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A FAMILY RESOLVENT COCYCLE AND HIGHER SPECTRAL FLOW

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Abstract. In this paper, we introduce a family resolvent cocycle and express the Chern Character of Dai-Zhang higher spectral flow as a pairing of a family resolvent cocycle and the odd Chern character of a unitary matrix, which generalize the odd index formula of Carey et al. to the family case.

1. Introduction

In [8], Connes and Moscovici established an odd local index formula in the noncommutative geometry framework, which may be thought of as a far reaching generalisation of the classical index theorem for Toeplitz operators. Motivated by the work due to Coburn et al. [6], Carey et al. introduced the resolvant cocycle and extended the Connes-Moscovici local index formula to the type II setting in [3] and [4]. In [10], Higson introduced the residue cocycle and gave another proof of the Connes-Moscovici local index formula. In [5], Carey et al. established the relations between the resolvent cocycle and the residue cocycle.

On the other hand, Dai and Zhang defined a higher spectral flow as a K-group element and showed that this higher spectral flow can be computed analytically by $\widehat{\eta}$-forms and is related to the family index in the same way as the spectral flow is related to the index [9]. In [13], Perrot gave a bivariant generalization of the Connes-Moscovici local index formula. In [11], Benameur and Carey defined an bivariant JLO cocycle for a smooth fibration of closed manifolds and a family of generalised Dirac operators along the fibres. Then they decomposed the Chern Character of the Dai-Zhang higher spectral flow as a pairing of the bivariant JLO cocycle and the Chern character of an idempotent matrix. Motivated by [3] and [11], in this paper, we introduce a family resolvent cocycle and express the Dai-Zhang higher spectral flow as a pairing of a family resolvent cocycle and the Chern Character of a unitary which generalize the odd index formula of Carey et al. to the family case.

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This paper is organized as follows: In Section 2, we recall the definitions of the Bismut superconnection and the higher spectral flow. In Section 3, we obtain some norm and trace estimates which will be used in the next Sections. In Section 4, we introduced a family resolvent cocycle and rewrite the formula of the Chern Character of the higher spectral flow. In Section 5, we prove our main theorem and compute the Chern Character of the higher spectral flow by a family resolvent cocycle.

2. Higher spectral flow

Firstly we recall the definition of the Bismut superconnection. Let M_0 be a $(q'+p)$ dimensional compact connected manifold, and B_0 be a q' dimensional compact connected manifold. We assume that $\pi : M_0 \to B_0$ is a submersion of M_0 onto B_0, which defines a fibration of M_0 with fibre Z. For $y \in B_0$, $\pi^{-1}(y)$ is then a submanifold Z_y of M_0. TZ denotes the p-dimensional vector bundle on M_0 whose fibre T_xZ is the tangent space at x to the fibre $Z_{\pi x}$. We assume that M_0 and B_0 are oriented.

We fix a connection for this fibration which amounts to a splitting of the tangent bundle TM_0 into the horizontal bundle T^HM_0 and the vertical bundle TZ, i.e., $TM_0 = T^HM_0 \oplus TZ$. Vector $X \in TB_0$ will be identified with their horizontal lifts $X \in T^HM_0$, and $T_x^HM_0$ is isomorphic to $T_{\pi(x)}B_0$ via π_*. Recall that B_0 is Riemannian, so we can lift the Euclidean scalar product g_{B_0} of TB_0 to T^HM_0. And we assume that TZ is endowed with a scalar product g_Z. Thus we can introduce in TM a new scalar product $g_{B_0} \oplus g_Z$, and denote by ∇^L the Levi-Civita connection on TM with respect to this metric. Let ∇^{B_0} denote the Levi-Civita connection on TB_0 and we still denote by ∇^{B_0} the pullback connection on T^HM_0. Let $\nabla^Z = P_Z(\nabla^L)$ where P_Z denotes the projection to TZ. Let $\nabla^{\oplus} = \nabla^{B_0} \oplus \nabla^Z$ and $S = \nabla^L - \nabla^{\oplus}$ and T is the torsion tensor of $\nabla^{\oplus}$. Let $SO(TZ)$ be the $SO(n)$ bundle of oriented orthonormal frames in TZ. Now we assume that bundle TZ is spin. Let $S(TZ)$ be the associated spinors bundle and ∇^Z can be lifted to give a connection ∇^S on $S(TZ)$. Let D be the tangent Dirac operator. Let E be the vector bundle $\pi^*(\wedge T^*B_0) \otimes S(TZ)$. This bundle carries a natural action m_0 of the degenerate Clifford module $C_0(M_0)$. The Clifford action of a horizontal cotangent vector $e_i^* \in \Gamma(M_0, T_H^*M_0)$ is given by exterior multiplication $m_0(e_i^*) = \varepsilon(e_i^*)$ acting on the first factor $\bigwedge T_H^*M_0$ in E, while the Clifford action of a vertical cotangent vector simply equals its Clifford action on $S(TZ)$. Define the connection

$$\nabla^{E,\oplus} := \pi^*\nabla^{B_0} \otimes 1 + 1 \otimes \nabla^S, \tag{2.1}$$

$$\omega(X)(Y,Z) := g(\nabla_X^L Y, Z) - g(\nabla_X^{\oplus} Y, Z), \tag{2.2}$$

$$\nabla_X^{E,0} := \nabla_X^{E,\oplus} + \frac{1}{2} m_0(\omega(X)), \tag{2.3}$$

for $X, Y, Z \in TM_0$.

Let $\{e_i\}_{i=1}^{q'}$ be an oriented orthonormal basis of TB_0 and $\{f_j\}_{j=1}^{p}$ be an oriented orthonormal basis of TZ, let $c(e_i^*)$ and $c(f_j^*)$ denote the Clifford action. Then the Bismut superconnection acting on $\Gamma(M, \pi^* \wedge (B_0) \otimes S(TZ))$ is define by

$$B = \sum_{i=1}^{q'} c(e_i^*)\nabla_{e_i}^{E,0} + \sum_{j=1}^{p} c(f_j^*)\nabla_{f_j}^{E,0}. \tag{2.4}$$

By Theorem 10.17 in [1], we have

$$\mathrm{F} = B^2 = -\sum_{j=1}^{p}(\nabla_{f_j}^{E,0})^2 + \sum_{j=1}^{p}\nabla_{\nabla_{f_i}^{Z} f_i}^{E,0} + \frac{1}{4}r_Z = D^2 + F_{[+]}, \tag{2.5}$$

where D is a family of generalized Dirac operators over B_0, $F_{[+]}$ is an operator with coefficients in $\Omega_{\geq 1}(B_0)$ and r_Z is the scalar curvature of fibres.

Higher spectral flow, for a family of fiberwise self-adjoint elliptic operators $D = (D_b)_{b\in B_0}$ introduced in [9], is only well defined under the assumption that the K^1 class defined by the family is trivial. Assume that the index bundle of D_0 vanishes and let Q_0, Q_1 be spectral sections of D_0, D_1 respectively. If we consider the total family $\widetilde{D} = \{D_{b,u}\}$ parametrized by $B_0 \times I$, then there is also a total spectral section $\widetilde{P} = \{P_{b,u}\}$. Let P_u be the restriction of P over $B \times \{u\}$. The (higher) spectral flow sf $\{(D_0, Q_0), (D_1, Q_1)\}$ between the pairs $(D_0, Q_0), (D_1, Q_1)$ is an element in $K(B_0)$ defined by

$$sf\{(D_0, Q_0), (D_1, Q_1)\} = [Q_1 - P_1] - [Q_0 - P_0] \in K(B_0). \tag{2.6}$$

It is easy to check that $sf\{(D_0, Q_0), (D_1, Q_1)\}$ does not depend on the choice of the global spectral section $\widetilde{P}$. In this paper we are mainly interested in the affine path $D_t := D + tU^{-1}[D, U]$ where D is a family of generalized Dirac operators over B_0 whose index class in $K_1(B_0)$ is trivial, and U is a given element of $GL_N(C^\infty(M_0))$. In this case, the endpoints are conjugate and we consider the higher spectral flow with respect to the spectral sections $P_0 = P$ and $P_1 = U^{-1}PU$, where P is a fixed spectral section for D. It turns out that the higher spectral flow does not depend on P either and is an invariant of the principal symbol of D and of the homotopy class of U. We denote it $sf(D, U)$. Indeed, Dai and Zhang proved the following proposition (for related notations, see [9]).

Proposition 2.1 ([9])**.** *We have in $K^0(B_0)$, $Ind(T_U) = -sf(D, U)$.*

The next proposition is an easy rephrasing of a result of Dai and Zhang:

Proposition 2.2 ([9])**.** *Let B be the Bismut superconnection associated with σD, then the cohomology class of the differential form $\frac{-1}{\pi^{\frac{1}{2}}}\int_0^1 Tr_\sigma(B_t e^{-B_t^2})dt$ coincides with the Chern character of the higher spectral flow, i.e.,*

$$Ch(sf(D, U)) = \frac{-1}{\pi^{\frac{1}{2}}}\int_0^1 Tr_\sigma(B_t e^{-B_t^2})dt. \tag{2.7}$$

3. Norm and trace estimates

Throughout this section, let $D : domD \subsetneqq \mathcal{H} \to \mathcal{H}$ be an unbounded self-adjoint operator on the Hilbert space $\mathcal{H}$. In a number of estimates, we will also consider a bounded self-adjoint operator A. The operators A that are of interest satisfy $s^2+sA+D^2 \geq 0$ for all real $s \geq 0$. However it is also convenient at times to assume that $\| A \|$ is relatively small: $\| A \| < \sqrt{2}$, for example. This can be achieved by scaling A: see Observation 2 of Section 5 in [3].

Lemma 3.1 (Lemma 5.1 in [3])**.** *Let D be an unbounded self-adjoint operator,*

(a) *For $\lambda = a + iv \in \mathbb{C}, 0 < a < 1/2, s \geq 0$ we have the estimate*

$$\|(\lambda - (1 + D^2 + s^2))^{-1}\| \leq (v^2 + (1 + s^2 - a)^2)^{-1/2} \leq \frac{1}{1-a}; \tag{3.1}$$

(b) *If A is bounded, self-adjoint and $s^2 + sA + D^2 \geq 0$ we have*

$$\|(\lambda - (1 + D^2 + s^2 + sA))^{-1}\| \leq (v^2 + (1 - a)^2)^{-1/2} \leq \frac{1}{1-a}; \tag{3.2}$$

(c) *If A is bounded, self-adjoint and $c = \|A\| < \sqrt{2}$ we have*

$$\|(\lambda - (1 + D^2 + s^2 + sA))^{-1}\| \leq (v^2 + (1 + s^2 - a - sc)^2)^{-1/2} \leq \frac{1}{1/2-a}. \tag{3.3}$$

Let $Q = (1 + s^2 + F)$, where F is B^2 as defined in Section 2 and $s \in [0, \infty)$. Define

$$\begin{aligned}(\lambda - Q)^{-1} &= \Big(\lambda - (1 + s^2 + D^2)\Big)^{-1} \\ &\quad + \sum_{k>0} \Big[\Big(\lambda - (1 + s^2 + D^2)\Big)^{-1} F_{[+]}\Big]^k \Big(\lambda - (1 + s^2 + D^2)\Big)^{-1}.\end{aligned} \tag{3.4}$$

Then for $z \in \mathbb{C}$ and $\mathrm{Re}(z) > p/2$, we write Q^{-z} using Cauchy's formula (see [12])

$$Q^{-z} = \frac{1}{2\pi i} \int_l \lambda^{-z} (\lambda - Q)^{-1} d\lambda, \tag{3.5}$$

where l is a vertical line $\lambda = a + iv$ parametrized by $v \in \mathbb{R}$ with $0 < a < 1/2$ fixed. Let $T^{(n)} = [F, \ldots [F, \ldots [F, T]]]$, then:

Lemma 3.2 (Compare with Lemma 6.9 in [3])**.** *Let m, n, k be non-negative integers and $T \in OP^m$. Then*

$$\begin{aligned}(\lambda - Q)^{-n} T &= T(\lambda - Q)^{-n} + nT^{(1)}(\lambda - Q)^{-(n+1)} + \frac{n(n+1)}{2} T^{(2)}(\lambda - Q)^{-(n+2)} \\ &\quad + \cdots + \begin{pmatrix} n+k-1 \\ k \end{pmatrix} T^{(k)}(\lambda - Q)^{-(n+k)} + P(\lambda) \\ &= \sum_{j=0}^{k} \begin{pmatrix} n+j-1 \\ j \end{pmatrix} T^{(j)}(\lambda - Q)^{-(n+j)} + P(\lambda),\end{aligned} \tag{3.6}$$

where the remainder $P(\lambda)$ *has order* $-(2n+k-m+1)$ *and is given by*

$$P(\lambda)=\sum_{j=1}^{k}\begin{pmatrix} j+k-1 \\ k \end{pmatrix}(\lambda-Q)^{j-n-1}T^{(k+1)}(\lambda-Q)^{-j-k}. \tag{3.7}$$

Corollary 3.3 (Compare with Corollary on Page 484 in [3])**.** *Let* n, M *be positive integers and* $A\in OP^k$*, let* $R=(\lambda-Q)^{-1}$*. Then*

$$R^nAR^{-n}=\sum_{j=0}^{M}\begin{pmatrix} n+j-1 \\ j \end{pmatrix}A^{(j)}R^j+P, \tag{3.8}$$

where

$$P=\sum_{j=0}^{n}\begin{pmatrix} j+M-1 \\ M \end{pmatrix}R^{n+1-j}A^{(M+1)}R^{M+j-n} \tag{3.9}$$

has order $k-M-1$.

We recall Lemma 6.10 in [3], let OP^k denote the set of order $\leq k$ vertical pseudodifferential operators along the fiber.

Lemma 3.4 (Compare with Lemma 6.10 in [3])**.** *Let* k, n *be non-negative integers, and suppose* $\lambda\in\mathbb{C}, a=\mathrm{Re}(\lambda)$ *with* $0<a<1/2$*. For* $A\in OP^k$ *and* $R_s(\lambda)=(\lambda-(1+D^2+s^2))^{-1}$*, we have*

$$\|R_s(\lambda)^{n/2+k/2}AR_s(\lambda)^{-n/2}\|\leq C_{n,k} \text{ and } \|R_s(\lambda)^{-n/2}AR_s(\lambda)^{n/2+k/2}\|\leq C_{n,k}, \tag{3.10}$$

where $C_{n,k}$ *is constant independent of* s *and* λ (*square roots use the principal branch of* log).

Then for the family case, we have:

Lemma 3.5 (Compare with Lemma 6.11 in [3])**.** *Let* $A_j\in OP^{n_j}$ *for* $j=1,\ldots,m$ *and let* $0<a=\mathrm{Re}(\lambda)<1/2$ *as above. We consider the operator*

$$R_s^F(\lambda)A_1R_s^F(\lambda)A_2\cdots R_s^F(\lambda)A_m\widetilde{R}_s^F(\lambda), \tag{3.11}$$

where $R_s^F(\lambda)=(\lambda-(1+s^2+F))^{-1}$, $\widetilde{R}_s^F(\lambda)=\big(\lambda-(1+s^2+sX+F)\big)^{-1}$ *and* X *is self-adjoint, bounded and* $s^2+sX+D^2\geq 0$*. Then for all* $M\geq 0$

$$\begin{aligned} &R_s^F(\lambda)A_1R_s^F(\lambda)A_2\cdots A_m\widetilde{R}_s^F(\lambda) \\ &=\sum_{|k|=0}^{M}C(k)A_1^{(k_1)}\cdots A_m^{(k_m)}R_s^F(\lambda)^{m+|k|}\widetilde{R}_s^F(\lambda)+P_{M,m,F}, \end{aligned} \tag{3.12}$$

where $P_{M,m,F}$ *is of order* (*at most*) $-2m-M+3+|n|$, *and* k, n *are multi-indices with* $|k|=k_1+\cdots+k_m$ *and* $|n|=n_1+\cdots+n_m$*. The constant* $C(k)$ *is given by*

$$C(k)=\frac{(|k|+m)!}{k_1!k_2!\cdots k_m!(k_1+1)(k_1+k_2+2)\cdots(|k|+m)}=(|k|+m)!\alpha(k). \tag{3.13}$$

Lemma 3.6. *With the assumptions and notation of the last lemma including the assumption that $A_i \in OP^{n_i}$ for each i, there is a positive constant C such that*

$$\|(\lambda - (1 + D^2 + s^2))^{m+M/2+1/2-|n|/2} P_{M,m,F}\| \leq C \tag{3.14}$$

independent of s and λ (though it depends on M, m and the A_i).

Proof. The remainder $P_{M,m,F}$ in the previous lemma obtained after applying the pseudodifferential expansion has terms of two kinds. The first kind we consider are the bookkeeping terms at the end of the proof of the last lemma. They are of the form

$$P = A_1^{(k_1)} \cdots A_m^{(k_m)} R_F^{m+|k|} \widetilde{R}_F \tag{3.15}$$

with $|k| > M$.

Next we prove the family term

$$R^{-(m+M/2+1/2)+|n|/2} A_1^{(k_1)} \cdots A_m^{(k_m)} R_F^{m+|k|} \widetilde{R}_F \tag{3.16}$$

is uniformly bounded. We know that

$$R_F = R + \sum_{d \geq 1} R(F_{[+]}R)^d. \tag{3.17}$$

In order to estimate (3.15), we only need estimate the following term. By Lemma 3.4 we have

$$\begin{aligned} &\| R^{-(m+M/2+1/2)+|n|/2} A_1^{(k_1)} \cdots A_m^{(k_m)} \big[R^{l_1} R(F_{[+]}R)^d R^{(m+|k|-l_1-1)}\big] \| \\ &= \| R^{-(m+M/2+1/2)+|n|/2} A_1^{(k_1)} \\ &\qquad \cdots A_m^{(k_m)} R^{m+|k|} R^{-(m+|k|)+l_1+1} (F_{[+]}R)^d R^{(m+|k|-l_1-1)} \| \\ &\leq C_0 \| R^{-(m+|k|)+l_1+1} F_{[+]} R \cdots F_{[+]} R R^{(m+|k|-l_1-1)} \| \\ &\leq C_0 \| R^{-(m+|k|)+l_1+1} F_{[+]} R^{m+|k|-l_1} R^{-(m+|k|)+l_1+1} F_{[+]} R^{m+|k|-l_1} \\ &\qquad \cdots F_{[+]} R^{m+|k|-l_1} \| \\ &\leq C_1, \end{aligned} \tag{3.18}$$

where $1 \leq j \leq i + k_1 + k_2 + \cdots + k_{i-1}$ and $0 \leq k_1, k_2, \ldots, k_{i-1} \leq M$. Then this term is bounded.

On the other hand, we will prove $\widetilde{R}_F$ is bounded. By Observation 2 on page 497 in [5], without loss of generality, we assume $\| X \| \leq \sqrt{2}$, then

$$\begin{aligned} \widetilde{R}_s^F(\lambda) &= (\lambda - (1 + s^2 + sX + F))^{-1} \\ &= \big(\lambda - (1 + s^2 + sX + D^2) - F_{[+]}\big)^{-1} \\ &= (\lambda - (1 + s^2 + sX + D^2))^{-1} + \sum_{k \geq 1} (\lambda - (1 + s^2 + sX + D^2))^{-1} \\ &\qquad \times \big[- F_{[+]}(\lambda - (1 + s^2 + sX + D^2))^{-1}\big]^k. \end{aligned} \tag{3.19}$$

By Lemma 3.1(c), we obtain

$$\|(\lambda-(1+s^2+sX+D^2))^{-1}\| \le \frac{1}{1/2-a}. \tag{3.20}$$

Since $\|X\| \le \sqrt{2}$, we have $(1+s^2+sX+D^2))^{-1} \le (1+s^2+s\|X\|+D^2))^{-1}$, then

$$\begin{aligned}
&\|F_{[+]}\big(\lambda-(1+s^2+sX+D^2)\big)^{-1}\| \\
&= \|F_{[+]}(1+s^2+sX+D^2)^{-1}\frac{(1+s^2+sX+D^2)}{\lambda-(1+s^2+sX+D^2)}\| \\
&\le \|F_{[+]}(1+s^2+sX+D^2)^{-1}\| \sup_{x\in[0,+\infty)}\Big|\frac{1+x}{\lambda-(1+x)}\Big| \\
&\le \|F_{[+]}(1+D^2)^{-\frac12}\|\|(1+D^2)^{\frac12}(1+s^2+sX+D^2)^{-1}\| \\
&\quad \times \frac{1+x}{\sqrt{(1+x-a)^2+v^2}} \\
&\le \widetilde{C}_0\|\frac{(1+D^2)^{1/2}}{1+s^2-s\|X\|+D^2}\|\frac{1+x}{1+x-a} \\
&\le \widetilde{C}_1. \qquad (3.21)
\end{aligned}$$

Combining these assertions, we see

$$\|\widetilde{R}^F(\lambda)\| \le \frac{\overline{C}}{\frac12-a}. \tag{3.22}$$

The other terms are the ones $P_1, P_2, \ldots, P_m$ obtained in the proof of the last lemma. Recall:

$$P_1 = R_F A_1^{(M+1)} R_F^{M+1} R_F A_2 R_F \cdots R_F A_m \widetilde{R}_F, \tag{3.23}$$

while a typical summand of P_2 is an integer multiple of:

$$A_1^{(k_1)} R_F^{3+k_1-j} A_2^{(M+1)} R_F^{M+j} R_F A_3 R_F \cdots R_F A_m \widetilde{R}_F, \tag{3.24}$$

where $1 \le j \le 2+k_1$ and $0 \le k_1 \le M$.

We work with the typical summand of P_i above, and let

$$B = A_1^{(k_1)} A_2^{(k_2)} \cdots A_{i-1}^{(k_i-1)}$$

which has order $(k_1+k_2+\cdots+k_{i-1})+(n_1+n_2+\cdots+n_{i-1}) = |k|+|n|_{i-1}$, where we have used the notation $|n|_j = n_1+n_2+\cdots+n_j$. We will also use the notation $|n|^{j+1} = n_{j+1}+\cdots+n_m$, then $|n| = |n|_j + |n|^{j+1}$. We need to show that

$$R^{-(m+M/2+1/2)+|n|/2} B R_F^{i+1+|k|-j} A_i^{(M+1)} R_F^{m+j} R_F A_{i+1} R_F \cdots R_F A_m \widetilde{R}_F \tag{3.25}$$

is bounded independent of s and λ. So, we calculate

$$\begin{aligned}
&R^{-(m+M/2+1/2)+|n|/2} B R_F^{i+1+|k|-j} A_i^{(M+1)} R_F^{M+j} R_F A_{i+1} R_F \cdots R_F A_m \widetilde{R}_F \\
&= \Big(R^{-(m+M/2+1/2-|n|/2)} B R^{(|k|+|n|_{i-1})/2} R^{(m+M/2+1/2-|n|/2)}\Big) R^{|k|/2}
\end{aligned}$$

$$
\begin{aligned}
&\times\Big(R^{-(|k|+|n|_{i-1})/2}R^{-|k|/2}R^{-(m+M/2+1/2-|n|/2)}R_F^{i+1+|k|-j}R^{(m+M/2+j-i-1/2-|n|^i/2)}\Big)\\
&\times\Big(R^{-(m+M/2+j-i-1/2)+|n|^i/2}A_i^{(M+1)}R^{((m+M/2+j-i-1/2)-|n|^i/2)+(n_i+M+1)/2}\Big)\\
&\times\Big(R^{-(m+M/2+j-i-1/2)+|n|^i/2-(n_i+M+1)/2}R_F^{m+j+1}R^{(m-i-1)-|n|^{i+1}/2}\Big)\\
&\times\Big(R^{-(m-i-1)+|n|^{i+1}/2}A_{i+1}R^{(m-i-1)-|n|^{i+1}/2+n_{i+1}/2}\Big)\\
&\times\Big(R^{-(m-i-1)+|n|^{i+1}/2-n^{i+1}/2}R_F R^{(m-i-2)-|n|^{i+2}/2}\Big)\\
&\times\Big(R^{-(m-i-2)+|n|^{i+2}/2}A_{i+2}R^{(m-i-2)-|n|^{i+2}/2+n_{i+2}/2}\Big)\times\cdots\\
&\times\Big(R^{-1+(n_{m-1}+n_m)/2}A_{m-1}R^{-(n_{m-1}+n_m)/2+n_{m-1}/2+1}\Big)\\
&\times\Big(R^{(n_{m-1}+n_m)/2-n_{m-1}/2-1}R_F R^{-n_m/2}\Big)R^{n_m/2}A_m\widetilde{R}_F. \qquad (3.26)
\end{aligned}
$$

Then each bracketed term in the last expression is bounded independent of s and λ by an application of Lemma 3.4, (3.22) and

$$\|R^{|k|/2}\| \le (\frac{1}{1-a})^{|k|/2}. \qquad (3.27)$$

□

4. A family resolvent cocycle

4.1. Resolvent expansion of the higher spectral flow

We require two estimates to guarantee that various operators which arise from the Cauchy formula and the resolvent expansion are trace class. We present these as separate lemmas as we will use them repeatedly. The techniques we use in these lemmas are indicative of the methods employed in the remainder of the proof. The computation begins by recalling the definition of q and $\{B, q\}$.

Definition 4.1. Form the Hilbert space $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes L^2(M_0, S(TZ) \otimes \pi^* \wedge^* (T^*B_0))$ acted by $\mathcal{A} = M^2 \otimes M^2 \otimes M_{k\times k}(C^\infty(M_0))$, where M^2 denotes the set of the 2×2 order matrix. Introduce the two dimensional Clifford algebra in the form

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad (4.1)$$

and define the grading on $\mathcal{H}$ by $\Gamma = \sigma_2 \otimes \sigma_3 \otimes 1 \in \mathcal{A}$.

Let $u \in \mathcal{A}$ be unitary and we introduce the following even operators (i.e., they commute with Γ):

$$
\begin{aligned}
&q = \sigma_3 \otimes \begin{pmatrix} 0 & -iu^{-1} \\ iu & 0 \end{pmatrix},\ \widetilde{B} = \sigma_2 \otimes 1_2 \otimes B,\\
&\widetilde{B}q + q\widetilde{B} = \sigma_1 \otimes \begin{pmatrix} 0 & [B, u^{-1}] \\ -[B, u] & 0 \end{pmatrix}.
\end{aligned} \qquad (4.2)
$$

Define $Str^{even}(a) = \frac{1}{2}Tr^{even}(\Gamma(a))$, then

$$\widetilde{B}^2_{r,s} = \widetilde{B}^2_r + s(1-2r)\sigma_1 \otimes \begin{pmatrix} 0 & [B, u^{-1}] \\ -[B, u] & 0 \end{pmatrix} + s^2. \tag{4.3}$$

From Lemma 3.5 and Lemma 3.6, we have:

Lemma 4.2 (Compare with Lemma 7.1 in [3]). *Let* $\dim Z = p$ *and* $\pi : M_0 \to B_0$ *be a fibration. Let* m *be a nonnegative integer, and for* $j = 0, \dots, m$ *let* $A_j \in OP^0$. *Define* $\widetilde{F} = \sigma_2 \otimes 1_2 \otimes F$ *on* $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \Gamma(\wedge(T^*B_0) \otimes S(Z))$. *Let* l *be the vertical line* $v \mapsto \lambda = a + iv$ *for* $v \in R$ *and* $0 \leq a \leq 1$, $R_s^{\widetilde{F}}(\lambda) = (\lambda - (1 + s^2 + \widetilde{F}))^{-1}$ *and* $\widetilde{R}_s^{\widetilde{F}}(\lambda) = \lambda - (1 + s^2 + \widetilde{F} + s\{\widetilde{B}, q\}))^{-1}$. *Then for* $r \in \mathbb{C}$ *and* $\mathrm{Re}(r) > 0$ *the operator*

$$B(s) = \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} A_0 R_s^{\widetilde{F}}(\lambda) A_1 R_s^{\widetilde{F}}(\lambda) A_2 \cdots R_s^{\widetilde{F}}(\lambda) A_m \widetilde{R}_s^{\widetilde{F}}(\lambda) d\lambda, \tag{4.4}$$

is trace class for $m > p/2$ *and the function* $s^m \|B(s)\|_l$ *is integrable on* $[0, \infty)$ *when*

$$p + \varepsilon < 1 + m \text{ and } 1 + \varepsilon < m + 2\mathrm{Re}(r). \tag{4.5}$$

Lemma 4.3 (Compare with Lemma 7.2 in [3]). *Let* m *be a nonnegative integer, and for* $j = 0, \dots, m$ *let* $A_j \in OP^{k_j}$, $k_j \geq 0$. *Let* l *be the vertical line* $v \mapsto \lambda = a + iv$ *for* $v \in R$ *and* $0 \leq a \leq 1$, $R_s^{\widetilde{F}}(\lambda) = (\lambda - (1 + s^2 + \widetilde{F}))^{-1}$. *Then for* $\mathrm{Re}(r) > 0$ *the operator*

$$B(s) = \frac{1}{2\pi i}\int_l \lambda^{-p/2-r} A_0 R_s^{\widetilde{F}}(\lambda) A_1 R_s^{\widetilde{F}}(\lambda) A_2 \cdots R_s^{\widetilde{F}}(\lambda) A_m R_s^{\widetilde{F}}(\lambda) d\lambda, \tag{4.6}$$

is trace class for $\mathrm{Re}(r) + m - \frac{|k|}{2} > 0$ *and the function* $s^\alpha \times \|B(s)\|_l$ *is integrable on* $[0, \infty)$ *when*

$$1 + \alpha + |k| - 2m < 2(\mathrm{Re}(r) - \varepsilon). \tag{4.7}$$

Lemma 4.4 (Compare with Lemma 7.3 in [3]). *With the notation as set out at the beginning of this section and with* $R_s^{\widetilde{F}}(\lambda) = (\lambda - (1 + s^2 + \widetilde{F}))^{-1}$, $\widetilde{R}_s^{\widetilde{F}}(\lambda) = (\lambda - (1 + s^2 + \widetilde{F} + s\{B, q\}))^{-1}$, *we have for* $\mathrm{Re}(r) > 0$ *and any positive integer* $M > p - 1$:

$$\begin{aligned} & Str^{even}\Big(q\big(1 + s^2 + \widetilde{F} + s\{\widetilde{B}, q\}\big)^{-\frac{p}{2}-r}\Big) \\ = & \sum_{m=1,odd}^{M} s^m Str^{even}\Big(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\big(R_s^{\widetilde{F}}(\lambda)\{\widetilde{B}, q\}\big)^m R_s^{\widetilde{F}}(\lambda) d\lambda\Big) \\ & + s^{M+1} Str^{even}\Big(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} q\big(R_s^{\widetilde{F}}(\lambda)\{\widetilde{B}, q\}\big)^{M+1} \widetilde{R}_s^{\widetilde{F}}(\lambda) d\lambda\Big), \end{aligned} \tag{4.8}$$

where Str^{even} *denotes taking supertrace with value in* $\Omega^{even}(B_0)$.

Let $\widetilde{B}_r = (1-r)\widetilde{B} - rq\widetilde{B}q$ and $\widetilde{B}_{r,s} = \widetilde{B}_r + sq$, then

$$\widetilde{B}_r \equiv \widetilde{B}_{r,0} = \sigma_2 \otimes \begin{pmatrix} B + ru^{-1}[B,u] & 0 \\ 0 & B + ru[B,u^{-1}] \end{pmatrix}. \tag{4.9}$$

Theorem 4.5. *Let $n > p = \dim Z$ and $l \leq \dim B_0$. Then*

$$\begin{aligned} & Ch\Big(sf(D,u^*Du)\Big)^{[l]} \\ &= \frac{\Gamma(n/2)}{\Gamma((n+l-1)/2)\sqrt{\pi}} \int_0^1 Tr^{even}\Big(u^*[B,u](1+(B+ru^*[B,u])^2)^{-\frac{n}{2}}\Big)^{[l]} dr \\ &= \frac{\Gamma(n/2)}{2\Gamma((n+l-1)/2)\sqrt{\pi}} \int_0^1 Str^{even}\Big(\frac{d\widetilde{B}_r}{dr}(1+\widetilde{B}_r^2)^{-\frac{n}{2}}\Big)^{[l]} dr. \end{aligned} \tag{4.10}$$

Proof. From (4.19), (4.29) and (4.41) in [9], we have

$$Ch\Big(sf(D,u^*Du)\Big) = \frac{-1}{\sqrt{\pi}} \int_0^1 Tr^{even}\Big(\frac{\partial B_t(r)}{\partial r}\exp(-B_t^2(r))\Big), \tag{4.11}$$

where $B_t(r) = B_t(0) + ru[B_t(0),u^{-1}]$ and $B_t(0) = \sqrt{t}\psi_t B$, $\psi_t : dy_j \to \frac{dy_j}{\sqrt{t}}$. Taking the l-degree component of $\Omega(B_0)$ on the last Eq., we obtain

$$\begin{aligned} & Ch\Big(sf(D,u^*Du)\Big)^{[l]} \\ &= \frac{-1}{\sqrt{\pi}} t^{-\frac{l}{2}} \int_0^1 Tr^{[l]}\Big(\sqrt{t}u^{-1}[B,u]\exp(-t(B+ru^{-1}[B,u])^2)\Big) dr. \end{aligned} \tag{4.12}$$

By the definition of the Gamma function, we have

$$1 = \frac{1}{\Gamma(\frac{n+l-1}{2})} \int_0^{+\infty} t^{\frac{n+l-3}{2}} e^{-t} dt. \tag{4.13}$$

Combining the above equations, and by Lemma 1 in [12] and the following Lemma 4.6, we have

$$\begin{aligned} & Ch\Big(sf(D,u^*Du)\Big)^{[l]} \\ &= \frac{-1}{\Gamma(\frac{n+l-1}{2})} \int_0^{+\infty} t^{\frac{n-3}{2}} e^{-t} \sqrt{\frac{t}{\pi}} \int_0^1 Tr^{[l]}\Big(u^{-1}[B,u]e^{-t(B+ru^{-1}[B,u])^2}\Big) dr dt \\ &= \frac{-1}{\Gamma(\frac{n+l-1}{2})\sqrt{\pi}} \int_0^1 \int_0^{+\infty} Tr^{[l]}\Big(t^{\frac{p-2}{2}} u^{-1}[B,u]e^{-t(1+(B+ru^{-1}[B,u])^2)}\Big) dr dt \\ &= \frac{-\Gamma(\frac{n}{2})}{\Gamma(\frac{n+l-1}{2})\sqrt{\pi}} \int_0^1 Tr^l\Big(u^{-1}[B,u](1+(B+ru^{-1}[B,u])^2)^{-\frac{n}{2}}\Big) dr. \end{aligned} \tag{4.14}$$

Then similarly to the computations in [3, p. 73], we get Theorem 4.5. $\square$

Lemma 4.6. *The following expression*

$$\int_0^{+\infty}\int_0^1 t^{\frac{n-3}{2}}e^{-t}\sqrt{t}Tr^{[l]}\Big(u^{-1}[B,u]e^{-t(B+ru^{-1}[B,u])^2}\Big)drdt \tag{4.15}$$

is uniformly convergent.

Proof. Since $n>\dim Z\geq 1$, then $\frac{n-3}{2}\geq -\frac{1}{2}$. By Theorem 4.8 in [9], we have when $t\to 0$,

$$t^{\frac{n-3}{2}}e^{-t}t^{-\frac{l}{2}}Tr^{[l]}\Big(\sqrt{t}u^{-1}[B,u]e^{-t(B+ru^{-1}[B,u])^2}\Big)\to O(t^{-\frac{1}{2}}). \tag{4.16}$$

So the above integral is convergent at $t=0$.

Let $(B+ru^{-1}[B,u])^2=D_u^2+F'_{[+]}$, where $F'_{[+]}$ is a first order differential operator with coefficient in $\Omega_{\geq 1}(B_0)$. By the Duhamel's principle and the Hölder equality, we have

$$\begin{aligned}
&\| Tr^{[l]}\Big(u^{-1}[B,u]e^{-t(B+ru^{-1}[B,u])^2}\Big)\|_1\\
\leq &\sum_{k\geq 0}^{\dim B_0} t^k\int_{\triangle_k}\| u^{-1}[B,u]e^{-t\sigma_0 D_u^2}F'_{[+]}\cdots F'_{[+]}e^{-t\sigma_k D_u^2}\|_1\, d\triangle_k\\
\leq &\sum_{k\geq 0}^{\dim B_0} t^k\int_{\triangle_k}\| u^{-1}[B,u]\|\| e^{-\frac{t}{2}\sigma_0 D_u^2}\|_{\sigma_0^{-1}}\| e^{-\frac{t}{2}\sigma_0 D_u^2}(\varepsilon+D_u^2)^{\frac{1}{2}}\\
&\|\| (\varepsilon+D_u^2)^{-\frac{1}{2}}F'_{[+]}\|\cdots\| e^{-\frac{t}{2}\sigma_{k-1}D_u^2}\|_{\sigma_{k-1}^{-1}}\| e^{-\frac{t}{2}\sigma_{k-1}D_u^2}(\varepsilon+D_u^2)^{\frac{1}{2}}\|\\
&\| (\varepsilon+D_u^2)^{-\frac{1}{2}}F'_{[+]}\|\| e^{-t\sigma_k D_u^2}\|_{\sigma_k^{-1}}\, d\triangle_k.
\end{aligned} \tag{4.17}$$

By

$$\sup_{x\in[0,+\infty)}\Big(e^{-cx}(\varepsilon+x)^{\frac{1}{2}}\Big)=e^{-\frac{1}{2}+c\varepsilon}(\frac{1}{2c})^{\frac{1}{2}}, \tag{4.18}$$

then

(4.19)

$$\begin{aligned}
&t^{\frac{n}{2}-1}e^{-t}\| Tr^l\Big(u^{-1}[B,u]e^{-t(B+ru^{-1}[B,u])^2}\Big)\|_1\\
\leq\ & t^{\frac{n}{2}-1}e^{-t}\sum_{k\geq 0}^{\dim B_0}t^k\int_{\triangle_k}L_0(tre^{-\frac{t}{2}D_u^2})e^{-\frac{1}{2}(k-1)}e^{t\varepsilon}(\frac{1}{2})^{\frac{k}{2}}\frac{1}{t^{\frac{k}{2}}}\frac{1}{\sqrt{\sigma_0\cdots\sigma_{k-1}}}d\triangle_k,
\end{aligned}$$

where L_0 is constant. By Theorem 8 in [2], then

$$t^{\frac{n-3}{2}}e^{-t}t^{-\frac{l}{2}}Tr^{[l]}\Big(\sqrt{t}u^{-1}[B,u]e^{-t(B+ru^{-1}[B,u])^2}\Big) \tag{4.20}$$

exponentially decays and is bounded independently of u, and is uniformly convergent, when $t\to+\infty$. □

Define

$$\mathcal{C}_{p/2+r,l} = \frac{\Gamma(p/2+r+l/2-1/2)\sqrt{\pi}}{\Gamma(p/2+r)}. \tag{4.21}$$

And then, the main result of this section is the following lemma. By the Mellin transform, we may prove the family case of Lemmas 5.6, 5.8 in [3], by the associated Lemmas 5.6 and 5.12 in [11], similar to the proof of Lemma 7.4 in [3], we get:

Lemma 4.7. *Let $N = [(p+q')/2] + 1$ be the least positive integer strictly greater than $(p+q')/2$. Then there is a δ', $0 < \delta < 1$ such that up to an exact form on B_0,*

$$\begin{aligned} &Ch\Big(sf(D, u^*Du)\Big)^{[l]} \mathcal{C}_{p/2+r,l} \\ &= \int_0^{+\infty} Str^{even}\Big(q(1+s^2+\widetilde{F}+s\{\widetilde{B},q\})^{-\frac{p}{2}-r}\Big)^{[l]} ds \\ &= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^{+\infty} s^m Str^{even}\Big(\int_l \lambda^{-p/2-r} q\big(R_s^{\widetilde{F}}(\lambda)\{\widetilde{B},q\}\big)^m \\ &\quad \times R_s^{\widetilde{F}}(\lambda)d\lambda\Big)^{[l]} ds + holo, \end{aligned} \tag{4.22}$$

where holo is a function of r holomorphic for $\mathrm{Re}(r) > -(p+q')/2 + \delta'/2$.

4.2. A resolvent cocycle

At this point it is interesting to perform the supertrace, so that we have an expression which only depends on our original spectral triple $(A, \mathcal{H}, D)$.

Definition 4.8. For $m \geq 0$, operators $A_0, \ldots, A_m$, $A_j \in OP^{k_j}$ and $2\mathrm{Re}(r) > k_0 + \cdots + k_m - 2m$ define

$$\begin{aligned} &\langle A_0, \ldots, A_m\rangle_{m,s,r} \\ &= Tr^{even}\Big(\frac{1}{2\pi i}\int_l \lambda^{-p/2-r} A_0 R_s^F(\lambda) A_1 \cdots A_m R_s^F(\lambda) d\lambda\Big). \end{aligned} \tag{4.23}$$

The conditions on the orders and on r are sufficient for the trace to be well-defined, then:

Lemma 4.9 (Compare with Lemma 7.6 in [3])**.** *For any integers $m \geq 0, k \geq 1$ and operators $A_0, \ldots, A_m$ with $A_j \in OP^{k_j}$, and $2\mathrm{Re}(r) > k + \sum k_j - 2m$, we may choose r with $\mathrm{Re}(r)$ sufficiently large such that*

$$\begin{aligned} &k\int_0^\infty s^{k-1}\langle A_0, \ldots, A_m\rangle_{m,s,r} ds \\ &= -2\sum_{j=0}^m \int_0^\infty s^{k+1}\langle A_0, \ldots, A_j, 1, A_{j+1}, \ldots, A_m\rangle_{m+1,s,r} ds. \end{aligned} \tag{4.24}$$

Lemma 4.10 (Compare with Lemma 7.7 in [3])**.** *For any integers* $m \geq 0$, $k \geq 1$*and* $\mathrm{Re}(r)$ *sufficiently large, and* $A_j \in OP^{k_j}, j = 0, \dots, m$, *we have*

$$\int_0^\infty s^k \langle A_0, \dots, A_m \rangle_{m,s,r} ds = \int_0^\infty s^k \langle A_m, A_0, \dots, A_{m-1} \rangle_{m,s,r} ds. \tag{4.25}$$

Lemma 4.11. *For operators* $A_0, \dots, A_m$, $A_j \in OP^{k_j}$, $k_j \geq 0$, *and* $\mathrm{Re}(r)$ *sufficiently large we have*

$$\begin{aligned} & - \langle A_0, \dots, [F, A_j], \dots, A_m \rangle_{m,s,r} \\ &= \langle A_0, \dots, A_{j-1}A_j, \dots, A_m \rangle_{m-1,s,r} - \langle A_0, \dots, A_j A_{j+1}, \dots, A_m \rangle_{m-1,s,r}, \end{aligned} \tag{4.26}$$

and for $k \geq 1$

$$\begin{aligned} & \int_0^\infty s^k \Big[\langle BA_0, A_1, \dots, A_m \rangle_{m,s,r} - \langle A_0, A_1, \dots, A_m B \rangle_{m,s,r} \Big] ds \\ &= d_{B_0} \int_0^\infty s^k \langle A_0, A_1, \dots, A_m \rangle_{m,s,r} ds. \end{aligned} \tag{4.27}$$

Proof. The first identity follows from observing that

$$-[F, A_j] = R_s(\lambda)^{-1} A_j - A_j R_s(\lambda)^{-1}, \tag{4.28}$$

then

$$\begin{aligned} & A_0 R_s^F(\lambda) A_1 R_s^F(\lambda) \cdots \big(R_s^F(\lambda)^{-1} A_j - A_j R_s^F(\lambda)^{-1} \big) \\ & R_s^F(\lambda) A_{j+1} R_s^F(\lambda) \cdots A_m R_s^F(\lambda) \\ &= A_0 R_s^F(\lambda) A_1 R_s^F(\lambda) \cdots A_{j-1} A_j R_s^F(\lambda) \cdots A_m R_s^F(\lambda) \\ & \quad - A_0 R_s^F(\lambda) A_1 R_s^F(\lambda) \cdots A_j A_{j+1} R_s^F(\lambda) \cdots A_m R_s^F(\lambda). \end{aligned} \tag{4.29}$$

We have the equalities

$$\begin{aligned} & \int_0^\infty s^k \langle BA_0, A_1, \dots, A_m \rangle_{m,s,r} ds \\ &= \int_0^\infty s^k Tr \Big(\frac{1}{2\pi i} \int_l \lambda^{-p/2-r} BA_0 R_s^F(\lambda) \cdots A_m R_s^F(\lambda) \Big) d\lambda, \end{aligned} \tag{4.30}$$

$$\begin{aligned} & \int_0^\infty s^k \langle A_0, A_1, \dots, A_m B \rangle_{m,s,r} ds \\ &= \int_0^\infty s^k Tr \Big(\frac{1}{2\pi i} \int_l \lambda^{-p/2-r} A_0 R_s^F(\lambda) \cdots A_m B R_s^F(\lambda) \Big) d\lambda ds. \end{aligned} \tag{4.31}$$

Combining the above equations, we obtain

$$\begin{aligned} & \int_0^\infty s^k \Big[\langle BA_0, A_1, \dots, A_m \rangle_{m,s,r} - \langle A_0, A_1, \dots, A_m B \rangle_{m,s,r} \Big] ds \\ &= \frac{1}{2\pi i} \int_0^\infty s^k Tr \Big[B, \int_l \lambda^{-p/2-r} A_0 R_s^F(\lambda) \cdots A_m R_s^F(\lambda) d\lambda \Big] ds \\ &= d_{B_0} \int_0^\infty s^k Tr \Big(\int_l \lambda^{-p/2-r} A_0 R_s^F(\lambda) \cdots A_m R_s^F(\lambda) d\lambda \Big) ds \end{aligned}$$

$$= d_{B_0} \int_0^\infty s^k \langle A_0, A_1, \dots, A_m \rangle_{m,s,r} ds. \tag{4.32}$$

□

Suspecting that the higher spectral flow is given by pairing a cocycle with the Chern character of a unitary, we remove the normalisation coming from $Ch_m(u)$ from our resolvent formula to define a cocycle. The factor of $\sqrt{2\pi i}$ is for compatability with the Kasparov product [7].

Definition 4.12. Let $\mathcal{C}(m)$ denote the constant $\frac{-2\sqrt{2\pi i}}{\Gamma\left((m+1)/2\right)}$. Then, for $Re(\lambda) > -m/2 + 1/2$ and $da = [B, a]$, we define $\phi_m^r : \mathcal{A}^{m+1} \to \mathbb{C}$ with $\mathcal{A} = C^\infty(M_0)$ by

$$\phi_m^r(a_0, \dots, a_m) = \mathcal{C}(m) \int_0^\infty s^m \langle a_0, da_1, \dots, da_m \rangle_{m,s,r} ds. \tag{4.33}$$

By Lemma 4.9, the condition on r ensures that the integral converges. We note that this constant $\mathcal{C}(m)$ is distinct from $C(k)$ which takes a multi-index k as its argument.

Proposition 4.13. *For $p \geq 1$, the collection of functionals $\phi^r = \{\phi_m^r\}_{m=1}^{2N-1}$, $m = 1, 3, \dots, 2N-1$ odd, such that*

$$\begin{aligned} &\left(B\phi_{m+2}^r + b\phi_m^r\right)(a_0, \dots, a_m) \\ =& -\mathcal{C}(m) d_{B_0} \int_0^\infty s^m \langle a_0, [B, a_1], \dots, [B, a_{m+1}] \rangle_{m+1,s,r} ds, \end{aligned} \tag{4.34}$$

$$\left(B\phi_1^r + d_{B_0}\phi_0^r\right)(a_0) = 0, \tag{4.35}$$

where $a_i \in \mathcal{A}$. Moreover, there is a δ', $0 < \delta' < 1$, then $b\phi_{2N-1}^r(a_0, \dots, a_{2N})$ is a holomorphic function of r for $Re(r) > -(p + q')/2 + \delta'/2$.

Proof. We start with the computation of the coboundaries of the ϕ_m^r. Recall the definition $B : C^{k+1}(A) \to C^k(A)$,

$$Bc(a_0, \dots, a_m) = \sum_{j=0}^{k} c(1, a_j, \dots, a_{m+1}, a_0, \dots, a_{j-1}). \tag{4.36}$$

By ϕ_{m+2}^r, Lemma 4.9 and Lemma 4.11, we get

(4.37)

$$\begin{aligned} &(B\phi_{m+2}^r)(a_0, \dots, a_{m+1}) \\ =& \sum_{j=0}^{m+1} (-1)^{j(m+1)} \phi_{m+2}^r(1, a_j, \dots, a_{m+1}, a_0, \dots, a_{j-1}) \\ =& \sum_{j=0}^{m+1} \mathcal{C}(m+2)(-1)^{j(m+1)} \int_0^\infty s^{m+2} \langle 1, [B, a_j], \dots, [B, a_{m+1}], [B, a_0], \dots, [B, a_{j-1}] \rangle_{m+2,s,r} ds \\ =& \sum_{j=0}^{m+1} \mathcal{C}(m+2) \int_0^\infty s^{m+2} \langle [B, a_0], \dots, [B, a_{j-1}], 1, [B, a_j], \dots, [B, a_{m+1}] \rangle_{m+2,s,r} ds \end{aligned}$$

$$= -\mathcal{C}(m+2)\frac{m+1}{2}\int_0^\infty s^m\langle [B,a_0],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}ds$$
$$= -\mathcal{C}(m)\int_0^\infty s^m\langle [Ba_0-a_0B],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}ds$$
$$= -\mathcal{C}(m)\int_0^\infty s^m\Big(\langle Ba_0,[B,a_1],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}$$
$$-\langle a_0,[B,a_1],\ldots,[B,a_{m+1}]B\rangle_{m+1,s,r}\Big)ds$$
$$-\mathcal{C}(m)\int_0^\infty s^m\sum_{j=1}^{m+1}(-1)^j\langle a_0,[B,a_1],\ldots,[B^2,a_j],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}ds$$
$$= -\mathcal{C}(m)\int_0^\infty s^m\sum_{j=1}^{m+1}(-1)^j\langle a_0,[B,a_1],\ldots,[B^2,a_j],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}ds$$
$$-\mathcal{C}(m)d_{B_0}\int_0^\infty s^m\langle a_0,[B,a_1],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}\Big)ds.$$

Observe that for ϕ_1^r, we have

$$\begin{aligned}(B\phi_1^r)(a_0) &= \frac{\mathcal{C}(1)}{2\pi i}\int_0^\infty sTr\Big(\int_l \lambda^{-p/2-r}R_s^F(\lambda)[B,a_0]R_s^F(\lambda)d\lambda\Big)ds\\ &= \frac{\mathcal{C}(1)}{2\pi i}\int_0^\infty sTr\Big(\int_l \lambda^{-p/2-r}[B,R_s^F(\lambda)a_0R_s^F(\lambda)]\Big)ds\\ &= \frac{\mathcal{C}(1)}{2\pi i}\int_0^\infty sTr\Big([B,\int_l \lambda^{-p/2-r}R_s^F(\lambda)a_0R_s^F(\lambda)ds]\Big)\\ &= d_{B_0}\frac{\mathcal{C}(1)}{2\pi i}\int_0^\infty s\int_l \lambda^{-p/2-r}R_s^F(\lambda)a_0R_s^F(\lambda)ds\\ &= d_{B_0}\mathcal{C}(1)\int_0^\infty s\langle 1,a_0\rangle_{1,s,r}ds. \end{aligned}\tag{4.38}$$

We now compute the Hochschild coboundary of ϕ_m^r. From the definitions ϕ_m^r and Lemma 4.11, we have

(4.39)

$$(b\phi_m^r)(a_0,\ldots,a_{m+1})$$
$$= \phi_m^r(a_0a_1,a_2,\ldots,a_{m+1}) + \sum_{j=1}^{k}(-1)^j\phi_m^r(a_0,\ldots,a_ja_{j+1},\ldots,a_{m+1})$$
$$+\phi_m^r(a_{m+1}a_0,a_1,\ldots,a_m)$$
$$= \mathcal{C}(m)\int_0^\infty s^m\Big(\langle a_0a_1,[B,a_2],\ldots,[B,a_{m+1}]\rangle_{m,s,r}$$

$$+\sum_{j=1}^{k}(-1)^j\langle a_0,[B,a_1],\ldots,a_j[B,a_{j+1}]+[\mathcal{B},a_j]a_{j+1},\ldots,[B,a_{m+1}]\rangle_{m,s,r}$$

$$+\langle a_{m+1}a_0,[B,a_1],\ldots,[B,a_m]\rangle_{m,s,r}\Big)ds$$

$$=\mathcal{C}(m)\int_0^\infty s^m\Big(\langle a_0a_1,[B,a_2],\ldots,[B,a_{m+1}]\rangle_{m,s,r}$$

$$-\langle a_0,a_1[B,a_2],\ldots,[B,a_{m+1}]\rangle_{m,s,r}\Big)ds$$

$$-\mathcal{C}(m)\int_0^\infty s^m\Big(\langle a_0,[B,a_1]a_2,\ldots,[B,a_{m+1}]\rangle_{m,s,r}$$

$$-\langle a_0,[B,a_1],a_2[B,a_3],\ldots,[B,a_{m+1}]\rangle_{m,s,r}\Big)ds$$

$$\vdots$$

$$-\mathcal{C}(m)\int_0^\infty s^m\Big(\langle a_0,[B,a_1],\ldots,[B,a_m]a_{m+1}\rangle_{m,s,r}$$

$$-\langle a_{m+1}a_0,[B,a_1],\ldots,[B,a_m]\rangle_{m,s,r}\Big)ds$$

$$=\sum_{j=1}^{m+1}(-1)^j\mathcal{C}(m)\int_0^\infty s^m\Big(\langle a_0,[B,a_1],\ldots,[B^2,a_j],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}\Big)ds.$$

For $m=1,3,5,\ldots,2N-3$, by Eqs. (4.37)-(4.39) we obtain

$$\begin{aligned}&\big(B\phi^r_{m+2}+b\phi^r_m\big)(a_0,\ldots,a_m)\\ &=\ -\mathcal{C}(m)d_{B_0}\int_0^\infty s^m\langle a_0,[B,a_1],\ldots,[B,a_{m+1}]\rangle_{m+1,s,r}ds.\end{aligned}\tag{4.40}$$

So we just need to check the claim that $b\phi^r_{2N-1}$ is holomorphic for $\mathrm{Re}(r)>-(p+q')/2+\delta'$ for some suitable δ'. The proof that (4.40) is holomorphic for $m=2N-1$ is similar to the analyticity proof in Lemma 4.7.

□

5. A family residue cocycle

In this section, we derive a new expression stated as Theorem 5.1 from the resolvent expansion of the higher spectral flow formula Theorem 4.5. This leaves us with a formula for higher spectral flow that involves an integral over the parameter s. By integrating out the s dependence in the formula of Theorem 5.1, we find a higher spectral flow formula which involves a sum of zeta functions. One immediately recognises that individual terms in this formula may be obtained from our resolvent cocycle by using the pseudodifferential calculus. Thus, from the resolvent cocycle we derive the residue cocycle in the final section. Our final formula for the higher spectral flow follows immediately by evaluating the residue cocycle on $Ch_*(u^*)$.

The aim of this section is to establish the family higher spectral flow formula which is summarised in the following result. We compute the Chern Character of the higher spectral flow from D to uDu^*, where $u \in A$ is unitary with $[D, u]$ bounded. Next we compute the residue of the gamma function $\Gamma(x)$. For the meromorphic function $\Gamma(x)$ which has simple poles at nonnegative integers, then

$$res(\Gamma(x), -n) = \frac{(-1)^n}{n!}. \tag{5.1}$$

The 'constant' $\mathcal{C}_{p/2+r,l}$ has simple poles at $r = \frac{1-(p+l)}{2}$, and $p = \dim Z$ is odd, l even. Therefore, $\Gamma(\frac{p}{2} + z)$ is holomorphic at $z = \frac{1-(p+l)}{2}$. For the simple poles $z = \frac{1-(p+l)}{2}$, by (5.1) we have

$$\begin{aligned} res_{\frac{1-(p+l)}{2}} \mathcal{C}_{\frac{p}{2}+z,l} &= res_{z=\frac{1-(p+l)}{2}} \Big(\Gamma(\frac{p+l}{2} + z - \frac{1}{2})\Big) \frac{\sqrt{\pi}}{\Gamma(\frac{p}{2} + \frac{1-(p+l)}{2})} \\ &= res_{z'=0}(\Gamma(z')) \frac{\sqrt{\pi}}{\Gamma(\frac{1-l}{2})} = \frac{1 \times 3 \times \cdots \times (l-1)}{(-1)^{\frac{l}{2}} 2^{\frac{l}{2}}}, \end{aligned} \tag{5.2}$$

where we have used $\Gamma(x + 1) = x\Gamma(x)$ and

$$\Gamma(\frac{1-l}{2}) = \frac{(-1)^{\frac{l}{2}} 2^{\frac{l}{2}}}{1 \times 3 \times \cdots \times (l-1)} \Gamma(\frac{1}{2}). \tag{5.3}$$

Set $A_l = \frac{(-1)^{\frac{l}{2}} 2^{\frac{l}{2}}}{1 \times 3 \times \cdots \times (l-1)}$, then we have

$$A_l \times res_{z=\frac{1-(p+l)}{2}} \mathcal{C}_{\frac{p}{2}+z,l} = 1. \tag{5.4}$$

With these preliminaries, we can state the main result of the paper.

Theorem 5.1. *In the cohomology of B_0, the following equality holds*

$$\begin{aligned} &Ch\big(sf(D, u^*Du)\big) \\ &= \frac{1}{\sqrt{2\pi i}} \sum_{l=0}^{\dim B_0} A_l \Big(res_{r=\frac{1-(p+l)}{2}} \Big(\sum_{m=1,odd}^{2N-1} \langle \phi_m^r, Ch_m(u) \rangle \Big)^{[l]} \Big). \end{aligned} \tag{5.5}$$

Proof. By Lemma 4.7, we have

$$\begin{aligned} &\mathcal{C}_{p/2+r,l} Ch\big(sf(D, u^*Du)\big)^{[l]} \\ &= \int_0^\infty Str^{even} \Big(q(1 + \widetilde{F} + s^2 + s[\widetilde{B}, q])^2\big)^{-p/2-r}\Big)^{[l]} ds. \end{aligned} \tag{5.6}$$

Then we take residues at $r = (1 - (p + l))/2$ of both sides, by Lemma 3.1 in [3], we have

(5.7)

$$Ch\big(sf(D, u^*Du)\big)^{[l]}$$

$$= A_l res_{r=\frac{1-(p+l)}{2}}\Big(\mathcal{C}_{p/2+r,l}Ch\big(sf(D,u^*Du)\big)^{[l]}\Big)$$

$$= \frac{A_l}{2} res_{r=\frac{1-(p+l)}{2}} \int_0^\infty Str^{even}\Big(q(1+\widetilde{F}+s^2+s[\widetilde{B},q])^2\big)^{-p/2-r}\Big)^{[l]} ds$$

$$= \frac{A_l}{2} res_{r=\frac{1-(p+l)}{2}} \Big[\frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^{+\infty} s^m Str^{even}$$

$$\times \Big(\int_l \lambda^{-p/2-r} q\big(R_s^F(\lambda)\{\widetilde{B},q\}\big)^M R_s^F(\lambda)d\lambda\Big)ds + holo\Big]^{[l]}$$

$$= \frac{A_l}{2} res_{r=\frac{1-(p+l)}{2}} \Big[\frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1} 2\times(-1)^{\frac{m+1}{2}} \int_0^{+\infty} s^m Tr^{even}\Big(\lambda^{-p/2-r}$$

$$\times \big(u^*R[B,u]R[B,u^*]\cdots[B,u]R - uR[B,u^*]R[B,u]\cdots[B,u^*]Bd\lambda\big)ds\Big)^{[l]}$$

$$= \frac{A_l}{2}\frac{1}{\sqrt{2\pi i}} res_{r=\frac{1-(p+l)}{2}} \sum_{m=1,odd}^{2N-1}\langle\phi_m^r, Ch_m u - Ch_m u^*\rangle^{[l]}$$

$$= \frac{A_l}{\sqrt{2\pi i}} res_{r=\frac{1-p}{2}}\Big(\sum_{m=1,odd}^{2N-1}\langle\phi_m^r, Ch_m u\rangle - \frac{1}{2}\sum_{m=1,odd}^{2N-1}\langle\phi_m^r, Ch_m u + Ch_m u^*\rangle\Big)^{[l]}$$

$$= \frac{A_l}{\sqrt{2\pi i}} res_{r=\frac{1-(p+l)}{2}}\Big[\Big(\sum_{m=1,odd}^{2N-1}\langle\phi_m^r, Ch_m(u)\rangle\Big)^{[l]}$$

$$+ d_{B_0} res_{r=\frac{1-p}{2}}\Big(\sum_{m=1,odd}^{2N-1}(-1)^{\frac{m+1}{2}}\int_0^\infty s^m\langle 1,[B,u^{-1}],[B,u],\ldots,[B,u^{-1}],$$

$$[B,u]\rangle_{m+1,s,r}ds\Big)^{[l-1]}\Big].$$

□

Then we have the pseudodifferential expansion of the higher spectral flow.

Proposition 5.2. *There is a δ', $0<\delta'<1$ and $0\le l\le \dim B_0$, l even, such that up to an exact form on B_0,*

(5.8)

$$Ch\big(sf(D,u^*Du)\big)^{[l]}\mathcal{C}_{p/2+r,l}$$

$$= \int_0^\infty Str^{even}\Big(q(1+\widetilde{F}+s^2+s\{\widetilde{B},q\})^{-p/2-r}\Big)^{[l]}ds$$

$$= \frac{1}{2\pi i}\sum_{m=1,odd}^{2N-1}\int_0^\infty s^m Str^{even}\Big(\int_l \lambda^{-p/2-r}\sum_{|k|=0}^{2N-1-m}\mathcal{C}_k q\{\widetilde{B},q\}^{(k_1)}\cdots$$

$$\cdots\{\widetilde{B},q\}^{(k_m)}R_s^{\widetilde{F}}(\lambda)^{m+1+|k|}d\lambda\Big)^{[l]}ds + holo$$

$$= \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} \mathcal{C}_k \frac{\Gamma(p/2+r+m+|k|)}{\Gamma(p/2+r)(m+|k|)!} \int_0^\infty s^m Str^{even}\Big(q\{\widetilde{B},q\}^{(k_1)}\cdots$$

$$\cdots\{\widetilde{B},q\}^{(k_m)}(1+\widetilde{F}+s^2)^{-(p/2+r+m+|k|)}\Big)^{[l]} ds + holo,$$

where holo is a function of r holomorphic for $Re(r) > -(p+q')/2 + \delta'/2$. *Consequently the sum of functions on the right-hand side has an analytic continuation to a deleted neighbourhood of* $r = (1-p-l)/2$ *(given by the left-hand side) with at worst a simple pole at* $r = (1-p-l)/2$.

Proposition 5.3. *There is a* δ', $0 < \delta' < 1$ *and* $0 \le l < \dim B$ *such that*

$$Ch\big(sf(D,u^*Du)\big)^{[l]}\mathcal{C}_{p/2+r,l}$$
$$= \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} \mathcal{C}_k(-1)^{|k|+m}\frac{\Gamma((m+1)/2)\Gamma(p/2+r+|k|+(m-1)/2)}{2(m+|k|)!\Gamma(p/2+r)}$$
$$\times Str^{even}\Big(q\{\widetilde{B},q\}^{(k_1)}\cdots\{\widetilde{B},q\}^{(k_m)}(1+\widetilde{F})^{-(p/2+r+|k|+(m-1)/2)}\Big)^{[l]}$$
$$+ holo, \tag{5.9}$$

where holo is a function of r holomorphic for $Re(r) > -(p+q')/2 + \delta'/2$. *Consequently the sum of functions on the right-hand side has an analytic continuation to a deleted neighbourhood of* $r = (1-p-l)/2$ *(given by the left-hand side) with at worst a simple pole at* $r = (1-p-l)/2$. *Moreover, if* $[p] = 2n$ *is even, each of the top terms with* $|k| = 2N-1-m$ *are holomorphic at* $r = (1-p-l)/2$, *including the one term with* $m = 2N-1$.

We denote the analytic continuation of a function analytic in a right half-plane to a deleted neighbourhood of the critical point by putting the function in boldface. Thus we define the functionals for each integer $j \ge 0$:

$$Str^{even}_{j,l}\Big(q\{\widetilde{B},q\}^{(k_1)}\cdots\{\widetilde{B},q\}^{(k_m)}(1+\widetilde{F})^{-(|k|+m/2)}\Big)^{[l]}$$
$$= res_{r=(1-(p+l))/2}(r-(1-p)/2)^j Str^{even}\Big(q\{\widetilde{B},q\}^{(k_1)}\cdots\{\widetilde{B},q\}^{(k_m)}$$
$$\times(1+\widetilde{F})^{-(p/2+r+|k|+(m-1)/2)}\Big)^{[l]}. \tag{5.10}$$

Recall that the $\sigma'_{h,j}s$ are the symmetric functions of the half-integers $1/2, 3/2, \ldots, h-1/2$. Now we start from the formula of Proposition 5.3 and take residues at $r = (1-(p+l))/2$ of both sides and multiply A_l,

(5.11)

$$Ch\big(sf(D,u^*Du)\big)^{[l]}$$
$$= A_l \sum_{m=1,odd}^{2N-1} \sum_{|k|=0}^{2N-1-m} \mathcal{C}_k(-1)^{|k|+m} res_{r=(1-(p+l))/2}\frac{\Gamma((m+1)/2)\Gamma(p/2+r+|k|+(m-1)/2)}{2(m+|k|)!\Gamma(p/2+r)}$$

$$\times Str^{even}\Big(q\{\widetilde{B},q\}^{(k_1)}\cdots\{\widetilde{B},q\}^{(k_m)}(1+\widetilde{F})^{-(p/2+r+|k|+(m-1)/2)}\Big)^{[l]}$$

$$= A_l\sum_{m=1,odd}^{2N-1}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|+m}\frac{\Gamma((m+1)/2)\alpha(k)}{2}\sum_{j=0}^{h}\sigma_{h,j}$$

$$\times Str^{even}_{j,l}\Big(q\{\widetilde{B},q\}^{(k_1)}\cdots\{\widetilde{B},q\}^{(k_m)}(1+\widetilde{F})^{-(|k|+m/2)}\Big)^{[l]}$$

$$= A_l\sum_{m=1,odd}^{2N-1}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|+(m+1)/2}\frac{\Gamma((m+1)/2)\alpha(k)}{2}\sum_{j=0}^{h}\sigma_{h,j}$$

$$\times Str^{even}_{j,l}\Big((u[\widetilde{B},u^*]^{(k_1)}\cdots[\widetilde{B},u^*]^{(k_m)}-u^*[\widetilde{B},u]^{(k_1)}\cdots[\widetilde{B},u]^{(k_m)})$$

$$\times(1+\widetilde{F})^{-(|k|+m/2)}\Big)^{[l]}.$$

To understand this formula in terms of cyclic (co)homology and Chern characters, we show that our higher spectral flow formula is obtained by pairing a cyclic cocycle with the Chern character of a unitary. We note that our resolvent cocycle ϕ^r pairs with normalised chains, so that by Lemma 3.1,

$$\phi^r(Ch_*(u)) = -\phi^r(Ch_*(u^*)) \tag{5.12}$$

is modulo functions holomorphic for $Re(r) > (1-(p+q'))/2-\delta$, then:

Theorem 5.4. *Assume that $(A,\mathcal{H},B)$ is a family spectral triple associated the fibration. For m odd, define functionals ϕ_m by*

$$\phi_m(a_0,\dots,a_m) := \sum_{l=0}^{\dim B_0}\phi_l(a_0,\dots,a_m) = \sqrt{2\pi i}\sum_{l=0}^{\dim B_0}\sum_{|k|=0}^{2N-1-m}(-1)^{|k|}\alpha(k)$$

$$\times\sum_{j=0}^{h}\sigma_{h,j}Str^{even}_{j,l}\Big(a_0[B,a_1]^{(k_1)}\cdots[B,a_m]^{(k_m)}(1+F)^{-|k|-m/2}\Big), \tag{5.13}$$

where $h=|k|+(m-1)/2$. Then $\phi=(\phi_m)$ is a (b,B)-cocycle in the cohomology of B_0 and

$$Ch(sf(D,u^*Du)) = \frac{1}{\sqrt{2\pi i}}\sum_{|k|=0}^{2N-1-m}\langle\phi_m, Ch_m(u)\rangle. \tag{5.14}$$

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