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MATRICES OF TOEPLITZ OPERATORS ON HARDY SPACES OVER BOUNDED DOMAINS

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Abstract. We compute explicitly the matrix represented by the Toeplitz operator on the Hardy space over a smoothly finitely connected bounded domain in the plane with respect to special orthonormal bases consisting of the classical kernel functions for the space of square integrable functions and for the Hardy space. The Fourier coefficients of the symbol of the Toeplitz operator are obtained from zeroth row vectors and zeroth column vectors of the matrix. And we also find some condition for the product of two Toeplitz operators to be a Toeplitz operator in terms of matrices.

1. Introduction

One of the reasons that theory about Toeplitz operators on the Hardy space over the unit disc U has been well-developed so far is that the set $\mathcal{A} = \{z^k \mid k \in \mathbb{Z}\}$ and the subset $\mathcal{B} = \{z^k \mid k \geq 0\}$ of monomials are orthonormal bases for the L^2 space $L^2(bU)$ and the Hardy space $H^2(bU)$ respectively under the normalized Lebesgue measure. In particular Brown and Halmos classified Toeplitz operators completely in terms of Toeplitz matrices in the case of the unit disc (See [4]). When $\varphi \in L^\infty(bU)$ is the symbol of the Toeplitz operator T_φ on $H^2(bU)$ represented by the Fourier series $\varphi(z) = \sum_{k=-\infty}^{\infty} \alpha_k z^k$ with $\alpha_k = \langle \varphi, z^k \rangle$, the matrix $[T_\varphi]$ of T_φ with respect to the orthonormal basis $\mathcal{B}$ is easily formulated as the identity

$$[T_\varphi]_{ml} = \alpha_{m-l}.$$

For general (even simply connected) domains, computation of the matrix is not that simple. The author in [5], for any C^∞ smoothly finitely connected bounded domain Ω, constructed a corresponding orthonormal basis $\mathcal{A}_\Omega$ for $L^2(b\Omega)$ explicitly which consists of the Szegő kernel, the Garabedian kernel and the Ahlfors map. In this case, the matrix $[T_\varphi]$ with respect to the basis $\mathcal{A}_\Omega$ turned out to be a Toeplitz matrix of order n where n is the finite connectivity of Ω. However recapturing the Fourier coefficients of the symbol φ from the

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matrix $[T_\varphi]$ is very difficult in general unlike the unit disc. On the other hand, the product of two Toeplitz operators is not a Toeplitz operator in general even in the unit disc and in fact such a problem depends heavily on symbols .

In this paper, we compute the matrix $[T_\varphi]$ over a bounded domain and expand each entry of the matrix explicitly as an infinite series whose summation runs from $-\infty$ to a specified number so that its corresponding term is non-zero possibly, in order to get a key relation between the symbol and the matrix. It will then turn out that the 0-th row vector and the 0-th column vector of the matrix represent all Fourier coefficients of the symbol of the Toeplitz operator. In the final section we find a necessary condition for the product $T_\varphi T_\psi$of two Toeplitz operators to be a Toeplitz operator in terms of matrices and this condition will imply, for the case of the unit disc, that the former symbol ψ becomes analytic and the latter symbol φ becomes co-analytic as we know.

2. Preliminaries and notes

Throughout the paper, we assume, unless otherwise specified, that Ω is a finitely connected bounded domain in the plane with C^∞ smooth boundary and T is the unit tangent vector function on $b\Omega$ pointing in the direction of the standard orientation of $b\Omega$.

Let $L^2(b\Omega)$ be the space of square integrable functions on $b\Omega$ with the inner product defined by

$$\langle u, v\rangle = \int_{b\Omega} u\,\overline{v}\ ds,$$

where ds is the differential element of arc length on the boundary $b\Omega$ of Ω. Let $H^2(b\Omega)$ denote the space of holomorphic functions on Ω with L^2-boundary values in $b\Omega$ which is called the classical Hardy space. Since $H^2(b\Omega)$ is a closed subspace of $L^2(b\Omega)$, there exists the orthogonal projection of $L^2(b\Omega)$ onto $H^2(b\Omega)$ called the Szegő projection which is denoted by

$$P : L^2(b\Omega) \to H^2(b\Omega).$$

Let $S(z, w)$ be the kernel for the Szegő projection which is called the Szegő kernel that has the reproducing property

$$Pu(a) = \langle u, S(\cdot, a)\rangle = \int_{b\Omega} S(a, z)u(z)ds_z$$

for $u \in L^2(b\Omega)$ and for $a \in \Omega$.

Let $a \in \Omega$ be fixed. Then it is easy to see from the Cauchy integral formula that

$$S(z, a) = P(C_a)(z),$$

where $C_a(z) = \overline{\frac{1}{2\pi i}\frac{T(z)}{z-a}}$ is the Cauchy kernel (where "bar" means the complex conjugate). We need another function $G(z, a)$ which is called the Garabedian kernel defined by

$$G(z,a) = \frac{1}{2\pi(z-a)} + P\left(\overline{iC_aT}\right)(z).$$

In this paper we often use the notations $S_a(z) = S(z,a)$ and $G_a(z) = G(z,a)$ for convenience when the functions $S(z,a)$ and $G(z,a)$ of two variables z and a are thought of functions of the first variable z for the second variable a fixed.

There are many important (but well-known) properties of the Szegő kernel and the Garabedian kernel to which are often referred in this article and so we list some of them. These properties can be found in [1] and [2]. For $a \in \Omega$ fixed, S_a is a holomorphic function in $H^2(b\Omega)$ and for $z \in \Omega$ fixed, $S(z,a)$ is anti-holomorphic in $a \in \Omega$. The function $S(z,a)$ is a C^∞ smooth function up to $\overline{\Omega} \times \overline{\Omega}$ minus the diagonal of the boundary $b\Omega$. For fixed $a \in \Omega$, $G(z,a)$ is a meromorphic function on Ω with a single simple pole at $z = a$ having residue $\frac{1}{2\pi}$ which extends C^∞ smoothly up to the boundary of Ω. Note that $G(z,a)$ never vanishes for all $(z,a) \in \overline{\Omega} \times \Omega$ with $z \neq a$. For a and b in Ω with $a \neq b$, $S(a,b) = \overline{S(b,a)}$ and $G(a,b) = -G(b,a)$. One of the most important properties about the Szegő kernel and the Garabedian kernel with which two kernel functions are interchangeable each other on the boundary is the identity

$$G(z,a) = i\,\overline{S(z,a)}\,\overline{T(z)}, \quad (z,a) \in \mathrm{b}\Omega \times \Omega. \tag{2.1}$$

Now in order to construct an orthonormal basis for L^2-space, we introduce a conformal map which plays a fundamental role such as the identity function on the unit disc. Suppose that Ω is n-connected and let a be in Ω. The function f_a defined on Ω by

$$f_a(z) = \frac{S(z,a)}{G(z,a)}$$

is an n to 1 proper holomorphic function mapping Ω onto the unit disc U which is called the Ahlfors map associated to the pair (Ω, a) (see [6]). The Ahlfors map f_a satisfies the properties $f_a(a) = 0, f_a'(a) > 0$ solving the extremal problem of maximizing $h'(a)$ among all holomorphic functions h mapping Ω into the unit disc making $h'(a)$ real valued. f_a has n zeroes in Ω and in fact, it turned out that for all but finitely many points a of Ω, all other $n-1$ zeroes of f_a become simple zeroes. Since $G(z,a)$ never vanishes in $\overline{\Omega}$ as a function of z for fixed a, all other $n-1$ zeroes are given by zeroes of S_a. (Refer to [2], [3]). For $a \in \Omega$ fixed, let $a = a_0, a_1, a_2, \ldots, a_{n-1}$ be the zeroes of f_a. When $n = 1$, i.e., if Ω is a simply connected domain, the function f_a is just the Riemann mapping function associated to (Ω, a).

For $\varphi \in L^\infty(b\Omega)$, the operator L_φ defined on $L^2(b\Omega)$ by

$$L_\varphi(u) = \varphi\, u, \quad u \in L^2(b\Omega).$$

is called the Laurent operator with symbol φ and the operator T_φ defined on $H^2(b\Omega)$ by

$$T_\varphi(h) = P(\varphi\, h), \quad h \in H^2(b\Omega).$$

is called the Toeplitz operator with symbol φ. Like the unit disc case, it is easy to see that $T_\varphi^* = T_{\overline{\varphi}}$ and $T_{\alpha_1\varphi+\alpha_2\psi} = \alpha_1 T_\varphi + \alpha_2 T_\psi$ for constants α_1, α_2 and for $\varphi, \psi \in L^\infty(b\Omega)$.

A two-way infinite matrix $A = [a_{ml}], m, l = 0, \pm 1, \pm 2, \ldots$ is called a Laurent matrix of order $k \in \mathbb{N}$ if

$$a_{m+k,l+k} = a_{m,l},\ m, l = 0, \pm 1, \pm 2, \ldots.$$

Similarly a one-way infinite matrix $B = [b_{ml}], m, l = 0, 1, 2, \ldots$ is called a Toeplitz matrix of order k if

$$b_{m+k,l+k} = b_{m,l},\ m, l = 0, 1, 2, \ldots.$$

3. Previous results on Toeplitz operators

In this section, we collect previous results which was mostly proved in [5] in order to refer to known results with consistent notations and to make a development further. The following proposition is about construction of an orthonormal basis for $L^2(b\Omega)$.

Proposition 3.1. *Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let f_a be the Ahlfors map associated to the pair (Ω, a). Let S_{a_j} and G_{a_j} be the Szegő kernels and Garabedian kernels associated to the points a_j, respectively for $j = 0, 1, \ldots, n-1$ with $a_0 = a$. Then*

1. *the set $\mathcal{BH} = \{S_{a_j} f_a^k \ :\ j = 0, 1, \ldots, n-1;\ k \geq 0\}$ is a basis for $H^2(b\Omega)$,*
2. *the set $\mathcal{BH}^\perp = \{G_{a_j} f_a^{-k} \ :\ j = 0, 1, \ldots, n-1,\ k \geq 0\}$ is a basis for $H^2(b\Omega)^\perp$,*
3. *the set $\mathcal{BL} = \{S_{a_j} f_a^k,\ G_{a_j} f_a^{-k} \ :\ j = 0, 1, \ldots, n-1;\ k \geq 0\}$ is a basis for $L^2(b\Omega)$.*

We can apply the Gram-Schmidt orthonormalization to the bases in Proposition 3.1 to get an orthonormal basis for $L^2(b\Omega)$.

Proposition 3.2. *Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let f_a be the Ahlfors map associated to the pair (Ω, a). Let S_{a_j} and G_{a_j} be the Szegő kernels and Garabedian kernels associated to the points a_j, respectively for $j = 0, 1, \ldots, n-1$ with $a_0 = a$. Then*

1. *the set $\mathcal{OL} = \{E_{kn+j} \ :\ j = 0, 1, \ldots, n-1;\ k \geq 0\} \cup \{E_{-kn-j} \ :\ j = 0, 1, \ldots, n-1;\ k \geq 0, -kn-j \leq -1\}$ is an orthonormal basis for $L^2(b\Omega)$ where*

$$E_{kn+j} = \sum_{i=0}^{j} c_{ij} S_{a_i} f_a^k \quad \text{for } kn+j \geq 0,$$

$$E_{-kn-j} = (1-\delta_0^j)\sum_{i=0}^{j-1} \overline{c_{i,j-1}} G_{a_i} f_a^{-k} + \delta_0^j \sum_{i=0}^{n-1} \overline{c_{i,n-1}} G_{a_i} f_a^{-k+1}$$
$$\text{for } -kn-j \leq -1$$

and c_{ij}, $0 \leq i \leq j \leq n-1$ *are constants with* $c_{00} = \sqrt{S(a,a)}^{-1}$ *obtained by the Gram-Schmidt orthonormalization to the basis* $\mathcal{BH}$ *for the order of* $S_a f_a^k, S_{a_1} f_a^k, \ldots, S_{a_{n-1}} f_a^k$, $k = 0, 1, \ldots$ *and* δ_0^j *is the Kronecker delta and*

(2) *the set* $\mathcal{OH} = \{E_{kn+j} \ : \ k \geq 0;\ j = 0, 1, \ldots, n-1\}$ *is an orthonormal basis for* $H^2(b\Omega)$.

Observe from orthogonality of S_a with S_{a_j} for $j \geq 1$ and the Gram-Schmidt orthonormalization that $c_{0j} = 0$ for $j = 1, 2, \ldots, n-1$ and $c_{kk} > 0$ for $k = 0, 1, \ldots, n-1$.

The Toeplitz operator T_{f_a} with symbol f_a as the Ahlfors map is then a shift operator of multiplicity n provided the base domain Ω is n-connected.

Proposition 3.3. *Suppose that* Ω *is a bounded* n*-connected domain with* C^∞ *smooth boundary. Let* $a \in \Omega$ *be fixed and let* f_a *be the Ahlfors map. Then*

$$T_{f_a}(E_m) = E_{m+n} \text{ for } m \geq 0.$$

A one-way infinite matrix $[a_{ij}], i, j \geq 0$ is called a Toeplitz matrix of order n if

$$\forall i, j \geq 0, \quad a_{i+n, j+n} = a_{ij}.$$

Then it is easy to see that given $\varphi \in L^\infty(b\Omega)$, the matrix $[T_\varphi]$ of T_φ with respect to the basis $\mathcal{OH}$ is a Toeplitz matrix of order n, i.e.,

$$\langle T_\varphi(E_{m+n}), E_{l+n}\rangle = \langle T_\varphi(E_m), E_l\rangle \text{ for } m, l \geq 0$$

when Ω is n-connected. On the other hand, suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary and suppose that given $a \in \Omega$, a bounded operator $A : H^2(b\Omega) \to H^2(b\Omega)$ satisfies the following commuting condition

$$\text{Cond(C)} \quad \begin{cases} A\ T_{S_{a_j}} = T_{S_{a_j}} A, & j = 0, 1, \ldots, n-1, \\ A\ T_{f_{a_j}} = T_{f_{a_j}} A, & j = 1, 2, \ldots, n-1. \end{cases}$$

Observe that the index $j = 0$ is excluded in the second identity. Then we have the following classification for the Toeplitz operators.

Proposition 3.4. *Suppose that* Ω *is a bounded* n*-connected domain with* C^∞ *smooth boundary. Let* a *be fixed in* Ω *and let* $a_1, a_2, \ldots, a_{n-1}$ *be distinct simple zeroes of* S_a. *If a bounded operator* $A : H^2(b\Omega) \to H^2(b\Omega)$ *satisfies the commuting condition* Cond(C) *and if* A *induces a Toeplitz matrix of order* n *with respect to* $\mathcal{OH}$, *then* A *is a Toeplitz operator.*

4. Matrices of Toeplitz operators

Now we are ready to work on matrices of Toeplitz operators. We sort all integers according to the Euclidean division divided by connectivity of the domain.

Definition. We say that an integer of the form $r = \alpha n + \beta$ with $\alpha \in \mathbb{Z}, |\beta| \leq n-1$ is written in the standard form if either both $\alpha \geq 0$ and $\beta \geq 0$ or both $\alpha < 0$ and $\beta \leq 0$.

We now compute inner products $\langle E_p E_l, E_m\rangle$ for various positive or negative integers p, l and m's. Since each case is mentioned in this contents, detailed computations are given case by case.

Lemma 4.1. *Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Suppose that $p = kn+j, l = k_l n + j_l, m = k_m n + j_m$ with $k, k_l, k_m, j, j_l, j_m \in \mathbb{Z}$, $|j|, |j_l|, |j_m| \leq n-1$ are all written in standard form. Let $c_{\alpha,\beta}$'s be the constants indicated in Proposition 3.2.*

(1) *If $p \geq 0, l \geq 0, m \geq 0$,*

$$\langle E_p E_l, E_m\rangle = \sum_{i=0}^{j}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{j_m} c_{ij}c_{\mu j_l}\overline{c_{\nu j_m}}\langle S_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle.$$

(2) *If $p \geq 0, l \geq 0, m < 0$,*

$$\begin{aligned}&\langle E_p E_l, E_m\rangle\\ &= (1-\delta_0^{j_m})\sum_{i=0}^{j}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{-j_m-1} c_{ij}c_{\mu j_l}c_{\nu,-j_m-1}\langle S_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, G_{a_\nu}\rangle\\ &\quad + \delta_0^{j_m}\sum_{i=0}^{j}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{n-1} c_{ij}c_{\mu j_l}c_{\nu,n-1}\langle S_{a_i}S_{a_\mu}f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle.\end{aligned}$$

(3) *If $p \geq 0, l < 0, m \geq 0$,*

$$\begin{aligned}&\langle E_p E_l, E_m\rangle\\ &= (1-\delta_0^{j_l})\sum_{i=0}^{j}\sum_{\mu=0}^{-j_l-1}\sum_{\nu=0}^{j_m} c_{ij}\overline{c_{\mu,-j_l-1}}\ \overline{c_{\nu j_m}}\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle\\ &\quad + \delta_0^{j_l}\sum_{i=0}^{j}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{j_m} c_{ij}\overline{c_{\mu,n-1}}\ \overline{c_{\nu j_m}}\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle.\end{aligned}$$

(4) *If $p \geq 0, l < 0, m < 0$,*

$\langle E_p E_l, E_m\rangle$

$$= (1-\delta_0^{j_l})(1-\delta_0^{j_m}) \sum_{i=0}^{j} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{-j_m-1} c_{ij}\overline{c_{\mu,-j_l-1}}c_{\nu,-j_m-1} \cdot \langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m}, G_{a_\nu}\rangle$$

$$+ (1-\delta_0^{j_l})\delta_0^{j_m} \sum_{i=0}^{j} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{n-1} c_{ij}\overline{c_{\mu,-j_l-1}}c_{\nu,n-1}\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$

$$+ \delta_0^{j_l}(1-\delta_0^{j_m}) \sum_{i=0}^{j} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{-j_m-1} c_{ij}\overline{c_{\mu,n-1}}c_{\nu,-j_m-1}\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$

$$+ \delta_0^{j_l}\delta_0^{j_m} \sum_{i=0}^{j} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} c_{ij}\overline{c_{\mu,n-1}}c_{\nu,n-1}\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+2}, G_{a_\nu}\rangle.$$

(5) *If* $p<0, l\geq 0, m\geq 0$

$$\langle E_pE_l, E_m\rangle$$

$$= (1-\delta_0^{j}) \sum_{i=0}^{-j-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{j_m} \overline{c_{i,-j-1}}c_{\mu j_l}\overline{c_{\nu j_m}}\langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle$$

$$+ \delta_0^{j} \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{j_m} \overline{c_{i,n-1}}c_{\mu j_l}\overline{c_{\nu j_m}}\langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle.$$

(6) *If* $p<0, l\geq 0, m<0$,

$$\langle E_pE_l, E_m\rangle$$

$$= (1-\delta_0^{j})(1-\delta_0^{j_m}) \sum_{i=0}^{-j-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{-j_m-1} \overline{c_{i,-j-1}}c_{\mu j_l}c_{\nu,-j_m-1} \cdot \langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, G_{a_\nu}\rangle$$

$$+ (1-\delta_0^{j})\delta_0^{j_m} \sum_{i=0}^{-j-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{n-1} \overline{c_{i,-j-1}}c_{\mu j_l}c_{\nu,n-1}\langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$

$$+ \delta_0^{j}(1-\delta_0^{j_m}) \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{-j_m-1} \overline{c_{i,n-1}}c_{\mu j_l}c_{\nu,-j_m-1}\langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$

$$+ \delta_0^{j}\delta_0^{j_m} \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{n-1} \overline{c_{i,n-1}}c_{\mu j_l}c_{\nu,n-1}\langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m+2}, G_{a_\nu}\rangle.$$

(7) *If* $p<0, l<0, m\geq 0$,

$$\langle E_pE_l, E_m\rangle$$

$$= (1-\delta_0^j)(1-\delta_0^{j_l}) \sum_{i=0}^{-j-1} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{j_m} \overline{c_{i,-j-1}}\ \overline{c_{\mu,-j_l-1}} c_{\nu j_m} \cdot \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m}, S_{a_\nu}\rangle$$
$$+ (1-\delta_0^j)\delta_0^{j_l} \sum_{i=0}^{-j-1} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{j_m} \overline{c_{i,-j-1}}\ \overline{c_{\mu,n-1}} c_{\nu,j_m} \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle$$
$$+ \delta_0^j(1-\delta_0^{j_l}) \sum_{i=0}^{n-1} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{j_m} \overline{c_{i,n-1}}\ \overline{c_{\mu,-j_l-1}} c_{\nu,-j_m-1} \cdot \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle$$
$$+ \delta_0^j\delta_0^{j_l} \sum_{i=0}^{n-1} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{j_m} \overline{c_{i,n-1}}\ \overline{c_{\mu,n-1}} c_{\nu j_m} \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+2}, S_{a_\nu}\rangle.$$

(8) *If* $p<0, l<0, m<0$,

$$\langle E_p E_l, E_m\rangle$$
$$= (1-\delta_0^j)(1-\delta_0^{j_l})(1-\delta_0^{j_m}) \sum_{i=0}^{-j-1} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{-j_m-1} \overline{c_{i,-j-1}}\ \overline{c_{\mu,-j_l-1}} c_{\nu,-j_m-1} \cdot \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m}, G_{a_\nu}\rangle$$
$$+ (1-\delta_0^j)(1-\delta_0^{j_l})\delta_0^{j_m} \cdot \sum_{i=0}^{-j-1} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{n-1} \overline{c_{i,-j-1}}\ \overline{c_{\mu,-j_l-1}} c_{\nu,n-1} \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$
$$+ (1-\delta_0^j)\delta_0^{j_l}(1-\delta_0^{j_m}) \cdot \sum_{i=0}^{-j-1} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{-j_m-1} \overline{c_{i,-j-1}}\ \overline{c_{\mu,n-1}} c_{\nu,n-1} \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$
$$+ (1-\delta_0^j)\delta_0^{j_l}\delta_0^{j_m} \cdot \sum_{i=0}^{-j-1} \sum_{\mu=0}^{n-1} \sum_{\nu=0}^{n-1} \overline{c_{i,-j-1}}\ \overline{c_{\mu,n-1}} c_{\nu,n-1} \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+2}, G_{a_\nu}\rangle$$
$$+ \delta_0^j(1-\delta_0^{j_l})(1-\delta_0^{j_m}) \sum_{i=0}^{n-1} \sum_{\mu=0}^{-j_l-1} \sum_{\nu=0}^{-j_m-1} \overline{c_{i,n-1}}\ \overline{c_{\mu,-j_l-1}} c_{\nu,-j_m-1} \cdot \langle G_{a_i} G_{a_\mu} f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle$$
$$+ \delta_0^j(1-\delta_0^{j_l})\delta_0^{j_m} \cdot$$

$$\sum_{i=0}^{n-1}\sum_{\mu=0}^{-j_l-1}\sum_{\nu=0}^{n-1}\overline{c_{i,n-1}}\;\overline{c_{\mu,-j_l-1}}c_{\nu,n-1}\langle G_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+2},G_{a_\nu}\rangle$$

$$+\,\delta_0^j\delta_0^{j_l}(1-\delta_0^{j_m})\;\cdot$$

$$\sum_{i=0}^{n-1}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{-j_m-1}\overline{c_{i,n-1}}\;\overline{c_{\mu,n-1}}c_{\nu,-j_m-1}\langle G_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+2},G_{a_\nu}\rangle$$

$$+\,\delta_0^j\delta_0^{j_l}\delta_0^{j_m}\;\cdot$$

$$\sum_{i=0}^{n-1}\sum_{\mu=0}^{n-1}\sum_{\nu=0}^{n-1}\overline{c_{i,n-1}}\;\overline{c_{\mu,n-1}}c_{\nu,n-1}\langle G_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+3},G_{a_\nu}\rangle.$$

Proof. The proof is straightforward by using that $f_a(z)^{-1}=\overline{f_a(z)}$ on $b\Omega$. Note also that on the boundary of Ω, $G_bf_b=S_b$ for $b\in\Omega$. □

Since we often use inqualities of integers in the form of Euclidean division by n, we need the following lemma which is easily proved case by case.

Lemma 4.2. *Let n be a positive integer. Suppose that $k,\widetilde{k},j,\widetilde{j}$ are all integers with $|j|,|\widetilde{j}|\le n-1$ that satisfy the inequality*

$$kn+j\ge\widetilde{k}n+\widetilde{j}.$$

Then

$$k-\widetilde{k}\begin{cases}\ge 2 & \text{for } -2n+2\le j-\widetilde{j}\le -n-1,\\ \ge 1 & \text{for } -n\le j-\widetilde{j}\le -1,\\ \ge 0 & \text{for } 0\le j-\widetilde{j}\le n-1,\\ \ge -1 & \text{for } n\le j-\widetilde{j}\le 2n-2.\end{cases}$$

Theorem 4.3. *Let n be a positive integer and suppose that $p=kn+j, l=k_ln+j_l, m=k_mn+j_m$ are all integers written in standard form. Then*

(1) *if r_1 and r_2 are integers for which $(k+r_1)n+j$, $(k_l+r_2)n+j_l$, $(k_m+r_1+r_2)n+j_m$ are all in standard form,*

$$\langle E_pE_l,E_m\rangle=\langle E_{p+r_1n}E_{l+r_2n},E_{m+(r_1+r_2)n}\rangle,$$

(2) $\langle E_pE_l,E_m\rangle=0$ *for* $k\ge k_m-k_l+3$,

(3) $\langle E_pE_l,E_m\rangle=0$ *for* $k\ge k_m-k_l+2$ *and* $m\ge 0, l\ge 0$.

Proof. For (1), suppose that $(k+r_1)n+j,(k_l+r_2)n+j_l,(k_m+r_1+r_2)n+j_m$ are all in standard form. Since adding $r_1n,r_2n,(r_1+r_2)n$ to p,l,m, respectively does not change remainders, the quotients and remainders of $p+r_1n,l+r_1n,m+(r_1+r_2)n$ have the same signs (positiveness or negativeness) as those of p,l,m, respectively. Thus when the inner product $\langle E_{p+r_1n}E_{l+r_2n},E_{m+(r_1+r_2)n}\rangle$ is expanded according to Lemma 4.1, all expressions except for the terms containing the Ahlfors map f_a are exactly identical to those for $\langle E_pE_l,E_m\rangle$. On the other hand, the exponents for the Ahlfors map f_a in the expansion of $\langle E_pE_l, E_m\rangle$ contain $k+k_l-k_m$ and the exponents in the inner product $\langle E_{p+r_1n}E_{l+r_2n}, E_{m+(r_1+r_2)n}\rangle$ also contain $(k+r_1)+(k_l+r_2)-(k_m+r_1+r_2) = k+k_l-k_m$ in all cases and hence eventually two expansions of $\langle E_pE_l, E_m\rangle$ and $\langle E_{p+r_1n}E_{l+r_2n}, E_{m+(r_1+r_2)n}\rangle$ are the same.

For (2), first observe that S_{a_μ} is holomorphic in $H^2(b\Omega)$ and G_{a_μ} is meromorphic with single simple pole at $z=a_\mu$ for $\mu=0,1,\ldots,n-1$. Note also that f_{a_ν} has zeroes at $z=a$ and $z=a_\nu$ for $\nu=0,1,\ldots,n-1$. Since $a=a_0,a_1,\ldots,a_{n-1}$ are all zeroes of f_a, $G_{a_\mu}f_a$ can be extended to be holomorphic at $z=a_\mu$ for all $\mu=0,1,\ldots,n-1$ and in particular $G_{a_\mu}f_{a_\mu}=S_{a_\mu}$ for $\mu=0,1,\ldots,n-1$. We now look at inner products expressed in Lemma 4.1 carefully and compute selected ones for the sake of simplicity.

$$\begin{aligned}
&\langle S_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle\\
&= S_{a_i}(a_\nu)S_{a_\mu}(a_\nu)f_a^{k+k_l-k_m}(a_\nu)=0 \text{ if } k+k_l-k_m\geq 1,\\
&\langle S_{a_i}S_{a_\mu}f_a^{k+k_l-k_m+1}, G_{a_\nu}\rangle = \langle S_{a_i}S_{a_\mu}f_{a_\nu}f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle\\
&= S_{a_i}(a_\nu)S_{a_\mu}(a_\nu)f_{a_\nu}(a_\nu)f_a^{k+k_l-k_m+1}(a_\nu)=0 \text{ if } k+k_l-k_m+1\geq 0,\\
&\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle = S_{a_i}(a_\nu)(G_{a_\mu}f_a)(a_\nu)f_a^{k+k_l-k_m}(a_\nu)\\
&= 0 \text{ if } k+k_l-k_m\geq 1,\\
&\langle S_{a_i}G_{a_\mu}f_a^{k+k_l-k_m+2}, G_{a_\nu}\rangle = \langle S_{a_i}(G_{a_\mu}f_a)f_{a_\nu}f_a^{k+k_l-k_m+1}, S_{a_\nu}\rangle\\
&= S_{a_i}(a_\nu)(G_{a_\mu}f_a)(a_\nu)f_{a_\nu}(a_\nu)f_a^{k+k_l-k_m+1}(a_\nu)\\
&= 0 \text{ if } k+k_l-k_m+1\geq 0,\\
&\langle G_{a_i}G_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle = \langle (G_{a_i}f_a)(G_{a_\mu}f_a)f_a^{k+k_l-k_m-2}, S_{a_\nu}\rangle\\
&= (G_{a_i}f_a)(a_\nu)(G_{a_\mu}f_a)(a_\nu)f_a^{k+k_l-k_m-2}(a_\nu)=0 \text{ if } k+k_l-k_m-2\geq 1.
\end{aligned}$$

Hence all cases vanish if $k\geq k_m-k_l+3$.

Similarly, (3) is proved easily using Lemma 4.1-(1),(5). In fact, suppose that $m,l\geq 0$ and that $p=kn+j$ and $k\geq k_m-k_l+2$. If $kn+j\geq 0$, then

$$\begin{aligned}
\langle E_pE_l, E_m\rangle &= \sum_{i=0}^{j}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{j_m} c_{ij}c_{\mu j_l}\overline{c_{\nu j_m}}\langle S_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle\\
&= \sum_{i=0}^{j}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{j_m} c_{ij}c_{\mu j_l}\overline{c_{\nu j_m}}S_{a_i}(a_\nu)S_{a_\mu}(a_\nu)f_a^{k+k_l-k_m}(a_\nu)=0
\end{aligned}$$

because $k+k_l-k_m\geq 1$ and $f_a(a)=0$. If $kn+j<0$, then again

$$\begin{aligned}
&\langle E_pE_l, E_m\rangle\\
&= (1-\delta_0^j)\sum_{i=0}^{-j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{j_m}\overline{c_{i,-j-1}}c_{\mu j_l}\overline{c_{\nu j_m}}\langle G_{a_i}S_{a_\mu}f_a^{k+k_l-k_m}, S_{a_\nu}\rangle
\end{aligned}$$

$$+ \delta_0^j \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{j_m} \overline{c_{i,n-1}} c_{\mu j_l} \overline{c_{\nu j_m}} \langle G_{a_i} S_{a_\mu} f_a^{k+k_l-k_m+1}, S_{a_\nu} \rangle$$

$$= (1-\delta_0^j) \sum_{i=0}^{-j-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{j_m} \overline{c_{i,-j-1}} c_{\mu j_l} \overline{c_{\nu j_m}} (G_{a_i} f_a)(a_\nu) S_{a_\mu}(a_\nu) f_a^{k+k_l-k_m-1}(a_\nu)$$

$$+ \delta_0^j \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{j_m} \overline{c_{i,n-1}} c_{\mu j_l} \overline{c_{\nu j_m}} (G_{a_i} f_a)(a_\nu) S_{a_\mu}(a_\nu) f_a^{k+k_l-k_m}(a_\nu) = 0$$

because both $k + k_l - k_m - 1$ and $k + k_l - k_m$ are bigger that 0 and $f_a(a) = 0$. Hence for either case, the inner product $\langle E_p E_l, E_m \rangle$ must be zero. Thus the proof is done. □

The vanishing property (2) and (3) in Theorem 4.3 immediately proves the following theorem which computes the matrices $[L_\varphi]$ and $[T_\varphi]$ of the Laurent and the Toeplitz operators.

Theorem 4.4. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be in $L^\infty(b\Omega)$. Suppose that $l = k_l n + j_l, m = k_m n + j_m$ are integers written in standard form. Then*

(1) *the entry of m-th row and l-th column of the matrix $[L_\varphi]$ of the Laurent operator L_φ with respect to the orthonormal basis $\mathcal{OL}$ is equal to*

$$[L_\varphi]_{m,l} = \sum_{p=-\infty}^{(k_m-k_l+2)n+n-1} \alpha_p \langle E_p E_l, E_m \rangle,$$

(2) *for $m, l \geq 0$ the entry of m-th row and l-th column of the matrix $[T_\varphi]$ of the Toeplitz operator T_φ with respect to the orthonormal basis $\mathcal{OH}$ is equal to*

$$[T_\varphi]_{m,l} = \sum_{p=-\infty}^{(k_m-k_l+1)n+n-1} \alpha_p \langle E_p E_l, E_m \rangle.$$

Now we want to find a relation between the one-way infinite matrix $[T_\varphi]$ and the Fourier coefficients $\langle \varphi, E_p \rangle$ for the symbol φ. First we compute the entry of the 0-th row and the 0-th column.

Corollary 4.5. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be in $L^\infty(b\Omega)$. Then the entry of 0-th row and 0-th column of the matrix $[T_\varphi]$ with* *respect to the orthonormal basis $\mathcal{OH}$ is equal to*

$$[T_\varphi]_{0,0} = \alpha_0 c_{00}^3 S_a(a)^2 + \sum_{j=1}^{n-1} \alpha_{-j} \sum_{i=0}^{j-1} \overline{c_{i,j-1}} c_{00}^2 \langle G_{a_i} S_a, S_a \rangle$$

$$+ \sum_{p=-\infty}^{-n} \alpha_p \langle E_p E_0, E_0 \rangle.$$

Proof. We split the infinite sum of Theorem 4.4 into three parts as

$$[T_\varphi]_{0,0} = \sum_{p=0}^{1\cdot n+n-1} \alpha_p \langle E_p E_0, E_0 \rangle$$

$$+ \sum_{p=-(n-1)}^{-1} \alpha_p \langle E_p E_0, E_0 \rangle + \sum_{p=-\infty}^{-n} \alpha_p \langle E_p E_0, E_0 \rangle.$$

The first term is then by Lemma 4.1-(1) equal to

$$\sum_{k=0}^{1} \sum_{j=0}^{n-1} \alpha_{kn+j} \langle E_{kn+j} E_0, E_0 \rangle = \sum_{k=0}^{1} \sum_{j=0}^{n-1} \alpha_{kn+j} \sum_{i=0}^{j} c_{ij} c_{00}^2 \langle S_{a_i} S_a f_a^k, S_a \rangle$$

$$= \sum_{j=0}^{n-1} \alpha_j \sum_{i=0}^{j} c_{ij} c_{00}^2 S_{a_i}(a) S_a(a) = \sum_{j=0}^{n-1} \alpha_j c_{0j} c_{00}^2 S_a(a)^2 = \alpha_0 c_{00}^3 S_a(a)^2.$$

In the proof we used the properties that $f_a(a) = 0$ and $S_{a_i}(a) = 0, c_{0j} = 0$ for $i \geq 1$. The second term in the expression of $[T_\varphi]_{0,0}$ above is nothing but the case $k = k_l = k_m = 0$ in Lemma 4.1(5). Hence we are done. □

In order to differentiate entries of the 0-th column vector of the matrix $[T_\varphi]$ which correspond to remainders and quotients in the Euclidean division, we first compute $[T_\varphi]_{j_m,0}$ for $j_m = 1, 2, \ldots, n-1$ and $n \geq 2$.

Corollary 4.6. *Let $n \geq 2$ be an integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be in $L^\infty(b\Omega)$. Then the entry of j_m-th row and 0-th column of the matrix $[T_\varphi]$ with $1 \leq j_m \leq n-1$ with respect to the orthonormal basis $\mathcal{OH}$ is equal to*

$$[T_\varphi]_{j_m,0} = \sum_{j=1}^{n-1} \alpha_{-j} \sum_{i=0}^{j-1} \sum_{\nu=0}^{j_m} \overline{c_{i,j-1}} c_{00} \overline{c_{\nu j_m}} \langle G_{a_i} S_a, S_{a_\nu} \rangle$$

$$+ \sum_{p=-\infty}^{-n} \alpha_p \langle E_p E_0, E_m \rangle.$$

Proof. Let j_m be an integer with $1 \leq j_m \leq n-1$. Suppose that $p = kn + j$ is written in standard form and $0 \leq p \leq j_m + 2n - 1 = 2n + (j_m - 1)$. Since

$0 \le j \le n-1$ and $0 \le j_m - 1 \le n-2$, $-n+1 \le (j_m-1)-j \le n-2$. It thus follows from Lemma 4.2 and Lemma 4.1 that $2-k \ge 0$ and hence that

$$\begin{aligned}\sum_{p=0}^{j_m+2n-1} &\alpha_p \langle E_p E_0, E_{j_m}\rangle = \sum_{k=0}^{2}\sum_{j=0}^{n-1} \alpha_{kn+j} \sum_{i=0}^{j}\sum_{\nu=0}^{j_m} c_{ij} c_{00} \overline{c_{\nu j_m}} \langle S_{a_i} S_a f_a^k, S_{a_\nu}\rangle \\ &= \sum_{j=0}^{n-1} \alpha_j \sum_{i=0}^{j}\sum_{\nu=0}^{j_m} c_{ij} c_{00}\overline{c_{\nu j_m}} S_{a_i}(a_\nu) S_a(a_\nu) = \sum_{j=0}^{n-1} \alpha_j \sum_{i=0}^{j} c_{ij} c_{00} \overline{c_{0 j_m}} S_{a_i}(a) S_a(a) \\ &= \sum_{j=0}^{n-1} \alpha_j c_{0j} c_{00} \overline{c_{0j_m}} S_a(a)^2 = \alpha_0 c_{00}^2 \overline{c_{0j_m}} S_a(a) S_a(a) = \delta_0^{j_m} \alpha_0 c_{00}^3 S_a(a)^2\end{aligned}$$

which is equal to zero because $j_m \ge 1$. Hence the proof is done by Theorem 4.4 and Lemma 4.1. $\square$

Next we compute the entry $[T_\varphi]_{m,0}$ for $m \ge n$.

Corollary 4.7. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be in $L^\infty(b\Omega)$. Then the entry of m-th row and 0-th column of the matrix $[T_\varphi]$ with $m = k_m n + j_m, k_m \ge 1, 0 \le j_m \le n-1$ with respect to the orthonormal basis $\mathcal{OH}$ is equal to*

$$\begin{aligned}[T_\varphi]_{m,0} &= [T_\varphi]_{k_m n + j_m, 0} \\ &= \delta_0^{j_m} \alpha_{k_m n} c_{00}^3 S_a(a)^2 + \sum_{j=1}^{n-1} \alpha_{(k_m-1)n+j} \sum_{i=0}^{j}\sum_{\nu=0}^{j_m} c_{ij} c_{00} \overline{c_{\nu j_m}} \langle S_{a_i} G_a, S_{a_\nu}\rangle \\ &\quad + \sum_{p=-\infty}^{(k_m-1)n} \alpha_p \langle E_p E_0, E_m\rangle.\end{aligned}$$

Proof. Suppose that $p = kn + j$ is written in standard form and $(k_m+1)n \le p \le m + 2n - 1 = (k_m+2)n + j_m - 1$. Since $0 \le j \le n-1$, $k \ge k_m + 1$. Thus the exponent $k - k_m$ is at least 1 in Lemma 4.1-(1) and hence $\langle E_p E_0, E_m\rangle$ must be zero. It follows from Theorem 4.4 that

$$\begin{aligned}[T_\varphi]_{m,0} = &\sum_{j=0}^{n-1} \alpha_{k_m n + j} \sum_{i=0}^{j}\sum_{\nu=0}^{j_m} c_{ij} c_{00} \overline{c_{\nu j_m}} \langle S_{a_i} S_a, S_{a_\nu}\rangle \\ &+ \sum_{j=1}^{n-1} \alpha_{(k_m-1)n+j} \langle E_{(k_m-1)n+j} E_0, E_m\rangle + \sum_{p=-\infty}^{(k_m-1)n} \alpha_p \langle E_p E_0, E_m\rangle.\end{aligned}$$

Now the first term equals

$$\sum_{j=0}^{n-1} \alpha_{k_m n + j} \sum_{i=0}^{j}\sum_{\nu=0}^{j_m} c_{ij} c_{00} \overline{c_{\nu j_m}} S_{a_i}(a_\nu) S_a(a_\nu)$$

$$= \sum_{j=0}^{n-1} \alpha_{k_m n+j} \sum_{i=0}^{j} c_{ij} c_{00} \overline{c_{0j_m}} S_{a_i}(a) S_a(a) = \sum_{j=0}^{n-1} \alpha_{k_m n+j} c_{0j} c_{00} \overline{c_{0j_m}} S_a(a)^2$$
$$= \alpha_{k_m n} c_{00}^2 \overline{c_{0j_m}} S_a(a)^2 = \delta_0^{j_m} \alpha_{k_m n} c_{00}^3 S_a(a)^2$$

and the second term equals

$$\sum_{j=1}^{n-1} \alpha_{(k_m-1)n+j} \sum_{i=0}^{j} \sum_{\nu=0}^{j_m} c_{ij} c_{00} \overline{c_{\nu j_m}} \langle S_{a_i} S_a f_a^{k_m-1-k_m}, S_{a_\nu} \rangle$$
$$= \sum_{j=1}^{n-1} \alpha_{(k_m-1)n+j} \sum_{i=0}^{j} \sum_{\nu=0}^{j_m} c_{ij} c_{00} \overline{c_{\nu j_m}} \langle S_{a_i} G_a, S_{a_\nu} \rangle$$

and hence the proof completes. □

Similarly we can compute 0-th row vector of the matrix $[T_\varphi]$ which produce the Fourier coefficients with negative numbered indexes.

Corollary 4.8. *Let $n \geq 2$ be an integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be in $L^\infty(b\Omega)$. Then the entry of 0-th row and j_l-th column of the matrix $[T_\varphi]$ with $1 \leq j_l \leq n-1$ with respect to the orthonormal basis $\mathcal{OH}$ is equal to*

$$[T_\varphi]_{0,j_l} = \sum_{j=1}^{n-1} \alpha_{-j} \sum_{i=0}^{j-1} \sum_{\mu=0}^{j_l} \overline{c_{i,j-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu}, S_a \rangle$$
$$+ \sum_{p=-\infty}^{-n} \alpha_p \langle E_p E_{j_l}, E_0 \rangle.$$

Proof. Let j_l be an integer with $1 \leq j_l \leq n-1$. Suppose that $p = kn + j$ is written in standard form and $0 \leq p \leq -j_l + 2n - 1 = n + (n - j_l - 1)$. Since $0 \leq n - j_l - 1 \leq n - 2$ and $0 \leq j \leq n-1$, $-n+1 \leq (n - j_l - 1) - j \leq n - 2$. It thus follows from Lemma 4.2 and Lemma 4.1 that $0 \leq k \leq 1$ and hence that

$$\sum_{p=0}^{-j_l+2n-1} \alpha_p \langle E_p E_{j_l}, E_0 \rangle = \sum_{k=0}^{1} \sum_{j=0}^{n-1} \alpha_{kn+j} \sum_{i=0}^{j} \sum_{\mu=0}^{j_l} c_{ij} c_{\mu j_l} c_{00} \langle S_{a_i} S_{a_\mu} f_a^k, S_a \rangle$$
$$= \sum_{j=0}^{n-1} \alpha_j \sum_{i=0}^{j} \sum_{\mu=0}^{j_l} c_{ij} c_{\mu j_l} c_{00} S_{a_i}(a) S_{a_\mu}(a) = \sum_{j=0}^{n-1} \alpha_j \sum_{i=0}^{j} c_{ij} c_{0 j_l} c_{00} S_{a_i}(a) S_a(a) = 0$$

since $c_0 j_l = 0$ for $j_l \geq 1$. Hence the proof of this corollary is done by Theorem 4.4 and Lemma 4.1. □

Corollary 4.9. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let a be fixed in Ω and let $a_1, a_2, \ldots, a_{n-1}$ be distinct simple zeroes of S_a. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ be in*

$L^\infty(b\Omega)$. *Then the entry of* 0*-th row and* l*-th column of the matrix* $[T_\varphi]$ *with* $l = k_l n + j_l, k_l \geq 1, 0 \leq j_l \leq n-1$ *with respect to the orthonormal basis* $\mathcal{OH}$ *is equal to*

$$\begin{aligned}
[T_\varphi]_{0,l} &= [T_\varphi]_{0,k_l n + j_l} \\
&= \chi_{[2,\infty)}(k_l)\ \alpha_{-(k_l-1)n-1}\ \delta_0^{j_l} c_{00}^3 S_a(a)^2 \\
&\quad + \alpha_{-k_l n}\delta_1^n \delta_0^{j_l} c_{00}^3 S_a(a)^2 + \sum_{j=1}^{n-1} \alpha_{-k_l n - j} \sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l} \overline{c_{i,j-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu}, S_a\rangle \\
&\quad + \sum_{p=-\infty}^{-(k_l+1)n} \alpha_p \langle E_p E_l, E_0\rangle.
\end{aligned}$$

Proof. Let $l = k_l n + j_l$ be in standard form with $l \geq 1$. Suppose also that $p = kn + j$ is written in standard form and $-(k_l-1)n \leq p \leq -l + 2n - 1 = -(k_l-2)n - (j_l+1)$.

If $j_l = n-1$, then $p = -(k_l-1)n$ and so in this case

$$\begin{aligned}
\langle E_p E_l, E_0\rangle &= \langle E_{-(k_l-1)n} E_{k_l n + (n-1)}, E_0\rangle = \langle E_n E_{n-1}, E_0\rangle \\
&= \sum_{\mu=0}^{n-1} c_{00} c_{\mu,n-1} c_{00} \langle S_a S_{a_\mu} f_a, S_a\rangle = 0.
\end{aligned}$$

Here for equality of the second and third identities Theorem 4.3 is used.

Suppose that $0 \leq j_l \leq n-2$ with $n \geq 2$. If $k_l = 1$, then $0 \leq p = j \leq n - j_l - 1 \leq n-1$, so

$$\langle E_p E_l, E_0\rangle = \langle E_j E_{n+j_l}, E_0\rangle = \sum_{i=0}^{j}\sum_{\mu=0}^{j_l} c_{ij} c_{\mu j_l} c_{00} \langle S_{a_i} S_{a_\mu} f_a, S_a\rangle = 0.$$

and if $k_l \geq 2$, then $p < 0$ and it follows from the inequality $-(k_l-1)n \leq p$ that $k + k_l - 1 \geq 0$. Since $0 \leq j_l \leq n-2$ and $-n+1 \leq j \leq 0$, $-n+1 \leq -j_l - 1 - j \leq n-2$, so it follows from the inequality $p \leq -(k_l-2)n - (j_l+1)$ and Lemma 4.2 that $-k_l + 2 - k \geq 0$ and hence $1 \leq k + k_l \leq 2$. Thus for this case,

$$\begin{aligned}
\langle E_p E_l, E_0\rangle &= (1-\delta_0^j) \sum_{i=0}^{-j-1}\sum_{\mu=0}^{j_l} \overline{c_{i,-j-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu} f_a^{k+k_l}, S_a\rangle \\
&\quad + \delta_0^j \sum_{i=0}^{n-1}\sum_{\mu=0}^{j_l} \overline{c_{i,n-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu} f_a^{k+k_l+1}, S_a\rangle
\end{aligned}$$

and the second term must be zero because the function $G_{a_i} S_{a_\mu} f_a^{k+k_l+1} = (G_{a_i} f_a) S_{a_\mu} f_a^{k+k_l}$ has a removable singularity at $z = a_i$ and $k + k_l \geq 1$. And the first term is equal to

$$(1-\delta_0^j) \sum_{i=0}^{-j-1}\sum_{\mu=0}^{j_l} \overline{c_{i,-j-1}} c_{\mu j_l} c_{00} (G_{a_i} f_a)(a) S_{a_\mu}(a) f_a^{k+k_l-1}(a)$$

$$
\begin{aligned}
&= (1-\delta_0^j) \sum_{i=0}^{-j-1} \overline{c_{i,-j-1}} c_{0j_l} c_{00} (G_{a_i} f_a)(a) S_a(a) f_a^{k+k_l-1}(a) \\
&= (1-\delta_0^j) \overline{c_{0,-j-1}} c_{0j_l} c_{00} S_a(a)^2 f_a^{k+k_l-1}(a) \\
&= \delta_{1-k_l}^k (1-\delta_0^j) \overline{c_{0,-j-1}} c_{0j_l} c_{00} S_a(a)^2 \\
&= \delta_{1-k_l}^k \delta_{-1}^j \delta_0^{j_l} c_{00}^3 S_a(a)^2.
\end{aligned}
$$

In equality of the second and the third identities, we used the fact that $G_{a_i} f_a$ is holomorphic at $z = a_i$ and is also zero if $i \neq 0$ because of $f_a(a) = 0$. Hence

$$
\sum_{p=-(k_l-1)n}^{-l+2n-1} \langle E_p E_l, E_0 \rangle = \chi_{[2,\infty)}(k_l)\ \alpha_{-(k_l-1)n-1}\ \delta_0^{j_l} c_{00}^3 S_a(a)^2. \tag{4.1}
$$

Next we compute $\langle E_p E_l, E_0 \rangle$ for the case $k = -k_l$. It then follows from Theorem 4.3 and Lemma 4.1 that

$$
\begin{aligned}
\sum_{j=0}^{n-1} \langle E_{-k_l n - j} E_{k_l n + j_l}, E_0 \rangle &= \sum_{j=0}^{n-1} \langle E_{-j} E_{j_l}, E_0 \rangle \\
&= \sum_{j=1}^{n-1} \sum_{i=0}^{j-1} \sum_{\mu=0}^{j_l} \overline{c_{i,j-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu}, S_a \rangle \\
&\quad + \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \overline{c_{i,n-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu} f_a, S_a \rangle.
\end{aligned}
$$

And the second term from above is as before equal to

$$
\begin{aligned}
\sum_{i=0}^{n-1} \overline{c_{i,n-1}} c_{0j_l} c_{00} (G_{a_i} f_a)(a) S_a(a) &= \overline{c_{0,n-1}} c_{0j_l} c_{00} (G_a f_a)(a) S_a(a) \\
&= \delta_1^n \delta_0^{j_l} c_{00}^3 S_a(a)^2.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\sum_{j=0}^{n-1} \alpha_{-k_l n - j} \langle E_{-k_l n - j} E_{k_l n + j_l}, E_0 \rangle \\
&= \alpha_{-k_l n} \delta_1^n \delta_0^{j_l} c_{00}^3 S_a(a)^2 + \sum_{j=1}^{n-1} \alpha_{-k_l n - j} \sum_{i=0}^{j-1} \sum_{\mu=0}^{j_l} \overline{c_{i,j-1}} c_{\mu j_l} c_{00} \langle G_{a_i} S_{a_\mu}, S_a \rangle.
\end{aligned} \tag{4.2}
$$

Therefore from identities (4.1) and (4.2) and from Theorem 4.4, the proof of Corollary 4.9 completes. □

5. Matrices of products of Topeplitz operators

In the final section we work on products of Toeplitz operators which may not be Toeplitz operators in general even for the case of the unit disc. Here we find a necessary condition for the products to be Toeplitz operators.

Theorem 5.1. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty} \beta_q E_q$ be symbols in $L^\infty(b\Omega)$. Then given nonnegative integers $m = k_m n + j_m, l = k_l n + j_l$ in standard form,*

(5.1)
$$\begin{aligned}
&\Big([T_\varphi][T_\psi]\Big)_{m+n,l+n} \\
&= \Big([T_\varphi][T_\psi]\Big)_{m,l} \\
&\quad + \sum_{r=0}^{n-1} \Bigg[\Bigg(\sum_{j=0}^{n-1} \alpha_{(k_m+1)n+j} \sum_{i=0}^{j} \sum_{\mu=0}^{r} \sum_{\nu=0}^{j_m} c_{ij} c_{\mu r} \overline{c_{\nu j_m}} S_{a_i}(a_\nu) S_{a_\mu}(a_\nu) \\
&\qquad\qquad + \sum_{p=-\infty}^{k_m n+n-1} \alpha_p \langle E_p E_r, E_{m+n} \rangle \Bigg) \cdot \\
&\qquad \Bigg(\sum_{j=0}^{n-1} \alpha_{-k_l n-j} (1-\delta_0^j) \sum_{i=0}^{j-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{r} \overline{c_{i,j-1}} c_{\mu j_l} \overline{c_{\nu r}} (G_{a_i} f_a)(a_\nu) S_{a_\mu}(a_\nu) \\
&\qquad + \sum_{j=0}^{n-1} \alpha_{-(k_l+1)n-j} (1-\delta_0^j) \sum_{i=0}^{j-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{r} \overline{c_{i,j-1}} c_{\mu j_l} \overline{c_{\nu r}} \langle G_{a_i} S_{a_\mu}, S_{a_\nu} \rangle \\
&\qquad + \sum_{j=0}^{n-1} \alpha_{-(k_l+1)n-j} \delta_0^j \sum_{i=0}^{n-1} \sum_{\mu=0}^{j_l} \sum_{\nu=0}^{r} \overline{c_{i,n-1}} c_{\mu j_l} \overline{c_{\nu r}} (G_{a_i} f_a)(a_\nu) S_{a_\mu}(a_\nu) \\
&\qquad + \sum_{q=-\infty}^{-(k_l+2)n} \beta_q \langle E_q E_{l+n}, E_r \rangle \Bigg) \Bigg].
\end{aligned}$$

Proof. Let $m = k_m n + j_m, l = k_l n + j_l$ be nonnegative integers in standard form. Then it follows from Theorem 4.4 that

(5.2)
$$\begin{aligned}
&([T_\varphi][T_\psi])_{m+n,l+n} = \sum_{r=0}^{\infty} [T_\varphi]_{m+n,r} [T_\psi]_{r,l+n} \\
&= \sum_{r=0}^{\infty} \left(\sum_{p=-\infty}^{(k_m-k_r+2)n+n-1} \alpha_p \langle E_p E_r, E_{m+n} \rangle \sum_{q=-\infty}^{(k_r-k_l)n+n-1} \beta_q \langle E_q E_{l+n}, E_r \rangle \right) \\
&= \sum_{r=0}^{n-1} \left(\sum_{p=-\infty}^{(k_m+2)n+n-1} \alpha_p \langle E_p E_r, E_{m+n} \rangle \sum_{q=-\infty}^{-k_l n+n-1} \beta_q \langle E_q E_{l+n}, E_r \rangle \right) \\
&\quad + \sum_{r=n}^{\infty} \left(\sum_{p=-\infty}^{(k_m-k_r+2)n+n-1} \alpha_p \langle E_p E_r, E_{m+n} \rangle \sum_{q=-\infty}^{(k_r-k_l)n+n-1} \beta_q \langle E_q E_{l+n}, E_r \rangle \right)
\end{aligned}$$

and as letting $s = r - n$, the second term from above is, by Theorem 4.3 equal to

$$\sum_{s=0}^{\infty}\left(\sum_{p=-\infty}^{(k_m-k_s+1)n+n-1}\alpha_p\langle E_pE_{s+n},E_{m+n}\rangle\cdot\sum_{q=-\infty}^{(k_s-k_l+1)n+n-1}\beta_q\langle E_qE_{l+n},E_{s+n}\rangle\right)$$

$$=\sum_{s=0}^{\infty}\left(\sum_{p=-\infty}^{(k_m-k_s+1)n+n-1}\alpha_p\langle E_pE_s,E_m\rangle\sum_{q=-\infty}^{(k_s-k_l+1)n+n-1}\beta_q\langle E_qE_l,E_s\rangle\right)$$

$$=\left([T_\varphi][T_\psi]\right)_{m,l}.$$

On the other hand, for $0 \le r, j \le n-1$, it is easy to see from Lemma 4.1 that

$$\begin{aligned}&\langle E_{(k_m+2)n+j}E_r,E_{m+n}\rangle\\ &=\sum_{i=0}^{j}\sum_{\mu=0}^{r}\sum_{\nu=0}^{j_m}c_{ij}c_{\mu r}\overline{c_{\nu j_m}}\langle S_{a_i}S_{a_\mu}f_a^{k_m+2-k_m-1},S_{a_\nu}\rangle\\ &=\sum_{i=0}^{j}\sum_{\mu=0}^{r}\sum_{\nu=0}^{j_m}c_{ij}c_{\mu r}\overline{c_{\nu j_m}}\langle S_{a_i}S_{a_\mu}f_a,S_{a_\nu}\rangle=0\end{aligned}\tag{5.3}$$

and

$$\begin{aligned}&\langle E_{-k_ln-j}E_{l+n},E_r\rangle\\ &=(1-\delta_0^j)\sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,j-1}}c_{\mu j_l}\overline{c_{\nu r}}\langle G_{a_i}S_{a_\mu}f_a^{-k_l+k_l+1},S_{a_\nu}\rangle\\ &\quad+\delta_0^j\sum_{i=0}^{n-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,n-1}}c_{\mu j_l}\overline{c_{\nu r}}\langle G_{a_i}S_{a_\mu}f_a^{-k_l+k_l+1+1},S_{a_\nu}\rangle\\ &=(1-\delta_0^j)\sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,j-1}}c_{\mu j_l}\overline{c_{\nu r}}(G_{a_i}f_a)(a_\nu)S_{a_\mu}(a_\nu)\end{aligned}\tag{5.4}$$

because in the second term the function $G_{a_i}S_{a_\mu}f_a^2 = (G_{a_i}f_a)S_{a_\mu}f_a$ is holomorphic at a_i and has a zero at a_ν.

And similarly it follows that

$$\begin{aligned}\langle E_{(k_m+1)n+j}E_r,E_{m+n}\rangle&=\sum_{i=0}^{j}\sum_{\mu=0}^{r}\sum_{\nu=0}^{j_m}c_{ij}c_{\mu r}\overline{c_{\nu j_m}}\langle S_{a_i}S_{a_\mu},S_{a_\nu}\rangle\\ &=\sum_{i=0}^{j}\sum_{\mu=0}^{r}\sum_{\nu=0}^{j_m}c_{ij}c_{\mu r}\overline{c_{\nu j_m}}S_{a_i}(a_\nu)S_{a_\mu}(a_\nu)\end{aligned}\tag{5.5}$$

and

$$
\begin{aligned}
&\langle E_{-(k_l+1)n-j}E_{l+n}, E_r\rangle\\
&= (1-\delta_0^j)\sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,j-1}}c_{\mu j_l}\overline{c_{\nu r}}\langle G_{a_i}S_{a_\mu}, S_{a_\nu}\rangle\\
&\quad + \delta_0^j\sum_{i=0}^{n-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,n-1}}c_{\mu j_l}\overline{c_{\nu r}}\langle G_{a_i}S_{a_\mu}f_a, S_{a_\nu}\rangle\\
&= (1-\delta_0^j)\sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,j-1}}c_{\mu j_l}\overline{c_{\nu r}}\langle G_{a_i}S_{a_\mu}, S_{a_\nu}\rangle\\
&\quad + \delta_0^j\sum_{i=0}^{n-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,n-1}}c_{\mu j_l}\overline{c_{\nu r}}(G_{a_i}f_a)(a_\nu)S_{a_\mu}(a_\nu).
\end{aligned}
\tag{5.6}
$$

Thus it follows from (5.3), (5.4), (5.5), and (5.6) that the first term in the last identity of (5.2) is equal to the remainder term of (5.1) and hence the proof is finished. □

The following corollary is an immediate consequence of Theorem 5.1.

Corollary 5.2. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let $\varphi = \sum_{p=-\infty}^{\infty}\alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty}\beta_q E_q$ be symbols. If, for all nonnegative integers $m = k_m n + j_m, l = k_l n + j_l$ written in standard form,*

(5.7)

$$
\begin{aligned}
\sum_{r=0}^{n-1}\Bigg[&\Bigg(\sum_{j=0}^{n-1}\alpha_{(k_m+1)n+j}\sum_{i=0}^{j}\sum_{\mu=0}^{r}\sum_{\nu=0}^{j_m}c_{ij}c_{\mu r}\overline{c_{\nu j_m}}S_{a_i}(a_\nu)S_{a_\mu}(a_\nu)\\
&\qquad + \sum_{p=-\infty}^{k_m n+n-1}\alpha_p\langle E_pE_r, E_{m+n}\rangle\Bigg)\cdot\\
&\Bigg(\sum_{j=0}^{n-1}\alpha_{-k_l n-j}(1-\delta_0^j)\sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,j-1}}c_{\mu j_l}\overline{c_{\nu r}}(G_{a_i}f_a)(a_\nu)S_{a_\mu}(a_\nu)\\
&+\sum_{j=0}^{n-1}\alpha_{-(k_l+1)n-j}(1-\delta_0^j)\sum_{i=0}^{j-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,j-1}}c_{\mu j_l}\overline{c_{\nu r}}\langle G_{a_i}S_{a_\mu}, S_{a_\nu}\rangle\\
&+\sum_{j=0}^{n-1}\alpha_{-(k_l+1)n-j}\delta_0^j\sum_{i=0}^{n-1}\sum_{\mu=0}^{j_l}\sum_{\nu=0}^{r}\overline{c_{i,n-1}}c_{\mu j_l}\overline{c_{\nu r}}(G_{a_i}f_a)(a_\nu)S_{a_\mu}(a_\nu)\\
&+\sum_{q=-\infty}^{-(k_l+2)n}\beta_q\langle E_qE_{l+n}, E_r\rangle\Bigg)\Bigg] = 0,
\end{aligned}
$$

then the product $T_\varphi T_\psi$ of two Toeplitz operators T_φ and T_ψ induces a Toeplitz matrix of order n.

Thus the equation (5.7) is a necessary condition for the product of two Toeplitz operators to be a Toeplitz operator and so by Proposition 3.4 we obtain the following result.

Corollary 5.3. *Let n be a positive integer. Suppose that Ω is a bounded n-connected domain with C^∞ smooth boundary. Let $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty} \beta_q E_q$ be symbols in $L^\infty(b\Omega)$. If the product $T_\varphi T_\psi$ of two Toeplitz operators T_φ and T_ψ on the Hardy space $H^2(b\Omega)$ satisfies* Cond(C) *and* (5.7)*, then it is a Toeplitz operator.*

In the case of simply connected domain Ω, either the former symbol ψ must be a function in the Hardy space $H^2(b\Omega)$ or some constant times the latter symbol φ plus the Szegő kernel S_a must be an element of the orthogonal complement $H^2(b\Omega)^\perp$ provided the product $T_\varphi T_\psi$ of two Toeplitz operators T_φ and T_ψ is a Toeplitz operator with some condition as follows.

Corollary 5.4. *Suppose that Ω is a simply connected bounded domain with C^∞ smooth boundary and suppose that the product $T_\varphi T_\psi$ of Toeplitz operators T_φ and T_ψ is a Toeplitz operator with $\varphi = \sum_{p=-\infty}^{\infty} \alpha_p E_p$ and $\psi = \sum_{q=-\infty}^{\infty} \beta_q E_q \in L^\infty(b\Omega)$. If, for all nonnegative integers m and l,*

$$\sum_{p=-\infty}^{-1} \alpha_{m+1+p} \langle E_p E_0, E_0 \rangle = \sum_{q=-\infty}^{-1} \beta_{-l-1+q} \langle E_q E_0, E_0 \rangle = 0, \tag{5.8}$$

then either

$$\varphi = \sum_{p=-\infty}^{0} \alpha_p E_p \ (\textit{which is in } H^2(b\Omega)^\perp)$$

or

$$\psi = \sum_{q=0}^{\infty} \beta_q E_q \ (\textit{which is in } H^2(b\Omega)).$$

In particular, in this case either $\varphi - \alpha_0 c_{00} S_a$ is in the orthogonal complement $H^2(b\Omega)^\perp$ or ψ is holomorphic in $H^2(b\Omega)$.

Remark 5.5. When Ω is the unit disc and the base point a is the origin $a = 0$ in the plane, this result is exactly the same as the one of Brown and Halmos [4]. In fact, for this case, $S_0 = \frac{1}{2\pi}, G_0 = \frac{1}{2\pi z}, f_0 = z, c_{00} = \sqrt{2\pi}$, $E_p = \frac{1}{\sqrt{2\pi}} z^p, p \in \mathbb{Z}$ and thus it is easy to see that for all $p \leq -1$, $\langle E_p E_0, E_0 \rangle = 0$ and hence the equation (5.8) trivially holds. Notice in particular that in this case, either the symbol φ becomes a co-analytic and the symbol ψ is analytic.

Proof of Corollary 5.4. When Ω is simply connected, for $m = k_m, l = k_l$, the equation (5.7) becomes

$$\left(\alpha_{m+1}c_{00}^3 S_a(a)^2 + \sum_{p=-\infty}^{m} \alpha_p \langle E_p E_0, E_{m+1}\rangle\right) \cdot \left(\alpha_{-l-1}c_{00}^3 S_a(a)^2 + \sum_{q=-\infty}^{-l-2} \beta_q \langle E_q E_{l+1}, E_0\rangle\right) = 0.$$

Thus by using $\langle E_p E_0, E_{m+1}\rangle = \langle E_{p-m-1} E_0, E_0\rangle$ and $\langle E_q E_{l+1}, E_0\rangle = \langle E_{q+l+1} E_0, E_0\rangle$ and letting $\widetilde{p} = p - m - 1$ and $\widetilde{q} = q + l + 1$, we obtain

$$\left(\alpha_{m+1}c_{00}^3 S_a(a)^2 + \sum_{\widetilde{p}=-\infty}^{-1} \alpha_{\widetilde{p}+m+1} \langle E_{\widetilde{p}} E_0, E_0\rangle\right) \cdot \left(\alpha_{-l-1}c_{00}^3 S_a(a)^2 + \sum_{\widetilde{q}=-\infty}^{-1} \beta_{\widetilde{q}-l-1} \langle E_{\widetilde{q}} E_0, E_0\rangle\right) = 0.$$

Hence if the equation (5.8) holds,

$$\alpha_{m+1}c_{00}^3 S_a(a)^2 \alpha_{-l-1}c_{00}^3 S_a(a)^2 = 0,$$

so for all $m, l \geq 0$, $\alpha_{m+1}\alpha_{-l-1} = 0$ and hence $\alpha_p = 0$ for $p \geq 1$ and $\beta_q = 0$ for $q \leq -1$, which proves the Corollary 5.4. $\square$

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