Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1373–1386
https://doi.org/10.4134/BKMS.b160593
pISSN: 1015-8634 / eISSN: 2234-3016

ORDER RELATED CONCEPTS FOR ARBITRARY GROUPOIDS

Hee Sik Kim, Joseph Neggers, and Keum Sook So

Abstract. In this paper, we introduce and explore suggested notions of 'above', 'below' and 'between' in general groupoids, $Bin(X)$, as well as in more detail in several well-known classes of groupoids, including groups, semigroups, selective groupoids (digraphs), d/BCK-algebras, linear groupoids over fields and special cases, in order to illustrate the usefulness of these ideas. Additionally, for groupoid-classes (e.g., BCK-algebras) where these notions have already been accepted in a standard form, we look at connections between the several definitions which result from our introduction of these ideas as presented in this paper.

1. Introduction

In the general description of the realm of mathematics one may recognize "kingdoms" such as logic, algebra, geometry, measure and quantity and derived "phyla" which may be considered hybrid creatures derived from elements belonging to these kingdoms. One such hybrid creature is the phylum which we shall call "order". In this phylum we describe possible "ordered structures" where the structure may be any structure whatsoever, which is then equipped with an order relation of some sort. Examples abound in the literature, the most elementary (but definitely not the most simple being the set $\mathbb{N}$ of natural numbers equipped with the order relation $1 < 2 < 3 < 4 < \cdots$; $2 < 3 < 4 < 5 < \cdots$; $n < n+1 < n+2 < n+3 < \cdots$, $\cdots$ et cetera. If we consider $\mathbb{N}$ as belonging to the "kingdom" of measure and quantity, then $(\mathbb{N}, <)$ can be thought of as a hybrid creature which generates a phylum of derived constructions. In the kingdom of algebra, there is the phylum of binary systems on a set X when equipped itself with a binary operation on the $Bin(X)$, to yield semigroups $(Bin(X), \square)$. The increased study of this phylum is a recent phenomenon ([7]) even though specific instance of elements $(X, *)$ of $Bin(X)$ have been studied to advantage for a very long time, with many of their properties discussed in books on groups, semigroups, rings, fields, etc., and even

Received July 15, 2016; Revised November 2, 2016; Accepted December 15, 2016.
2010 *Mathematics Subject Classification.* Primary 20N02, 20M10, 06A06.
Key words and phrases. below, above, between, d/BCK-algebra, poset, $Bin(X)$, order-preserving, β-medial.

in more general studies such as Bruck's "A survey of binary systems" ([4]). Given this phylum we can for example create the hybrid creature "topological binary systems" consisting of groupoids $(X, *, \tau)$, where τ is a topology on X such that the operation $*$ is bi-continuous. The study of this creature in full generality has not been done yet, but certainly appears to have possibilities.

In this paper the hybrid structure involves groupoids $(X, *)$ equipped with several order structures which we have termed *below, above* and as a consequence a relation $x \leq y$ ($x\beta y$ and $y\alpha x$) and further relations $z \in \langle x, y\rangle_\beta$, $z \in \langle x, y\rangle_\alpha$, with $x\beta z, z\beta y$ and $x\alpha z, z\alpha y$ and $x\alpha z, z\alpha y$ respectively, as *β-between-ness* and *α-between-ness*, from which further information can then be obtained, especially when one specializes to specific types of groupoids. The point is that in doing so, one provides a sense of "order" to the entire phylum of binary systems in what appears from the following to be a simple way.

Historically, the notion of the semigroup $(Bin(X), \square)$ was introduced by H. S. Kim and J. Neggers ([7]). H. Fayoumi ([5]) introduced the notion of the center $ZBin(X)$ in the semigroup $Bin(X)$ of all binary systems on a set X, and showed that if $(X, \bullet) \in ZBin(X)$, then $x \neq y$ implies $\{x, y\} = \{x\bullet y, y\bullet x\}$. Moreover, she showed that a groupoid $(X, \bullet) \in ZBin(X)$ if and only if it is a locally-zero groupoid. J. S. Han et al. ([6]) introduced the notion of hypergroupoids $(HBin(X), \square)$, and showed that $(HBin(X), \square)$ is a supersemigroup of the semigroup $(Bin(X), \square)$ via the identification $x \longleftrightarrow \{x\}$. They proved that $(HBin^*(X), \ominus, [\emptyset])$ is a BCK-algebra. S. J. Shin et al. ([11]) introduced the notion of abelian fuzzy subsets on a groupoid, and discussed diagonal symmetric relations, convex sets, and the fuzzy center on $Bin(X)$. In [12] they discussed properties of a class of real-valued functions on a groupoid $(X, *)$ and fuzzy subsets on X related to $(Bin(X), \square)$. S. S. Ahn et al. ([1]) studied fuzzy upper bounds in $Bin(X)$. J. Zhan et al. ([17]) generalized the left-zero semigroup by introducing the notions of a weak-zero groupoid and an (X, N)-zero groupoid. P. A. Allen et al. ([3]) studied several types of groupoids related to semigroups, i.e., twisted semigroups. P. J. Allen et al. ([2]) developed a theory of companion d-algebras, and they showed that if $(X, *, 0)$ is a d-algebra, then $(Bin(X), \oplus, \diamond_0)$ is also a d-algebra. S. Z. Song et al. ([13]) studied soft saturated values and soft dried values in BCK/BCI-algebras. Thus, it is clear that the study of groupoids (binary systems) is undergoing vigorous development at present to which this paper aims to make a further useful contribution as well.

2. Preliminaries

A *d-algebra* ([10]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying the following axioms:

(I) $x * x = 0$,
(II) $0 * x = 0$,
(III) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$.

For brevity, we also call X a *d-algebra.* In X we can define a binary relation "$\leq$" by $x \leq y$ if and only if $x * y = 0$.

If K is a field, then $(K, *, 0)$ with $x*y := x(x-y)$ is an example of d-algebra which is not a BCK-algebra.

A BCK-algebra ([8]) is a d-algebra X satisfying the following additional axioms:

(IV) $(x * y) * (x * z)) * (z * y) = 0$,
(V) $(x * (x * y)) * y = 0$

for all $x, y, z \in X$. It is well-known that if $(X, *, 0)$ is a BCK-algebra, then $(X, \leq)$ forms a partially ordered set with the least element 0. We refer to [9, 14] for further information on partially ordered sets and we refer to [15] for the graph theory.

Given a non-empty set X, we let $Bin(X)$ denote the collection of all groupoids $(X, *)$, where $* : X \times X \to X$ is a map and where $*(x, y)$ is written in the usual product form. Given elements $(X, *)$ and $(X, \bullet)$ of $Bin(X)$, define a product "$\square$" on these groupoids as follows:

$$(X, *) \square (X, \bullet) = (X, \square),$$

where

$$x \square y = (x * y) \bullet (y * x)$$

for any $x, y \in X$. Using that notion, H. S. Kim and J. Neggers proved the following theorem.

Theorem 2.2 ([7]). *$(Bin(X), \square)$ is a semigroup, i.e., the operation "$\square$" as defined in general is associative. Furthermore, the left- zero-semigroup is the identity for this operation.*

3. Below, above and between

Let $(X, *)$ be a groupoid and let $x, y, z \in X$. x is said to be *below* y, denoted by $x\beta y$, if $x * y = y$; x is said to be *above* y, denoted by $x\alpha y$, if $x * y = x$. An element $z \in X$ is said to be *β-between x and y*, denoted by $z \in \langle x, y\rangle_\beta$, if $x\beta z, z\beta y$; an element z is said to be *α-between x and y*, denoted by $z \in \langle x, y\rangle_\alpha$, if $x\alpha z, z\alpha y$.

Example 3.1. Let $D = (V, E)$ be a digraph and let $(V, *)$ be its associated groupoid, i.e., $*$ is a binary operation on V defined by

$$x * y := \begin{cases} x & \text{if } x \to y \notin E, \\ y & \text{otherwise.} \end{cases}$$

Let $D = (V, E)$ be a digraph with the following graph:

$$\begin{tikzcd}
3 \bullet \arrow[dr] & \bullet 4 \\
1 \bullet \arrow[u] & \bullet 2 \arrow[u]
\end{tikzcd}$$

Then its associated groupoid $(V, *)$ has the following table:

$*$	1	2	3	4
1	1	1	3	1
2	2	2	2	4
3	3	2	3	3
4	4	4	4	4

It is easy to see that there are no elements $x, y \in V$ such that both $x\alpha y$ and $x\beta y$ hold simultaneously. Note that the relations α and β need not be transitive. In fact, $1 \to 3, 3 \to 2$ in E, but not $1 \to 2$ in E imply that $1\beta 3, 3\beta 2$, but not $1\beta 2$. Similarly, $1\alpha 4, 4\alpha 3$, but not $1\alpha 3$.

Remark. In Example 3.1, $z \in \langle x, y\rangle_\beta$ means that $x\beta z, z\beta y$, i.e., $x \to z \to y$ in E. Similarly, $z \in \langle x, y\rangle_\alpha$ means that $x\alpha z, z\alpha y$, i.e., no arrow from x to z, and no arrow from z to y in E.

Proposition 3.2. *Let $(X, *)$ be a left-zero-semigroup. Then for any $x, y, z \in X$,*

(i) *$x\alpha y$ and $z \in \langle x, y\rangle_\alpha$,*
(ii) *$x\beta y$ implies $x = y$,*
(iii) *$z \in \langle x, y\rangle_\beta$ implies $x = y = z$.*

Proof. (i) Since $x * y = x$ for any $x, y \in X$, we have $x\alpha y$. Given $x, y, z \in X$, since $(X, *)$ is the left-zero-semigroup, $x * z = x, z * y = z$, which prove that $x\alpha z, z\alpha y$, i.e., $z \in \langle x, y\rangle_\alpha$.

(ii) If $x\beta y$, then $x * y = y$. Since $(X, *)$ is the left-zero-semigroup, we have $x = x * y = y$.

(iii) If $z \in \langle x, y\rangle_\beta$, then $x\beta z$ and $z\beta y$. By (ii), we obtain $x = z = y$. □

Example 3.3. Let $\mathbb{R}$ be the set of all real numbers and let $x, y \in \mathbb{R}$. Define a binary operation "$*$" on $\mathbb{R}$ by $x * y := xy$, the ordinary multiplication, for any $x, y \in \mathbb{R}$. Then it is easy to see that $1 \in \langle 1, y\rangle_\beta$, $z \in \langle 1, 0\rangle_\beta$, $0 \in \langle x, 0\rangle_\beta$ for all $x, y, z \in \mathbb{R}$.

Proposition 3.4. *Let $(X, *)$ be a semigroup. Then the relations α, β are transitive.*

Proof. Let $x\beta y$ and $y\beta z$. Then $x * y = y, y * z = z$. It follows that $x * z = x * (y * z) = (x * y) * z = y * z = z$, proving $x\beta z$.

Let $x\alpha y$ and $y\alpha z$. Then $x*y = x, y*z = y$. It follows that $x*z = (x*y)*z = x * (y * z) = x * y = x$, proving $x\alpha z$. □

The converse of Proposition 3.4 need not be true in general.

Example 3.5. Let $\mathbb{R}$ be the set of all real numbers and let $x, y \in \mathbb{R}$. If we define a binary operation "$*$" on $\mathbb{R}$ by $x*y := y^2$, then $(\mathbb{R}, *)$ is not a semigroup. In fact, $(x*y)*z = z^2$, while $x*(y*z) = z^4$. If $x\beta y$ and $y\beta z$, then $z = y*z = z^2$ and hence $z = 0$ or $z = 1$, which implies that $x * z = z$, i.e., $x\beta z$, proving that β is transitive.

Proposition 3.6. *Let $(X, *)$ be a groupoid. Then for any $x, y, z \in X$,*

(i) *if $x\beta y, x\alpha y$, then $x = y$;*
(ii) *if $(X, *)$ is commutative, i.e., $x * y = y * x$, then $x\beta y \iff y\alpha x$;*
(iii) *if $x\beta y, y\alpha x$, then $x * y = y * x = y$.*

Let $(X, *)$ be a groupoid and let $x, y \in X$. Define a binary relation "$\leq$" on X by $x \leq y \iff x\beta y, y\alpha x$. Then it is easy to see that $\leq$ is anti-symmetric.

Proposition 3.7. *Let $(X, *)$ be a groupoid. If α, β are transitive, then $\leq$ is transitive.*

Proof. Let $x \leq y, y \leq z$. Then $x\beta y, y\alpha x$ and $y\beta z, z\alpha y$. Since α, β are transitive, $x\beta z, z\alpha x$, i.e., $x \leq z$. □

Proposition 3.8. *If $(X, *)$ is a semigroup with $x * x = x$ for all $x \in X$, then $(X, \leq)$ is a poset.*

Proof. By Propositions 3.4 and 3.7, the relation $\leq$ is transitive. Since $x*x = x$ for all $x \in X$, we have $x\beta x, x\alpha x$, which implies $x \leq x$. This proves the proposition. □

Let $(X, *)$ be a groupoid and let $x, y \in X$. We define an interval as follows:

$$[x, y] := \{q \in X \mid x \leq q \leq y\}.$$

The following proposition can be easily proved.

Proposition 3.9. *Let $(X, *)$ be a groupoid and let $x, y \in X$. Then $z \in [x, y]$ if and only if $z \in \langle x, y\rangle_\beta$ and $z \in \langle y, x\rangle_\alpha$.*

Proposition 3.10. *Let $(X, *, 0)$ be a d-algebra. If $X \times X = \alpha \cup \beta$ as a relation, then $(x * (x * y)) * y = 0$ for all $x, y \in X$.*

Proof. Given $x, y \in X$, if $x\alpha y$, then $x * y = x$ and hence $(x * (x * y)) * y = (x * x) * y = 0 * y = 0$. If $x\beta y$, then $x * y = y$ and hence $(x * (x * y)) * y = (x * y) * y = y * y = 0$, proving the proposition. □

Proposition 3.11. *Let $(X, *, 0)$ be a d-algebra and let $x, z \in X$. If $x\alpha y, x\alpha z$, then $((x * y) * (x * z)) * (z * y) = 0$ for all $y \in X$.*

Proof. If $x\alpha y, x\alpha z$, then $x*y = x, x*z = x$ and hence $((x*y)*(x*z))*(z*y) = (x * x) * (z * y) = 0 * (z * y) = 0$. □

The converse of Propositions 3.11 and 3.12 need not be true in general.

Example 3.12. Let $X := \{0, 1, 2, 3, 4\}$ be a set. Define a binary operation "$*$" on X as follows:

$*$	0	1	2	3	4
0	0	0	0	0	0
1	1	0	0	0	0
2	2	1	0	1	0
3	3	1	1	0	0
4	4	1	1	1	0

Then $(X, *, 0)$ is a BCK-algebra ([8, p. 249]) and hence it is a d-algebra. Since $3 * 2 = 1$, we have $(3, 2) \notin \alpha \cup \beta$, i.e., the converse of Proposition 3.10 does not hold. Moreover, since $2 * 1 = 1 \neq 2, 2 * 3 = 1 \neq 2$, the converse of Proposition 3.11 does not hold either.

Example 3.13. Let $\mathbb{R}$ be the set of all real numbers and let $x, y \in \mathbb{R}$. If we define $x * y := x(x - y)$ for all $x, y \in \mathbb{R}$, then $(\mathbb{R}, *, 0)$ is a d-algebra, but not a BCK-algebra. If $x\beta 1$, then $x * 1 = 1$ and hence $x^2 - x - 1 = 0$, which means $x = \frac{1\pm\sqrt{5}}{2}$, i.e., the Fibonacci golden sections lie below 1. If $1\beta x$, then $x = \frac{1}{2}$. Assume that $x\beta y$ and $x\beta z$. Then $(z - y)(x - 1) = 0$, and hence either $z = y$ or $y = z = \frac{1}{2}$, proving that the left cancelation law holds. Also given $x \neq 1$, we have $x^2 = (1 + x)y$ and $y = \frac{x^2}{1+x}$ is uniquely determined as being related to x in the β relation. Thus, given $x \neq 1$, we have the sequence $\{x, \frac{x^2}{1+x}, \frac{x^4}{(1+x)(1+x+x^2)}, \frac{x^8}{(1+x)(1+x+x^2)[(1+x)(1+x+x^2)+x^4]}, \ldots\}$ Hence for this d-algebra $x\beta y$ and $x\beta z$ implies $y = z$.

Problem. Construct a d-algebra $(X, *, 0)$ such that $x\beta y, x\beta z, y \neq z$ is possible for "many" x, i.e., it is impossible for exceptional x, e.g., $x = 0$, where $0 * y = y = 0$, i.e., $0\beta 0$ and $a\alpha y$ for all $y \in X$.

4. Below, above and between in semigroups

Let $(X, *)$ be a groupoid and let $x, y \in X$. We define some notations as below:

$$\beta(y) := \{x \in X \mid x\beta y\}, \qquad \alpha(x) := \{y \in X \mid x\alpha y\}$$
$$(x)_\beta := \{y \in X \mid x\beta y\}, \qquad (y)_\alpha := \{x \in X \mid x\alpha y\}$$

Let $(X, *)$ be a finite groupoid and let $E_\beta := \{(x, y) \mid x * y = y\}$. Then $|E_\beta|$ is the β-*arrow number* of $(X, *)$. It follows that $\sum_{x\in X} |(x)_\beta| = \sum_{y\in X} |\beta(y)| = |E_\beta|$ is an analog of a well-known result in graph theory.

Similarly, if $E_\alpha := \{(x, y) \mid x * y = x\}$, then $|E_\alpha|$ is the α-*arrow number* of $(X, *)$. Again it follows that $\sum_{x\in X} |\alpha(x)| = \sum_{y\in X} |(y)_\alpha| = |E_\alpha|$ using the same observation, i.e., every "arrow" (x, y) has precisely one "initial" element x and precisely one "terminal" element y.

If $(X, *)$ is a finite left-zero-semigroup, then $E_\beta = \{(x, y) \mid x * y = x = y\}$ and E_β is the diagonal $\triangle(X) = \{(x, x) \mid x \in X\}$ of $X \times X$, so that $|E_\beta| = |X|$.

On the other hand $E_\alpha = \{(x,y) \,|\, x * y = x\} = X \times X$ and $E_\alpha| = |X|^2$, so that if $|X| \geq 2$, then $|E_\alpha| = |E_\beta|^2$.

Example 4.1. Let $\mathbb{R}$ be the set of all real numbers and let $x, y \in \mathbb{R}$. Define a binary operation "$*$" on $\mathbb{R}$ by $x * y := y^2$. Then

$$\beta(y) = \begin{cases} \mathbb{R} & \text{if } y = 0 \text{ or } y = 1, \\ \emptyset & \text{otherwise,} \end{cases}$$

and $\alpha(x) = \{\sqrt{x}, -\sqrt{x}\}$, $(x)_\beta = \{0, 1\}$, $(y)_\alpha = [0, \infty)$.

Example 4.2. Let $(A, +)$ be an abelian group and let $x, y \in A$. Then $x\beta y$ means $x + y = y$, i.e., $x = 0$. Hence $\beta(y) = \{0\}$. Similarly, we obtain $\alpha(x) = \{0\}$. If $x \neq 0, y \neq 0$, then $(x)_\beta = \emptyset = (y)_\alpha$ and $(0)_\beta = A = (0)_\alpha$.

Proposition 4.3. *Let $(X, *)$ be a semigroup and let $x, y \in X$. Then $\beta(y)$ and $\alpha(x)$ are subsemigroups of $(X, *)$.*

Proof. If $a, b \in \beta(y)$, then $a * y = y, b * y = y$ and hence $(a * b) * y = a * (b * y) = a * y = y$, proving $a * b \in \beta(y)$. If $a, b \in (x)_\beta$, then $a * y = a, b * y = b$ and hence $(a * b) * y = a * (b * y) = a * b$, proving $a * b \in \alpha(x)$. $\square$

Proposition 4.4. *Let $(X, *)$ be a semigroup and let $x, y \in X$. Then $(x)_\beta$ is a right ideal and $(y)_\alpha$ is a left ideal of $(X, *)$.*

Proof. If $a \in (x)_\beta$ and $q \in X$, then $x * (a * q) = (x * a) * q = a * q$, and hence $a * q \in (x)_\beta$.

If $a \in (y)_\alpha$ and $q \in X$, then $(q * a) * y = q * (a * y) = q * a$, and hence $q * a \in (y)_\alpha$. $\square$

Proposition 4.5. *Let $(X, *)$ be a groupoid and let $e \in X$. Then e is the right (resp., left) identity if and only if $e \in \cap_{x \in X} \alpha(x)$ (resp., $e \in \cap_{x \in X} \beta(x)$).*

Proof. It follows that

$$\begin{aligned} e : \text{right identity} &\Longleftrightarrow x * e = x, \forall x \in X \\ &\Longleftrightarrow e \in \alpha(x), \forall x \in X \\ &\Longleftrightarrow e \in \cap_{x \in X} \alpha(x). \end{aligned}$$

The proof for the left identity is similar to the right identity case, and we omit it. $\square$

Note that $\cap_{x \in X} \alpha(x)$ (resp., $\cap_{x \in X} \beta(x)$) is a left (resp., right)-zero-semigroup if it is a non-empty set.

Theorem 4.6. *Let $(X, *)$ be a groupoid and let $\emptyset \neq L \subseteq X$. Then the following are equivalent.*

(1) *$(L, *)$ is a left-zero-semigroup*;
(2) *$L \subseteq \cap_{x \in L} \alpha(x)$*;
(3) *$L \subseteq \cap_{x \in L} (y)_\alpha$.*

Proof. (1)$\Longrightarrow$(2). Since $(L,*)$ is a left-zero-semigroup, $x*y=x$ for all $x,y\in L$. It follows that $x\alpha y$, i.e., $y\in\alpha(x)$ for all $x,y\in L$. Hence $y\in\cap_{x\in L}\alpha(x)$ for any $y\in L$, proving that $L\subseteq\cap_{x\in L}\alpha(x)$.

(2)$\Longrightarrow$(1). If $L\subseteq\cap_{x\in L}\alpha(x)$, then $y\in\cap_{x\in L}\alpha(x)$ for any $y\in L$. It follows that $x*y=x$ for all $x,y\in L$, proving that $(L,*)$ is a left-zero-semigroup.

(1)$\Longrightarrow$(3). Since $(L,*)$ is a left-zero-semigroup, $x*y=x$ for all $x,y\in L$. It follows that $x\in(y)_\alpha$ for all $y\in L$, i.e., $x\in\cap_{x\in L}(y)_\alpha$. Hence $L\subseteq\cap_{x\in L}(y)_\alpha$.

(3)$\Longrightarrow$(1). For any $x,y\in L$, since $L\subseteq\cap_{x\in L}(y)_\alpha$, we have $x\in\cap_{x\in L}(y)_\alpha$, which means that $x\alpha y$ for any $x,y\in L$. Hence $x*y=x$ for all $x,y\in L$, proving that $(L,*)$ is a left-zero-semigroup. □

Let $(X,*)$ be a groupoid and let $\emptyset\neq K\subseteq L, u\in X$. We denote by $K\alpha\, u$ if $u\in\cap_{x\in K}\alpha(x)$, i.e., $u\in\alpha(x)$ for all $x\in K$, or equivalently $x*u=x$ for all $x\in K$, i.e., u is a right identity in K. The subset K is said to be *lower-α-closed* if $K\alpha\, u$ implies $u\in K$.

Example 4.7. Let $X:=\{e,a,b,c\}$ be a set. Define binary operations $*_i$ on X as follows:

$*_1$	e	a	b	c
e	e	e	b	e
a	a	a	b	e
b	b	b	b	b
c	b	b	b	c

$*_2$	e	a	b	c
e	e	a	b	e
a	a	a	a	a
b	b	a	b	b
c	c	c	c	c

If we let $K_1:=\{e,a,b\}$, then K_1 is lower-α-closed in $(X,*_1)$. If we let $K_2:=\{a,b,c\}$, then K_2 is not lower-α-closed in $(X,*_2)$, since $K_2\,\alpha\, e$, but $e\notin K_2$.

Proposition 4.8. *Let $(X,*)$ be a groupoid and let K be a lower-α-closed subset of X and $K\subseteq T$. Then T is also lower-α-closed in $(X,*)$.*

Proof. Let $u\in X$ such that $T\alpha\, u$. Then $x*u=x$ for all $x\in T$. Since $K\subseteq T$, $x*u=x$ for all $x\in K$. It follows that $K\alpha\, u$. Since K is lower-α-closed, $u\in K\subseteq T$, proving that T is also lower-α-closed in $(X,*)$. □

Similarly, we can define the notion of upper-α-closed subsets in $(X,*)$.

5. Below, above and between in $Bin(X)$

In this section, we discuss the notions of below, above and between in $Bin(X)$.

Proposition 5.1. *Let $(X,\cdot)$ be a left-zero-semigroup. Then $(X,\cdot)\,\beta\,(X,\star)$ and $(X,\star)\alpha\,(X,\cdot)$ for any $(X,\star)\in Bin(X)$.*

Proof. For any $(X,\star)\in Bin(X)$, since $(X,\cdot)$ is a left-zero-semigroup, $(x\cdot y)\star(y\cdot x)=x\star y$ for any $x,y\in X$, i.e., $(X,\cdot)\Box(X,\star)=(X,\star)$. Hence $(X,\cdot)\beta\,(X,\star)$.

For any $(X,\star)\in Bin(X)$, since $(X,\cdot)$ is a left-zero-semigroup, $(x\star y)\cdot(y\star x)=x\star y$ for any $x,y\in X$, i.e., $(X,\star)\Box(X,\cdot)=(X,\star)$. Hence $(X,\star)\alpha\,(X,\cdot)$. □

Given $(X,\star) \in Bin(X)$, we define a set $\beta_r[(X,\star)]$ by

$$\beta_r[(X,\star)] := \{(X,*) \in Bin(X) \mid (X,*)\beta\,(X,\star)\}.$$

By Proposition 5.1, every $\beta_r[(X,\star)]$ contains a left-zero-semigroup, and hence $\beta_r[(X,\star)]$ is not empty.

Proposition 5.2. *$(\beta_r[(X,\star)],\Box)$ is a subsemigroup of $(Bin(X),\Box)$.*

Proof. Given $(X,*),(X,\triangledown) \in (\beta_r[(X,\star)],\Box)$, we have $(X,*)\Box(X,\star) = (X,\star)$ and $(X,\triangledown)\Box(X,\star) = (X,\star)$. It follows that

$$\begin{aligned}[(X,*)\Box(X,\triangledown)]\Box(X,\star) &= (X,*)\Box[(X,\triangledown)\Box(X,\star)]\\ &= (X,*)\Box(X,\star)\\ &= (X,\star).\end{aligned}$$

Hence $(X,*)\Box(X,\triangledown)\beta(X,\star)$, proving that $(X,*)\Box(X,\triangledown) \in (\beta_r[(X,\star)],\Box)$. $\Box$

Similarly, given $(X,\star) \in Bin(X)$, we define a set $\beta_l[(X,\star)]$ by

$$\beta_l[(X,\star)] := \{(X,*) \in Bin(X) \mid (X,\star)\beta\,(X,*)\}.$$

We obtain that $\beta_l[(X,\star)]$ is non-empty and a subsemigroup of $(Bin(X),\,\Box)$ for any $(X,\star) \in Bin(X)$.

If $(X,\cdot)$ is a left-zero-semigroup, then it is easy to see that $\beta_l[(X,\cdot)] = Bin(X)$.

Two groupoids $(X,\cdot)$ and $(X,\star)$ are said to be *β_l-similar* (*β_r-similar*, resp.) if $\beta_l[(X,\cdot)] = \beta_l[(X,\star)]$ ($\beta_r[(X,\cdot)] = \beta_r[(X,\star)]$, resp.).

Example 5.3. Let $X := \mathbb{R}$ be the set of all real numbers. Define a binary operation $\cdot$ on X by $x\cdot y := x^2$ for all $x,y \in X$. Let $(X,*) \in \beta_l[(X,\cdot)]$. Then $(X,*)\Box(X,\cdot) = (X,\cdot)$. Hence $(x*y)\cdot(y*x) = x\cdot y$ for all $x,y \in X$. It follows that $(x*y)^2 = x^2$ and hence $x*y \in \{-x,x\}$ for all $x,y \in X$. Define a binary operation $\odot$ on X by $x\odot y := x^4$ for all $x,y \in X$. If $(X,*) \in \beta_l[(X,\odot)]$, then $(X,*)\Box(X,\odot) = (X,\odot)$. Hence $(x*y)\odot(y*x) = x^4$ for all $x,y \in X$. It follows that $[(x*y)^2 - x^2][(x*y)^2 + x^2] = 0$ and hence $x*y \in \{-x,x\}$ for all $x,y \in X$. Thus $\beta_l[(X,\cdot)] = \beta_l[(X,\odot)]$, i.e., $(X,\cdot)$ is β_l-similar to $(X,\odot)$.

Let $X := \mathbb{R}$ be the set of all real numbers. Define a binary operation $\oslash$ on X satisfying $(-x)\oslash(-y) = (-x)\oslash y = x\oslash(-y) = x\oslash y$ for all $x,y \in X$. We call such a groupoid $(X,\oslash)$ an *even groupoid.*

Proposition 5.4. *Let $(X,\oslash)$ be an even groupoid. If $(X,*) \in Bin(X)$ such that $x*y \in \{-x,x\}$ for all $x,y \in X$, then $(X,*) \in \beta_r[(X,\oslash)]$.*

Proof. Let $(X,\Box) := (X,*)\Box(X,\oslash)$. Then $x\Box y = (x*y)\oslash(y*x) \in \{(-x)\oslash(-y),(-x)\oslash y, x\oslash(-y), x\oslash y\} = \{x\oslash y\}$ for all $x,y \in X$. It follows that $x\Box y = x\oslash y$ for all $x,y \in X$. This proves that $(X,*)\Box(X,\oslash) = (X,\oslash)$, i.e., $(X,*) \in \beta_r[(X,\oslash)]$. $\Box$

6. Order preserving mappings and medial groupoids

Given groupoids $(X, *), (Y, \bullet)$, a mapping $F : X \to Y$ is said to be *β-order-preserving* if $x\beta y$ implies $F(x)\beta F(y)$, i.e., $x * y = y$ implies $F(x) \bullet F(y) = F(y)$.

Proposition 6.1. *Every groupoid homomorphism $F : (X, *) \to (Y, \bullet)$ is β-order-preserving.*

Proof. If $x\beta y$, then $x * y = y$. Since F is a groupoid homomorphism, we have $F(y) = F(x * y) = F(x) \bullet F(y)$, proving that F is β-order-preserving. $\square$

The converse of Proposition 6.1 need not be true in general.

Example 6.2. Let $\mathbb{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbb{R}$ by $x * y := \frac{x+y}{2}$. Then $x * x = x$ for all $x \in \mathbb{R}$, and if $x < y$, then $x < x * y < y$ for all $x, y \in \mathbb{R}$. Let $(X, \bullet)$ be any left-zero-semigroup and let $F : (X, \bullet) \to (\mathbb{R}, *)$ be any function. Then F is β-order-preserving. In fact, if $x\beta y$, then $x \bullet y = y$. Since $(X, \bullet)$ is the left-zero-semigroup, we have $x = x \bullet y = y$ and hence $F(x) * F(y) = F(x) * F(x) = F(x) = F(y)$, which shows that $F(x)\beta F(y)$. Assume that F is a groupoid homomorphism. If $m, n \in X$ such that $F(m) < F(n)$, then $F(m) < F(m) * F(n) < F(n)$. Since $(X, \bullet)$ is the left-zero-semigroup, we obtain $F(m) < F(m) * F(n) = F(m \bullet n) = F(m)$, a contradiction.

Note that if $(X, *), (Y, \bullet)$ and $(X, \diamond)$ are groupoids and if $F : (X, *) \to (Y, \bullet)$, $G : (Y, \bullet) \to (Z, \diamond)$ are β-order-preserving, then $G \circ F : (X, *) \to (Z, \diamond)$ is also β-order-preserving.

Proposition 6.3. *Let $F : (X, *) \to (Y, \bullet)$ be a β-order-preserving mapping.*

(i) *if $(X, *)$ is the left-zero-semigroup and $x\beta y$, then $F(x)$ is an idempotent on $(Y, \bullet)$,*

(ii) *if $(X, *)$ is the right-zero-semigroup, then $(ImF, \bullet)$ is a subgroupoid of $(Y, \bullet)$.*

Proof. (i) If $x\beta y$, then $x = x * y = y$, $(X, *)$ is the left-zero-semigroup. Since F is a β-order-preserving mapping, we have $F(x) = F(y) = F(x) \bullet F(y) = F(x) \bullet F(x)$.

(ii) If $(X, *)$ is the right-zero-semigroup, then $x\beta y = y$ for all $x, y \in X$. Since F is a β-order-preserving mapping, $F(x)\beta F(y)$, i.e., $F(x) \bullet F(y) = F(y)$. Hence $(ImF, \bullet)$ is a subgroupoid of $(Y, \bullet)$. $\square$

A groupoid mapping $F : (X, *) \to (Y, \bullet)$ is said to be a *$\langle\beta\rangle$-order-preserving* if $z \in \langle x, y\rangle_\beta$ then $F(z) \in \langle F(x), F(y)\rangle_\beta$.

Proposition 6.4. *If $F : (X, *) \to (Y, \bullet)$ is β-order-preserving, then it is $\langle\beta\rangle$-order-preserving.*

Proof. If $z \in \langle x, y\rangle_\beta$ then $x\beta z, z\beta y$. Since F is β-order-preserving, we obtain $F(x)\beta F(z), F(z)\beta F(y)$. It follows that $F(z) \in \langle F(x), F(y)\rangle_\beta$. $\square$

Proposition 6.5. *Let $(X,*) \in Bin(X)$ with the property (P) : for any $y \in X$, there exists a $z \in X$ such that $y\beta z$. If $F : (X,*) \to (Y,\bullet)$ is $\langle\beta\rangle$-order-preserving, then it is β-order-preserving.*

Proof. If $x\beta y$, then there exists a $z \in X$ such that $y\beta z$, since $(X,*)$ has the property (P). Since $x\beta y$, we have $y \in \langle x,z\rangle_\beta$. Since F is $\langle\beta\rangle$-order-preserving, we obtain $F(y) \in \langle F(x),F(z)\rangle_\beta$, proving that $F(x)\beta F(y)$. □

Proposition 6.6. *Let $(X,*)$ be a group with identity e and let $(Y,\bullet)$ be a group with identity $\widehat{e}$. If $F : (X,*) \to (Y,\bullet)$ is a map such that $F(e) = \widehat{e}$ (not necessarily a group homomorphism), then F is β-order-preserving.*

Proof. If $x\beta y$, then $x*y = y$. Since $(X,*)$ is a group, we obtain $x = e$. It follows that $F(x)\bullet F(y) = F(e)\bullet F(y) = \widehat{e}\bullet F(y) = F(y)$, proving that $F(x)\beta F(y)$. □

A groupoid $(X,*)$ is said to be *β-linear selective* if for all $x,y \in X$, $x\beta y$ or $y\beta x$. Let $(X,*)$ be a selective groupoid, i.e., $x*y \in \{x,y\}$ for all $x,y \in X$. If we assume $x*y = y*x$ for all $x,y \in X$, then it is β-linear selective.

A groupoid mapping $F : (X,*) \to (Y,\bullet)$ is said to be *α-order-preserving* if $x\alpha y$ implies $F(x)\alpha F(y)$, i.e., $x*y = x$ implies $F(x)\bullet F(y) = F(x)$.

Proposition 6.7. *Every groupoid homomorphism $F : (X,*) \to (Y,\bullet)$ is α-order-preserving.*

Proof. The proof is similar to the proof of Proposition 6.1, and we omit it. □

Note that, by Propositions 6.1 and 6.7, every groupoid homomorphism is both α-order-preserving and β-order-preserving. We obtain the exact analog of Proposition 6.6 for α-order-preserving mappings.

Proposition 6.6′. *Let $(X,*)$ be a group with identity e and let $(Y,\bullet)$ be a group with identity $\widehat{e}$. If $F : (X,*) \to (Y,\bullet)$ is a map such that $F(e) = \widehat{e}$ (not necessarily a group homomorphism), then F is α-order-preserving.*

Note that such a mapping $F : (X,*) \to (Y,\bullet)$ discussed in Propositions 6.6 and 6.6′ is both a β-order-preserving and an α-order-preserving mapping without being a group homomorphism necessarily.

Given a groupoid $(X,*)$, consider $[\beta] := \{(x,y) \in X \times X \mid x\beta y\}$ and the subset $[\beta]^* := \{x \in X \mid \exists\, y \in X \text{ such that } x\beta y\}$. If a groupoid $(X,*)$ has the property (P), then $X = [\beta]^*$ and $[\beta] = \cup_{x\in X}[\beta]_x$. The set $[\beta]$ acts as a natural set of edges ($x*y = y$ corresponds to $x \to y$ in digraphs).

A groupoid $(X,*)$ is said to be *medial* if $(x*u)*(y*v) = (x*y)*(u*v)$ for all $x,y,u,v \in X$. It is known that every p-semisimple BCI-algebra is medial ([16, p. 42]).

Proposition 6.8. *If $(X,*)$ is a medial groupoid, then $[\beta]^*$ is a subgroupoid of $(X,*)$.*

Proof. If $x, y \in [\beta]^*$, then there exist $u, v \in X$ such that $x\beta u, y\beta v$. It follows that $x * u = u, y * v = v$. Since $(X, *)$ is medial, we have $(x * y) * (u * v) = (x * u) * (y * v) = u * v$, proving that $x * y \in [\beta]^*$. □

A groupoid $(X, *)$ is said to be β-*medial* if $([\beta]^*, *)$ is a subgroupoid of $(X, *)$.

Example 6.9. Let $(X, *)$ be a group and let $x \in [\beta]^*$. Then there exists $y \in X$ such that $x\beta y$. It follows that $x * y = y$, which shows that $x = e$, an identity. Hence $[\beta]^* = \{e\}$ is a subgroupoid of $(X, *)$. Hence every group is β-medial.

Note that non-abelian group is β-medial, but not medial. In fact, assume that $(X, *)$ is a group and medial. Then, for all $x, y \in X$, $x*y = (e*x)*(y*e) = (e * y) * (x * e) = y * x$. Hence $(X, *)$ is abelian.

Given a groupoid $(X, *)$, consider $[\alpha] := \{(x, y) \in X \times X \mid x\alpha y\}$ and the subset $[\alpha]^* := \{x \in X \mid \exists\, y \in X \ \text{ such that } \ x\alpha y\}$.

Proposition 6.10. *If $(X, *)$ is a medial groupoid, then $[\alpha]^*$ is a subgroupoid of $(X, *)$.*

Proof. If $x, y \in [\beta]^*$, then there exist $u, v \in X$ such that $x\alpha u, y\alpha v$. It follows that $x * u = x, y * v = y$. Since $(X, *)$ is medial, we have $(x * y) * (u * v) = (x * u) * (y * v) = x * y$, proving that $x * y \in [\alpha]^*$. □

A groupoid $(X, *)$ is said to be α-*medial* if $([\alpha]^*, *)$ is a subgroupoid of $(X, *)$.

By Propositions 6.8 and 6.10, every medial groupoid is both β-medial and α-medial.

Note that the converse of Propositions 6.8 and 6.10 need not be true in general. In fact, in the groupoid $(X, *_1)$ of Example 4.7, we can see that $[\alpha]^* = [\beta]^* = X$, a subgroupoid of itself, but it is not medial, since $(a *_1 e) *_1 (c *_1 e) = a *_1 b = b$, $(a *_1 c) *_1 (e *_1 e) = e *_1 e = e$.

Example 6.11. Let $(X, *)$ be a group. If $x \in X$, then $x * e = x$, i.e., $x\alpha e$. Hence $x \in [\alpha]^*$, proving that $[\alpha]^* = X$ is a subgroupoid of $(X, *)$. Hence every group is α-medial.

By Examples 6.9 and 6.11, we conclude that every group is both β-medial and α-medial. It is easy to see that every non-abelian group is α-medial, but not medial.

7. Conclusion

In this paper we have addressed the question whether it is possible to construct a notion of order($\leq$) which is natural for arbitrary groupoids. Given the great generality needed in order to succeed in doing so, we have dispensed with transitivity as a fundamental idea in this setting, even though we have certainly considered it an important idea in this paper as well as elsewhere. As a consequence we were encouraged to consider the notion of below (smaller, weaker, ...) and above (larger, stronger, ...) as basic. Experience with examples in the real world, such as in children's games (rock, scissors, paper) indicating that above and below are not identical in their implication as opposites. Accepting this fact, we used a graph-theoretical analogue for directed graphs as translated to selective groupoids to define relations β(below) and α(above), whence the relation $\leq$ becomes $\alpha \cap \beta$ in a natural way. Testing the usefulness of this idea, we discover that in the semigroup $(Bin(X), \square)$ the identity element (i.e., the left-zero semigroup) is a unique least element in the $\leq$-relation. Other examples also correspond well to intuitive ideas of what should be the meaning of these terms.

We make no claims to the uniqueness of these notions. In fact, in many areas, e.g., in the study of BCK-algebras there are competing notions ($x \leq y$ if and only if $x * y = 0$) which have been in use for a considerable amount of time and which are accepted in the literature.

If we define a "new" operation $x \wedge y = x * (x * y)$ on the BCK-algebra $(X, *, 0)$, as is commonly done, then $x * y = 0$ implies $x \wedge y = x * 0 = x$, i.e., $x\alpha y$ in $(X, \wedge, 0)$, so that these distinct operations are in fact quite close. Therefore, because of these considerations, we offer these ideas on order as suggestions for their use in the development of a general theory of groupoids $Bin(X)$ for sets X, as a natural approach to the subject.

Acknowledgement. Authors are very grateful for referee's valuable suggestions and help.

References

[1] S. S. Ahn, Y. H. Kim, and J. Neggers, *Fuzzy upper bounds in groupoids*, Sci. World J. **2014** (2014), Article ID 697012, 6pages.

[2] P. J. Allen, H. S. Kim, and J. Neggers, *Companion d-algebras*, Math. Slovaca **57** (2007), no. 2, 93–106.

[3] ———, *Several types of groupoids induced by two-variable functions*, Springer Plus **5** (2016), 1715–1725; DOI:10.1186/s40064-016-3411-y.

[4] R. H. Bruck, *A Survey of Binary Systems*, Springer, New York, 1971.

[5] H. F. Fayoumi, *Locally-zero groupoids and the center of $Bin(X)$*, Commun. Korean Math. Soc. **26** (2011), no. 2, 163–168.

[6] J. S. Han, H. S. Kim, and J. Neggers, *The hypergroupoid semigroups as generalizations of the groupoid semigroups*, J. Appl. Math. **2012** (2012), Article ID 717698, 8 pages.

[7] H. S. Kim and J. Neggers, *The semigroups of binary systems and some perspectives*, Bull. Korean Math. Soc. **45** (2008), no. 4, 651–661.

[8] J. Meng and Y. B. Jun, *BCK-Algebras*, Kyung Moon Sa, Seoul, 1994.

[9] J. Neggers and H. S. Kim, *Basic Poset*, World Scientific, Singapore, 1998.

[10] ———, *On d-algebras*, Math. Slovaca **49** (1999), no. 1, 19–26.

[11] S. J. Shin, H. S. Kim, and J. Neggers, *On Abelian and related fuzzy subsets of groupoids*, Sci. World J. **2013** (2013), Article ID 476057, 5pages.

[12] ———, *The intersection between fuzzy subsets and groupoids*, Sci. World J. **2014** (2014), Article ID 246285, 6pages.

[13] S. Z. Song, H. S. Kim, and Y. B. Jun, *Soft saturated and dried values with applications in BCK/BCI-algebras*, J. Comput. Anal. Appl. **21** (2016), no. 3, 528–544.

[14] R. Stanley, *Enumerative Combinatorics. Vol. 1*, Wadsworth & Brooks/Cole, Monterey, 1986.

[15] D. B. West, *Introduction to Graph Theory*, Prentice Hall, London, 2001.

[16] H. Yisheng, *BCI-Algebra*, Science Press, Beijing, 2006.
[17] J. Zhan, H. S. Kim, and J. Neggers, *On (fuzzy) weak-zero groupoids and* (X, N)*-zero groupoids*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys. **78** (2016), no. 2, 99–110.

Hee Sik Kim
Research Institute for Natural Sciences
Department of Mathematics
Hanyang University
Seoul 04763, Korea
E-mail address: heekim@hanyang.ac.kr

Joseph Neggers
Department of Mathematics
University of Alabama
Tuscaloosa, AL 35487-0350, USA
E-mail address: jneggers@ua.edu

Keum Sook So
Department of Statistics and Financial Informatics
Hallym University
Chuncheon 24252, Korea
E-mail address: ksso@hallym.ac.kr