_0} P_{nj} \neq 0$. Notice that P_{nj} and P_{n_0} are all minimal projection in $\mathcal{V}^*(z^N)$. As in [7], we have P_{nj} is unitarily equivalent to P_{n_0}. Similarly, there is $m_0 \sim m$ such that P_{mi} is unitarily equivalent to P_{m_0}. Therefore, $[e_{nj}]$ is unitarily equivalent to $[e_{mi}]$ if and only if L_{n_0} is unitarily equivalent to L_{m_0}. By (i), we get the desired result. □

3. The results on $A^2_\alpha(\mathbb{B}_2)$ and $H^2(\mathbb{B}_2)$

In this section, we consider the reducing subspaces of $M_{z_1^{N_1} z_2^{N_2}}$ with $N_1, N_2 \geq 1$ and $N_1 \neq N_2$ on the weighted Bergman space $A^2_\alpha(\mathbb{B}_2)(\alpha > -1)$ and the Hardy space $H^2(\mathbb{B}_2)$. Let $n \in \mathbb{Z}^2_+$. Denote by $(n + hN)! = \prod_{i=1}^{2} (n_i + hN_i)!$ and $|n + hN| = \sum_{i=1}^{2} (n_i + hN_i)$. On $A^2_\alpha(\mathbb{B}_2)$, we have

$$\gamma_{n+hN} = \|z^{n+hN}\|^2_\alpha = \Gamma(\alpha+3)(n+hN)!/\Gamma(\alpha+3+|n+hN|)$$

for $\alpha > -1$. Obviously, $\{z^m/\sqrt{\gamma_m}\}_{m \in \mathbb{Z}^2_+}$ is an orthogonal basis of $A^2_\alpha(\mathbb{B}_2)$. Notice that on the Hardy space $H^2(\mathbb{B}_2)$, $\gamma_{n+hN} = \|z^{n+hN}\|^2 = (n+hN)!/(1 + |n+hN|)! = \Gamma(\alpha+3)(n+hN)!/\Gamma(\alpha+3+|n+hN|)$ with $\alpha = -1$.

By Proposition 2.2, we know that the unitarily equivalent of reducing subspaces is converted to the equivalence of some numbers. So the relevant research on Bergman space $A^2_\alpha(\mathbb{B}_2)$ and that on the Hardy space $H^2(\mathbb{B}_2)$ are similar. In the following, define

$$\gamma_{n+hN} = \Gamma(\alpha+3)(n+hN)!/\Gamma(\alpha+3+|n+hN|)$$

for $\alpha \geq -1$ and $n \in \mathbb{Z}^2_+$.

As in above section, define

$$\Omega = \{(n_1, n_2) \in \mathbb{Z}^2_+ : 0 \leq n_i < N_i \text{ for some } i\},$$

and

$$q \sim n \Leftrightarrow \frac{\gamma_{q+hN}}{\gamma_q} = \frac{\gamma_{n+hN}}{\gamma_n}, \forall h \geq 1$$

for $q, n \in \Omega$. Since

$$\lim_{h\to\infty} \frac{\gamma_{q+hN}}{\gamma_{n+hN}} = \lim_{h\to\infty} \frac{(q+hN)!\Gamma(\alpha+3+|n+hN|)}{(n+hN)!\Gamma(\alpha+3+|q+hN|)} = 1,$$

$q \sim n$ if and only if $\gamma_{q+hN} = \gamma_{n+hN}$, $\forall h \in \mathbb{Z}_+$.

If $m \in \Im_n$, then

$$\frac{\gamma_{n+hN}}{\gamma_{n+(h+1)N}} = \frac{\gamma_{m+hN}}{\gamma_{m+(h+1)N}}, \forall h \in \mathbb{Z}_+. \tag{1}$$

Since $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$, we get

$$\frac{\prod_{j=1}^{N_1+N_2}(\alpha+2+n_1+hN_1+n_2+hN_2+j)}{\prod_{i=1}^{2}\prod_{j=1}^{N_i}(n_i+hN_i+j)} = \frac{\prod_{j=1}^{N_1+N_2}(\alpha+2+m_1+hN_1+m_2+hN_2+j)}{\prod_{i=1}^{2}\prod_{j=1}^{N_i}(m_i+hN_i+j)}.$$

Let $g(\lambda) = \prod_{j=1}^{N_1+N_2}(\alpha+2+n_1+n_2+\lambda(N_1+N_2)+j)\prod_{i=1}^{2}\prod_{j=1}^{N_i}(m_i+\lambda N_i+j) - \prod_{j=1}^{N_1+N_2}(\alpha+2+m_1+m_2+\lambda(N_1+N_2)+j)\prod_{i=1}^{2}\prod_{j=1}^{N_i}(n_i+\lambda N_i+j)$. Obviously, g is a polynomial over $\mathbb{C}$ and $g(h) = 0$ for any $h \in \mathbb{Z}_+$. By fundamental theorem of algebra, $g(\lambda) \equiv 0$ for all $\lambda \in \mathbb{C}$. Set

$E_1 = \{\frac{n_1+j}{N_1} : j = 1, 2, \ldots, N_1\}$; $E_2 = \{\frac{n_2+j}{N_2} : j = 1, 2, \ldots, N_2\}$;
$E_3 = \{\frac{2+\alpha+n_1+n_2+j}{N_1+N_2} : j = 1, 2, \ldots, N_1+N_2\}$;
$F_1 = \{\frac{m_1+j}{N_1} : j = 1, 2, \ldots, N_1\}$; $F_2 = \{\frac{m_2+j}{N_2} : j = 1, 2, \ldots, N_2\}$;
$F_3 = \{\frac{2+\alpha+m_1+m_2+j}{N_1+N_2} : j = 1, 2, \ldots, N_1+N_2\}$.

Therefore,

$$E_1 \sqcup E_2 \sqcup F_3 = F_1 \sqcup F_2 \sqcup E_3. \tag{2}$$

Denote by $\delta = GCD(N_1, N_2)$, then $N_i = \delta q_i$ for $i = 1, 2$ and $GCD(q_1, q_2) = 1$.

Lemma 3.1. *Let $\alpha \geq -1$, $n, m \in \Omega$ such that $n \sim m$ and $n \neq m$. Then $n_1 + n_2 = m_1 + m_2$ or $n_1 + n_2 = m_1 + m_2 \pm 1$.*

Proof. Without lose of generality, assume $n_1+n_2 > m_1+m_2+1$ and $n_1 > m_1$. Denote by $\widetilde{E}_i = E_i \setminus F_i$ and $\widetilde{F}_i = F_i \setminus E_i$ for $i = 1, 2, 3$. Then $\widetilde{E}_i \cap \widetilde{F}_i = \emptyset$ and

$$\widetilde{E}_1 \sqcup \widetilde{E}_2 \sqcup \widetilde{F}_3 = \widetilde{F}_1 \sqcup \widetilde{F}_2 \sqcup \widetilde{E}_3. \tag{3}$$

Clearly, $\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1, \frac{1+\alpha+n_1+n_2}{N_1+N_2}+1 \in \widetilde{E}_1 \sqcup \widetilde{E}_2$ and $\frac{3+\alpha+m_1+m_2}{N_1+N_2}, \frac{4+\alpha+m_1+m_2}{N_1+N_2} \in \widetilde{F}_1 \sqcup \widetilde{F}_2$. Furthermore, for $i, j \in \{1, 2\}$ we claim that

(a) if $\frac{2+\alpha+n_1+n_2}{N_1+N_2} + 1 \in \widetilde{E}_i$, then $\frac{1+\alpha+n_1+n_2}{N_1+N_2} + 1 \in \widetilde{E}_j$ for $j \neq i$.

(b) if $\frac{3+\alpha+m_1+m_2}{N_1+N_2} \in \widetilde{F}_i$, then $\frac{4+\alpha+m_1+m_2}{N_1+N_2} \in \widetilde{F}_j$ for $j \neq i$.

In fact, if $\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1, \frac{1+\alpha+n_1+n_2}{N_1+N_2}+1 \in \widetilde{E}_i$ for some $i \in \{1,2\}$, there are integers $1 \leq p_i, q_i \leq N_i$ such that

$$\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1=\frac{n_i+p_i}{N_i}$$
$$\frac{1+\alpha+n_1+n_2}{N_1+N_2}+1=\frac{n_i+q_i}{N_i}.$$

Then $0 \neq \frac{1}{N_1+N_2}=\frac{p_i-q_i}{N_i}>\frac{p_i-q_i}{N_1+N_2} \geq \frac{1}{N_1+N_2}$, which is a contradiction. So (a) holds. Since the proof of (a) and (b) are similar, we omit the details of (b).

Next, we find the contradictions for three cases respectively.

(1) If $m_2>n_2$, then $\min \widetilde{F}_2>\max \widetilde{E}_2$. Since one of $\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1$ and $\frac{1+\alpha+n_1+n_2}{N_1+N_2}+1$ is in $\widetilde{E}_2$, $\lambda>\max \widetilde{E}_2 \geq \frac{1+\alpha+n_1+n_2}{N_1+N_2}+1>\max \widetilde{F}_3$ for $\lambda \in \widetilde{F}_2$. It means that $\widetilde{F}_3 \cap \widetilde{F}_2=\emptyset$, which is contradict with (b).

(2) If $m_2=n_2$, then $E_2=F_2$. Equality (3) implies that $\widetilde{F}_3=\widetilde{F}_1$, which is also contradict with (b).

(3) If $m_2<n_2$, we consider the maximum of equality (3), we have

$$\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1=\max\{\frac{n_1}{N_1}+1, \frac{n_2}{N_2}+1\}.$$

If $\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1=\frac{n_1}{N_1}+1 \in \widetilde{E}_1$, then $\frac{2+\alpha+n_2}{N_2}=\frac{n_1}{N_1}$. Since $\frac{1+\alpha+n_1}{N_1} \geq \frac{n_1}{N_1}=\frac{2+\alpha+n_2}{N_2}>\frac{n_2}{N_2}$, we have $\frac{1+\alpha+n_1+n_2}{N_1+N_2}+1 \notin \widetilde{E}_2$, which contradicts (a).

If $\frac{2+\alpha+n_1+n_2}{N_1+N_2}+1=\frac{n_2}{N_2}+1 \in \widetilde{E}_2$, by the symmetry of n_1 and n_2, we get $\frac{1+\alpha+n_1+n_2}{N_1+N_2}+1 \notin \widetilde{E}_1$, which also contradicts (a). So we finish the proof. $\square$

Lemma 3.2. *Let $\alpha \geq -1$, $n, m \in \Omega$ and $n \neq m$. Suppose $n_1+n_2=m_1+m_2$, then $n \sim m$ if and only if $n \in \Delta_1 \cup \widetilde{\Delta}_1$, where $\Delta_1=\{(kq_1, kq_2-1): 1 \leq k \leq \delta, k \in \mathbb{N}\}$ and $\widetilde{\Delta}_1=\{(kq_1-1, kq_2): 1 \leq k \leq \delta, k \in \mathbb{N}\}$.*

Proof. The sufficiency is easy to check, we only show the proof of necessity. If $n_1+n_2=m_1+m_2$, then $E_3=F_3$ and $E_1 \sqcup E_2=F_1 \sqcup F_2$. Since $n \neq m$, we have $n_1 \neq m_1$. Without lose of generality, let $n_1>m_1$, then $n_2<m_2$. Eq. $E_1 \sqcup E_2=F_1 \sqcup F_2$ shows that

$$\max\{\frac{n_1}{N_1}, \frac{n_2}{N_2}\}=\max\{\frac{m_1}{N_1}, \frac{m_2}{N_2}\};$$
$$\min\{\frac{n_1+1}{N_1}, \frac{n_2+1}{N_2}\}=\min\{\frac{m_1+1}{N_1}, \frac{m_2+1}{N_2}\}.$$

Thus

$$\begin{cases} \frac{n_1}{N_1}=\frac{m_2}{N_2} \\ \frac{m_1+1}{N_1}=\frac{n_2+1}{N_2}. \end{cases}$$

It follows that $\frac{m_1-n_1+1}{N_1}=\frac{n_2-m_2+1}{N_2}$. Since $m_1-n_1=n_2-m_2$ and $N_1 \neq N_2$, we get $m_1-n_1+1=n_2-m_2+1=0$. Further, $\frac{n_1}{N_1}=\frac{m_2}{N_2}$ implies that $m_2=n_2+1=\frac{q_2}{q_1} n_1$. Thus there exists k such that $n_1=kq_1$. Then $m_2=kq_2$,

$n_2 = kq_2 - 1$ and $m_1 = kq_1 - 1$. To satisfy $n, m \in \Omega$, there is a confine that $k \in \mathbb{Z}_+$ and $1 \leq k \leq \delta$. That is, $m = n + (-1, 1) \in \widetilde{\Delta}_1$ for $n \in \Delta_1$. $\square$

If $n_1+n_2 = m_1+m_2\pm 1$, there are three cases: (i) $n_1 = m_1$; (ii) $n_2 = m_2$; (iii) $n_1 \neq m_1$ and $n_2 \neq m_2$. We give the characterization of n and m, respectively.

Lemma 3.3. *Let $\alpha \geq -1$, $n \in \Omega$. There is $m \in \Omega$ such that $m \sim n$ and $m \neq n$. Then the following statements hold.*

- (i) *If $n_1 = m_1$, then $\alpha \in \mathbb{Q}$ and there is an integer $0 \leq i_0 < q_1$ such that $\frac{2+\alpha+i_0}{q_1} q_2 \in \mathbb{Z}_+$. In this case, $(n, m) \in \Delta_2 \cup \widetilde{\Delta}_2$, where $\Delta_2 = \{(kq_1 - 2 - \alpha, kq_2) : k = \frac{2+\alpha+i_0}{q_1} + i, 0 \leq i \leq \delta - 1\}$ and $\widetilde{\Delta}_2 = \{(kq_1 - 2 - \alpha, kq_2 - 1) : k = \frac{2+\alpha+i_0}{q_1} + i, 0 \leq i \leq \delta - 1\}$.*
- (ii) *If $n_2 = m_2$, then $\alpha \in \mathbb{Q}$ and there is integer $0 \leq j_0 < q_2$ such that $\frac{2+\alpha+j_0}{q_2} q_1 \in \mathbb{Z}_+$. In this case, $(n, m) \in \Delta_3 \cup \widetilde{\Delta}_3$, where $\Delta_3 = \{(kq_1, kq_2 - 2 - \alpha) : k = \frac{2+\alpha+j_0}{q_2} + j, 0 \leq j \leq \delta - 1\}$ and $\widetilde{\Delta}_3 = \{(kq_1 - 1, kq_2 - 2 - \alpha) : k = \frac{2+\alpha+j_0}{q_2} + j, 0 \leq j \leq \delta - 1\}$.*
- (iii) *If $n_1 \neq m_1$ and $n_2 \neq m_2$, then $\alpha \in \mathbb{N}$ and $q_1, q_2 \in \{1 + \alpha, 1\}$. Furthermore,*
  - (a) *if $q_1 = 1$, then $q_2 = 1 + \alpha$ and $(n, m) \in \Delta_4 \cup \widetilde{\Delta}_4$, where $\Delta_4 = \{(kq_1, kq_2 - 2 - \alpha) : 2 \leq k \leq \delta + 1, k \in \mathbb{N}\}$ and $\widetilde{\Delta}_4 = \{(kq_1 - 2, kq_2 - 1 - \alpha) : 2 \leq k \leq \delta + 1, k \in \mathbb{N}\}$.*
  - (b) *if $q_2 = 1$, then $q_1 = 1 + \alpha$ and $(n, m) \in \Delta_5 \cup \widetilde{\Delta}_5$, where $\Delta_5 = \{(kq_1 - 2 - \alpha, kq_2) : 2 \leq k \leq \delta + 1, k \in \mathbb{N}\}$ and $\widetilde{\Delta}_5 = \{(kq_1 - 1 - \alpha, kq_2 - 2) : 2 \leq k \leq \delta + 1, k \in \mathbb{N}\}$.*

Proof. By Lemma 3.1, we assume $n_1 + n_2 = m_1 + m_2 + 1$, or else exchanging (n_1, n_2) and (m_1, m_2). Therefore,

$$\widetilde{E}_1 \sqcup \widetilde{E}_2 \sqcup \{\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2}\} = \widetilde{F}_1 \sqcup \widetilde{F}_2 \sqcup \{\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} + 1\}. \tag{4}$$

(i) By $n_1 = m_1$, we have $n_2 = m_2 + 1$. Eq. (2) implies that

$$\frac{2 + \alpha + n_1 + n_2}{N_1 + N_2} = \frac{n_2}{N_2},$$

that is $\frac{2+\alpha+n_1}{N_1} = \frac{n_2}{N_2}$. So there exists $k \geq 0$ such that $n_2 = kq_2$, $n_1 = kq_1 - 2 - \alpha$. It follows that $n = (kq_1 - 2 - \alpha, kq_2)$ and $m = n + (0, -1) = (kq_1 - 2 - \alpha, kq_2 - 1) \in \Im_n$. By $n, m \in \Omega$, we have $kq_2 = \frac{2+\alpha+h}{q_1} q_2 \in \mathbb{N}$ for some nonnegative integer $h = i_0 + iq_1 (0 \leq i_0 < q_1, 0 \leq i \leq \delta - 1)$. That is, $k = \frac{2+\alpha+h}{q_1} = \frac{2+\alpha+i_0}{q_1} + i$. Since $0 \leq i_0 < q_1$, the choose of i_0 is unique. So we finish the proof of necessity. The sufficiency is easy to check. So (i) holds.

(ii) By the symmetry of n_1 and n_2, we have the statement (ii) holds.

(iii) First, if $n_1 > m_1$, $n_2 \neq m_2$ implies that $n_2 + 1 \leq m_2$ and $n_1 \geq m_1 + 2$. Considering the maximum and minimum of Eq. (4), it is easy to see

$$\begin{aligned} 1+\frac{n_1}{N_1} &= \max\{1+\frac{2+\alpha+n_1+n_2}{N_1+N_2}, 1+\frac{m_2}{N_2}\}, \\ \frac{m_1+1}{N_1} &= \min\{\frac{2+\alpha+n_1+n_2}{N_1+N_2}, \frac{n_2+1}{N_2}\}. \end{aligned} \tag{5}$$

We claim that

$$1+\frac{n_1}{N_1} = \frac{2+\alpha+n_1+n_2}{N_1+N_2}+1. \tag{6}$$

Or else, assume $\frac{n_1}{N_1}+1 = \frac{m_2}{N_2}+1 > \frac{2+\alpha+n_1+n_2}{N_1+N_2}+1$. Clearly

$$\frac{2+\alpha+m_2}{N_2} > \frac{m_2}{N_2} = \frac{n_1}{N_1} \geq \frac{m_1+1}{N_1}.$$

Therefore,

$$\frac{m_1+1}{N_1} < \frac{3+\alpha+m_1+m_2}{N_1+N_2} = \frac{2+\alpha+n_1+n_2}{N_1+N_2}.$$

So (5) implies that

$$\frac{n_2+1}{N_2} = \frac{m_1+1}{N_1}. \tag{7}$$

Since $n_1+n_2 = m_1+m_2+1$, we get $\frac{m_2-n_2-1}{N_2} = \frac{n_1-m_1-1}{N_1} = \frac{m_2-n_2}{N_1}$. Let $m_2 - n_2 = pq_1$, then $m_2 - n_2 - 1 = pq_2$. That is, $p \in \mathbb{N}$ and $1 = p(q_1 - q_2)$. Therefore, $p = 1$, $q_1 = q_2 + 1$, forcing $N_1 \geq 2$ and $N_1 > N_2$. Then $1 + \frac{n_1-1}{N_1} > 1 + \frac{m_2-1}{N_2}$. The Eq. (4) shows that

$$1+\frac{n_1-1}{N_1} = 1+\frac{2+\alpha+n_1+n_2}{N_1+N_2}. \tag{8}$$

If $\frac{2+\alpha+n_1+n_2}{N_1+N_2} \in \widetilde{F}_2$, then $\mathrm{Card}\widetilde{F}_2 = \mathrm{Card}\widetilde{E}_2 \geq 2$. By $N_1 > N_2$, we have $\frac{n_2+2}{N_2} > \frac{m_1+2}{N_1}$. The equalities (7) and (8) show that $\frac{m_1+2}{N_1} \notin \widetilde{E}_1 \sqcup \widetilde{E}_2 \sqcup \{\frac{2+\alpha+n_1+n_2}{N_1+N_2}\}$, which is a contradiction.

If $\frac{2+\alpha+n_1+n_2}{N_1+N_2} \in \widetilde{F}_1$, then

$$\frac{2+\alpha+n_1+n_2}{N_1+N_2} = \frac{m_1+2}{N_1}. \tag{9}$$

In fact, equality (8) implies that

$$\{z \in \widetilde{F}_1 : z < \frac{2+\alpha+n_1+n_2}{N_1+N_2}\} = \{z \in \widetilde{E}_2 : z < \frac{2+\alpha+n_1+n_2}{N_1+N_2}\}. \tag{10}$$

Since $N_1 \neq N_2$, we have $\mathrm{Card}\{z \in \widetilde{F}_1 : z < \frac{2+\alpha+n_1+n_2}{N_1+N_2}\} = 1$. So (9) holds.

Combining (8) and (9), we get $n_1 = m_1 + 3$. It means that $q_1 = m_2 - n_2 = 2$ and $q_2 = 1$. By (7) and (9), we have $\frac{2+\alpha}{N_2} = \frac{1}{N_1}$, i.e., $2(2+\alpha) = 1$, which is contradict with $\alpha > -1$. So we get the claim.

By (6), there is

$$\frac{2+\alpha+n_2}{N_2} = \frac{n_1}{N_1}. \tag{11}$$

It follows that

$$\begin{cases} n_1 = kq_1 \\ n_2 = kq_2 - 2 - \alpha \end{cases} \tag{12}$$

for some $k \geq 2+\alpha$. Therefore,

$$\frac{2+\alpha+n_1+n_2}{N_1+N_2} = \frac{k}{\delta} > \frac{kq_2-(1+\alpha)}{\delta q_2} = \frac{n_2+1}{N_2}. \tag{13}$$

Then Eq. (5) deduces that

$$\frac{m_1+1}{N_1} = \frac{n_2+1}{N_2}, \text{ i.e., } m_1+1 = kq_1 - \frac{(1+\alpha)q_1}{q_2}. \tag{14}$$

If $N_1 = 1$, then $N_2 > 1$. Since $m_2 - n_2 > 1$, we have $\frac{n_2+2}{N_2} \in \widetilde{E_2}$, but $\frac{n_2+2}{N_2} \notin \widetilde{F}_1 \sqcup \widetilde{F}_2 \sqcup \widetilde{E}_3$, which is a contradiction.

If $N_1 > 1$, then

$$\max\{1+\frac{n_1-1}{N_1}, \frac{2+\alpha+n_1+n_2}{N_1+N_2}\} = 1+\frac{m_2}{N_2}.$$

By Eq. (11), we have $\frac{2+\alpha+n_1+n_2}{N_1+N_2} < \frac{n_1-1}{N_1}+1$. Therefore,

$$\frac{n_1-1}{N_1} = \frac{m_2}{N_2}.$$

It follows that $m_2 = kq_2 - \frac{q_2}{q_1} \in \mathbb{Z}_+$. Combining $n_1+n_2 = m_1+m_2+1$ with (12), (14) and $N_1 \neq N_2$, we conclude that $\frac{q_2}{q_1} = 1+\alpha$. Therefore, $q_2 = 1+\alpha$, $q_1 = 1$, and $\alpha \in \mathbb{N}$. In this case, $(m_1, m_2) = (n_1-2, n_2+1)$ and $(n,m) \in \Delta_4$.

Next, if $n_1 < m_1$, we have $n_2 > m_2+1$ and $n_1+1 < m_1$. Since n_1 and n_2 are symmetric; m_1 and m_2 are symmetric, it is easy to check that $q_2 = 1$, $q_1 = 1+\alpha$, and $(n,m) \in \Delta_5$. So (iii) holds. □

Remark 3.4. In above lemma, the number k in condition (i) and (ii) is not always an integer. If n and m satisfy one of the conditions (i), (ii) and (iii), then $n \sim m$ and $n \neq m$.

Notice that $\Delta_1 \neq \emptyset$ and does not change with α. However, $\{\Delta_i\}(i=2,3,4,5)$ heavily depend on the α, and some of them may be empty. By careful computation, we know that each two of $\{\Delta_i, \widetilde{\Delta}_i : i = 1, \ldots, 5\}$ are either equal or disjoint. Therefore, we assert that the Card of $\Im_n$ heavily depend on the α.

For the case that $\alpha = -1$, it is easy to see that $\Delta_4 = \Delta_5 = \emptyset$, $\widetilde{\Delta}_1 = \Delta_2$ and $\widetilde{\Delta}_2 = \widetilde{\Delta}_3$. So we have the following result.

Lemma 3.5. *If $\alpha = -1$, then $\Im_n \neq \{n\}$ if and only if*

$$\Im_n = \{(kq_1, kq_2 - 1), (kq_1 - 1, kq_2), (kq_1 - 1, kq_2 - 1)\}$$

for some $1 \leq k \leq GCD(N_1, N_2)$.

For the case that $\alpha > -1$, we have the following statements hold.

1° If $\alpha \in (-1, +\infty) \setminus \mathbb{Q}$, then $\Delta_i = \emptyset$ for $i = 2, 3, 4, 5$. Therefore, $\mathrm{Card}\Im_n \neq 1$ if and only if $\mathrm{Card}\Im_n = 2$ for $n \in \Delta_1 \cup \widetilde{\Delta}_1$.

2° If $\alpha \in (\mathbb{Q} \cap (-1, +\infty)) \setminus \mathbb{Z}_+$, then $\Delta_4 = \Delta_5 = \emptyset$. Therefore, $\mathrm{Card}\Im_n \neq 1$ if and only if $\mathrm{Card}\Im_n = 2$, and $n \in \Delta_1 \cup \widetilde{\Delta}_1 \cup \Delta_2 \cup \widetilde{\Delta}_2 \cup \Delta_3 \cup \widetilde{\Delta}_3$. Moreover, Δ_2 and Δ_3 are not non-empty sets at the same time. In fact, let $\alpha = \frac{q}{p}$ where $p, q \in \mathbb{Z}$, $p > 1$, $q > -p$ and $\mathrm{GCD}(p, |q|) = 1$. By $\frac{2+\alpha+i_0}{q_1} q_2 \in \mathbb{Z}_+$, it is easy to see $(2 + \alpha + i_0)q_2 = \frac{(2+i_0)p+q}{p} q_2 \in \mathbb{Z}_+$. Since $\mathrm{GCD}(p, |q|) = 1$, we have $\mathrm{GCD}((2 + i_0)p + q, p) = 1$. So $p \mid q_2$. Similar, $\frac{2+\alpha+j_0}{q_2} q_1 \in \mathbb{Z}_+$ implies that $p \mid q_1$. Thus we get $p = 1$, which is a contradiction.

3° If $\alpha \in \mathbb{Z}_+$, then Δ_2 and Δ_3 are not empty.

(1) If $N_2 \neq (1 + \alpha)N_1$ and $N_1 \neq (1 + \alpha)N_2$, then $\delta_4 = \delta_5 = \emptyset$. Therefore, $\mathrm{Card}\Im_n \neq 1$ if and only if $\mathrm{Card}\Im_n = 2$, for $n \in \Delta_1 \cup \widetilde{\Delta}_1 \cup \Delta_2 \cup \widetilde{\Delta}_2 \cup \Delta_3 \cup \widetilde{\Delta}_3$.

(2) If $N_2 = (1 + \alpha)N_1, \alpha \neq 0$, then $\Delta_5 = \emptyset$, $\Delta_1 = \widetilde{\Delta}_3$, $\widetilde{\Delta}_1 = \widetilde{\Delta}_4$ and $\Delta_3 = \Delta_4$. $\mathrm{Card}\Im_n \neq 1$ if and only if $\mathrm{Card}\Im_n = 2$ or $\mathrm{Card}\Im_n = 3$. Moreover, $\mathrm{Card}\Im_n = 2$ if and only if $n \in \Delta_2 \cup \widetilde{\Delta}_2$; $\mathrm{Card}\Im_n = 3$ if and only if $n \in \Delta_1 \cup \widetilde{\Delta}_1 \cup \Delta_3$. In this case, $n \sim n + (-1, 1) \sim n + (1, 0)$ for $n \in \Delta_1$.

(3) If $N_1 = (1 + \alpha)N_2, \alpha \neq 0$, then $\Delta_4 = \emptyset$, $\widetilde{\Delta}_1 = \widetilde{\Delta}_2$, $\Delta_2 = \Delta_5$ and $\widetilde{\Delta}_5 = \Delta_1$. $\mathrm{Card}\Im_n \neq 1$ if and only if $\mathrm{Card}\Im_n = 2$ or $\mathrm{Card}\Im_n = 3$. Moreover, $\mathrm{Card}\Im_n = 2$ if and only if $n \in \Delta_3 \cup \widetilde{\Delta}_3$; $\mathrm{Card}\Im_n = 3$ if and only if $n \in \Delta_1 \cup \widetilde{\Delta}_1 \cup \Delta_2$. In this case, $n \sim n + (-1, 1) \sim n + (-1, 2)$ for $n \in \Delta_1$.

Combining above analysis and the results in section two, we have the following results. Recall that $\delta = GCD(N_1, N_2)$.

Theorem 3.6. *On the Bergman space $A^2_\alpha(\mathbb{B}_2)$ with $\alpha \in (-1, +\infty) \setminus \mathbb{Q}$, $\mathcal{V}^*(z^N)$ is $*$-isomorphic to*

$$\bigoplus_{i=1}^{\delta} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C},$$

where $N = (N_1, N_2)$ and $N_1 \neq N_2$.

Theorem 3.7. *On the Bergman space $A^2_\alpha(\mathbb{B}_2)$ with $\alpha \in (\mathbb{Q} \cap (-1, +\infty)) \setminus \mathbb{Z}_+$, $\mathcal{V}^*(z^N)$ is $*$-isomorphic to*

$$\bigoplus_{i=1}^{s} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C},$$

where $s \in \{\delta, 2\delta\}$, *where* $N = (N_1, N_2)$ *and* $N_1 \neq N_2$.

Example 3.8. Let $\alpha = \frac{2}{5}$, $N_1 = 6$, $N_2 = 9$. Then $\Delta_1 = \{(2,2),(4,5),(6,8)\}$, $\Delta_2 = \Delta_3 = \Delta_4 = \Delta_5 = \emptyset$. So on the Bergman space $A^2_{\frac{2}{5}}(\mathbb{B}_2)$, $\mathcal{V}^*(z_1^6 z_2^9)$ is $*$-isomorphic to

$$\bigoplus_{i=1}^{3} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C}.$$

Example 3.9. Let $\alpha = \frac{2}{3}$, $N_1 = 6$, $N_2 = 9$. It is easy to check that (1)$\Delta_1 = \{(2,2),(4,5),(6,8)\}$; (2) $\Delta_3 = \Delta_4 = \Delta_5 = \emptyset$; (3) $\Delta_2 = \{(0,4),(2,7),(4,10)\}$ with $k = 1 + \frac{1}{3}, 2 + \frac{1}{3}, 3 + \frac{1}{3}$, respectively. Then on the Bergman space $A^2_{\frac{2}{3}}(\mathbb{B}_2)$, $\mathcal{V}^*(z_1^6 z_2^9)$ is $*$-isomorphic to

$$\bigoplus_{i=1}^{6} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C}.$$

Theorem 3.10. *Let* $N = (N_1, N_2)$ *and* $N_1 \neq N_2$. *On the Bergman space* $A^2_\alpha(\mathbb{B}_2)$ *with* $\alpha \in \mathbb{Z}_+$, *the following statements hold*:

(i) *if* $N_1 \neq (1+\alpha)N_2$ *and* $N_2 \neq (1+\alpha)N_1$, *then* $\mathcal{V}^*(z^N)$ *is* $*$*-isomorphic to*

$$\bigoplus_{i=1}^{3\delta} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C};$$

(ii) *if* $N_1 = (1+\alpha)N_2$ *or* $N_2 = (1+\alpha)N_1$, *then* $\mathcal{V}^*(z^N)$ *is* $*$*-isomorphic to*

$$\bigoplus_{i=1}^{\delta} M_3(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{\delta} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C}.$$

Example 3.11. If $\alpha = 4$, $N_1 = 6$, $N_2 = 9$, then $\Delta_1 = \{(2,2),(4,5),(6,8)\}$, $\Delta_2 = \{(0,9),(2,12),(4,15)\}$, $\Delta_3 = \{(4,0),(6,3),(8,6)\}$, $\Delta_4 = \Delta_5 = \emptyset$. On the Bergman space $A^2_4(\mathbb{B}_2)$, $\mathcal{V}^*(z_1^6 z_2^9)$ is $*$-isomorphic to

$$\bigoplus_{i=1}^{9} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C}.$$

Example 3.12. If $\alpha = 2$, $N_1 = 3$, $N_2 = 9$, then $\Delta_1 = \widetilde{\Delta}_3 = \{(1,2),(2,5),(3,8)\}$, $\Delta_2 = \{(0,12),(1,15),(2,18)\}$, $\Delta_3 = \Delta_4 = \{(2,2),(3,5),(4,8)\}$ and $\Delta_5 = \emptyset$. On the Bergman space $A^2_2(\mathbb{B}_2)$, $\mathcal{V}^*(z_1^6 z_2^9)$ is $*$-isomorphic to

$$\bigoplus_{i=1}^{3} M_3(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{3} M_2(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C}.$$

Theorem 3.13. *On the Hardy space* $H^2(\mathbb{B}_2)$, $\mathcal{V}^*(z^N)$ *is* $*$*-isomorphic to*

$$\bigoplus_{i=1}^{\delta} M_3(\mathbb{C}) \bigoplus \bigoplus_{i=1}^{+\infty} \mathbb{C},$$

where $N = (N_1, N_2)$ *and* $N_1 \neq N_2$.

Acknowledgments. The authors thank the reviewer very much for his helpful suggestions which led to the present version of this paper.

References

[1] M. Albaseer, Y. Lu, and Y. Shi, *Reducing subspaces for a class of Toeplitz Operators on the Bergman space of the bidisk*, Bull. Korean Math. Soc. **52** (2015), no. 5, 1649–1660.

[2] H. Dan and H. Huang, *Multiplication operators defined by a class of polynomials on* $L_a^2(\mathbb{D}^2)$, Integral Equations Operator Theory **80** (2014), no. 4, 581–601.

[3] K. Guo and H. Huang, *Multiplication operators on the Bergman space*, Lecture Notes in Mathematics, 2145, Springer, 2015.

[4] Y. Lu and X. Zhou, *Invariant subspaces and reducing subspaces of weighted Bergman space over bidisk*, J. Math. Soc. Japan **62** (2010), no. 3, 745–765.

[5] L. Shan, *Reducing subspaces for a class of analytic Toeplitz operators on the bidisc*, J. Fudan Univ. Nat. Sci. **42** (2003), no. 2, 196–200.

[6] Y. Shi and Y. Lu, *Reducing subspaces for Toeplitz operators on the polydisk*, Bull. Korean Math. Soc. **50** (2013), no. 2, 687–696.

[7] X. Wang, H. Dan, and H. Huang, *Reducing subspaces of multiplication operators with the symbol* $\alpha z^k + \beta w^l$ *on* $L_a^2(\mathbb{D}^2)$, Sci. China Math. **58** (2015), no. 10, 2167–2180.

Bin Liu
School of Mathematical Sciences
Ocean University of China
Qingdao 266100, P. R. China
E-mail address: liubin_taixi@163.com

Yanyue Shi
School of Mathematical Sciences
Ocean University of China
Qingdao 266100, P. R. China
E-mail address: shiyanyue@163.com