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UNIFIED (α, β)-FLOWS ON TRIANGULATED MANIFOLDS WITH TWO AND THREE DIMENSIONS

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Abstract. In this paper, we introduce a framework of (α, β)-flows on triangulated manifolds with two and three dimensions, which unifies several discrete curvature flows previously defined in the literature.

1. Background and preliminaries

In his famous book [15], Thurston introduced the circle packing metric, which was used to study low dimensional topology. In [12], Hamilton introduced the well-known Ricci flow, which is a powerful tool to deform Riemannian metrics. More important to our subject, as a pioneering work, Chow and Luo [1] established the intrinsic connection between Hamilton's surface Ricci flow and Thurston's circle patterns. They first introduced the combinatorial (discrete) Ricci flows on triangulated surface. Inspired by their work, Glickenstein [10] first introduced the combinatorial Yamabe flows on triangulated 3-dimensional manifolds. Since then, discrete curvature flow has been becoming popular for its usefulness in engineering fields, especially in the Graphics and Image Processing areas.

Suppose M^n is a closed manifold with dimension $n = 2$ or 3. Given a triangulation $\mathcal{T} = \{\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n\}$ on M, where $\mathcal{T}_i$ represents the set of i-simplices ($0 \leq i \leq n$). In what follows, $(M^n, \mathcal{T})$ will be referred to as a triangulated n-manifold. All the vertices are ordered one by one, marked by $v_1, \ldots, v_N$, where $N = \mathcal{T}_0^\sharp$ is the number of vertices. We use $i \sim j$ to denote that the vertices i and j are adjacent if there is an edge $\{ij\} \in \mathcal{T}_1$. Throughout this paper, all functions $f : \mathcal{T}_0 \to \mathbb{R}$ will be regarded as column vectors in $\mathbb{R}^N$. And we denote $C(\mathcal{T}_0)$ as the set of functions defined on $\mathcal{T}_0$.

The most natural way to define a discrete metric on a triangulated manifold $(M^n, \mathcal{T})$ is to evaluate a length l_{ij} for each edge $i \sim j$ directly. Alternatively, discrete metrics can also be defined on all vertices, based on which the length l_{ij} can be derived indirectly. Thurston introduced the circle packing metric for

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$n = 2$, while Cooper and Rivin considered the sphere packing metric for $n = 3$. Now we review these two definitions:

Definition 1.1 (Thurston's circle packing metric)**.** Given a triangulated surface $(M^2, \mathcal{T})$. Let $\Phi : \mathcal{T}_1 \to [0, \frac{\pi}{2}]$ be a function assigning each edge $\{ij\}$ a weight Φ_{ij}. Each map $r : \mathcal{T}_0 \to (0, +\infty)$ is called a circle packing metric.

For given $(M^2, \mathcal{T}, \Phi)$, we attach each edge $\{ij\}$ a length

$$l_{ij} = \sqrt{r_i^2 + r_j^2 + 2r_ir_j \cos \Phi_{ij}}. \tag{1.1}$$

Thurston proved [15] that the lengths $\{l_{ij}, l_{jk}, l_{ik}\}$ satisfy the triangle inequality on each face $\{ijk\} \in \mathcal{T}_2$, which ensures that the face $\{ijk\}$ could be realized as an Euclidean triangle with lengths $\{l_{ij}, l_{jk}, l_{ik}\}$. Thus the space of all circle packing metrics is exactly

$$\mathbb{R}^N_{>0} \triangleq (0, +\infty)^N.$$

The triangulated surface $(M^2, \mathcal{T}, \Phi)$ can be seen as gluing many Euclidean triangles coherently. However, this gluing produces singularities at vertices, which is described as discrete curvature.

Definition 1.2 (Discrete Gaussian curvature)**.** Suppose θ_i^{jk} is the inner angle of triangle $\triangle v_iv_jv_k$ at vertex i, the classical discrete Gauss curvature at i is defined as

$$K_i = 2\pi - \sum_{\{ijk\} \in \mathcal{T}_2} \theta_i^{jk}, \tag{1.2}$$

where the sum is taken over all the triangles with v_i as one of their vertices.

For discrete Gaussian curvature K_i, there is a discrete version of Gauss-Bonnet identity

$$\sum_{i \in \mathcal{T}_0} K_i = 2\pi\chi(M). \tag{1.3}$$

Definition 1.3 (Cooper & Rivin's ball packing metric)**.** Given a triangulated surface $(M^3, \mathcal{T})$. Each map $r : \mathcal{T}_0 \to (0, +\infty)$ derives a length

$$l_{ij} = r_i + r_j \tag{1.4}$$

for each $i \sim j$. If for each $\{i, j, k, l\} \in \mathcal{T}_3$, $l_{ij}, l_{ik}, l_{il}, l_{jk}, l_{jl}, l_{kl}$ determines an Euclidean tetrahedron, then call r a (non-degenerate) ball packing metric.

It is pointed out [10] that a tetrahedron $\{i, j, k, l\} \in \mathcal{T}_3$ generated by four positive radii r_i, r_j, r_k, r_l can be realized as an Euclidean tetrahedron if and only if

$$Q_{ijkl} = \left(\frac{1}{r_i} + \frac{1}{r_j} + \frac{1}{r_k} + \frac{1}{r_l}\right)^2 - 2\left(\frac{1}{r_i^2} + \frac{1}{r_j^2} + \frac{1}{r_k^2} + \frac{1}{r_l^2}\right) > 0. \tag{1.5}$$

Thus the space of all ball packing metrics is exactly

$$\mathfrak{M}_{\mathcal{T}} = \left\{ r \in \mathbb{R}^N_{>0} \;\middle|\; Q_{ijkl} > 0,\ \forall\{i,j,k,l\} \in \mathcal{T}_3 \right\}.$$

Cooper and Rivin [2] proved that $\mathfrak{M}_{\mathcal{T}}$ is a simply connected open subset of $\mathbb{R}^N_{>0}$, but not convex. The ball packing metric induces a piecewise linear metric which makes the curvature flat everywhere on $M^3 - \mathcal{T}_1$ while singular on $\mathcal{T}_1$. Thus Cooper and Rivin defined a discrete scalar curvature concentrated on all vertices.

Definition 1.4 (Cooper & Rivin's discrete scalar curvature)**.** Denote α_{ijkl} as the solid angle at a vertex i in a single Euclidean tetrahedron $\{i,j,k,l\} \in \mathcal{T}_3$, then the discrete scalar curvature at vertex i is

$$K_i = 4\pi - \sum_{\{i,j,k,l\}\in\mathcal{T}_3} \alpha_{ijkl}, \tag{1.6}$$

where the sum is taken over all $\{j,k,l\} \in \mathcal{T}_2$ such that $\{i,j,k,l\} \in \mathcal{T}_3$.

In what follows we call Cooper & Rivin's discrete curvature as CR-curvature for short. To study the CR-curvature K_i, the first of the authors and Xu [9] introduced the following 3-dimensional α-flow,

$$\frac{dr_i}{dt} = s_\alpha r_i^\alpha - K_i, \tag{1.7}$$

where $s_\alpha = \frac{\sum_i K_i r_i}{\sum_i r_i^{\alpha+1}}$. They proved that if the α-flow (1.7) converges, there exists a metric r^* whose α-order curvature K_i/r_i^α is a constant for each $i \in V$. On the contrary, assume r^* is a metric whose α-order curvature is a constant, and the first positive eigenvalue of $-\Delta_\alpha$ (see (4.2) for a definition) at r^* is bigger than αs_α^*, then r^* is a asymptotically stable point of the α-flow (1.7). In this paper, we shall generalize the α-flow (1.7) to one with the following more universal form

$$\frac{dg_i}{dt} = s_\alpha r_i^\alpha - K_i,$$

where $g_i = \ln r_i$ or $g_i = r_i^\sigma$, $\sigma \in \mathbb{R}$. We explain the idea behind this generalization. If we consider the conformal deformation of the discrete metric r_i, we may take $g_i = \ln r_i$. This is inspired by Chow and Luo's pioneer work [1]. If we consider the deformation of the conical metric, we may take take $g_i = r_i^\sigma$ as a metric (of σ-order). This is inspired by the viewpoint of Riemannian geometry. A piecewise flat metric is a singular Riemannian metric on M, which produces conical singularities at all vertices. For any $\sigma \in \mathbb{R}$, a metric g with conical singularity at a point can be expressed as

$$g(z) = e^{f(z)} \frac{dz \wedge d\bar{z}}{|z|^{2(1-\sigma)}} \tag{1.8}$$

locally. Choosing $f(z) = -\ln \sigma^2$, then $g(z) = |dz^\sigma|^2$. Comparing r^σ with $|dz^\sigma|$, the σ-order metric r^σ may be considered as a discrete analogue of conical metric to some extent. In this paper, we will express the flow (2.2) as $\frac{dr_i}{dt} = s_\alpha r_i^\beta -$

$K_i r_i^{\beta-\alpha}$ by leaving out a constant σ and taking $\beta = 1 + \alpha - \sigma$ (see Definition 2.1 below). We will prove that the flow (2.2) exhibits similar convergence properties as the α-flow (1.7) in Section 4.

2. Definition of unified (α, β)-flow

For any $\alpha \in \mathbb{R}$, the first of the authors and Xu [8, 9] studied discrete circle (ball) packing metrics with all curvature K_i/r_i^α equal to a constant. It is easy to see that, if $\frac{K_i}{r_i^\alpha} \equiv s_\alpha$ for each vertex i, where s_α is a constant, then

$$s_\alpha = \frac{\sum K_i r_i^{n-2}}{\|r\|_{\alpha+n-2}^{\alpha+n-2}}. \tag{2.1}$$

Following the definition in [6], we call this type of metric as "discrete α-quasi-Einstein metric" or "constant α-curvature metric".

One of the most important problems we concern is to understand whether there are discrete α-quasi-Einstein metrics in $\mathfrak{M}_{\mathcal{T}}$ for $n = 3$, or in $\mathbb{R}^N_{>0}$ for $n = 2$. For the $n = 2$ case, this problem is basically resolved in [8,9]. Especially for triangulated surfaces with $\alpha\chi(M) \leq 0$, both the combinatorial-topological conditions and the analytical conditions are given for the existence of discrete α-quasi-Einstein metrics. Furthermore, discrete α-quasi-Einstein metrics can be obtained by evolving discrete curvature flows or minimizing discrete Ricci potentials. For the $n = 3$ case, very few combinatorial-topological conditions are known for the existence of discrete α-quasi-Einstein metrics. We want to study discrete α-quasi-Einstein metrics by introducing an unified (α, β)-flow:

Definition 2.1. Let s_α be defined in (2.1). If $n = 2$, let K_i be defined in (1.2). If $n = 3$, let K_i be defined in (1.6). For any $\alpha, \beta \in \mathbb{R}$, the unified (α, β)-flow is defined as

$$\frac{dr_i}{dt} = s_\alpha r_i^\beta - K_i r_i^{\beta-\alpha}. \tag{2.2}$$

The prototype of the unified (α, β)-flow (2.2) is Chow-Luo's combinatorial Ricci flow [1] in dimension two, which can be expressed as $\frac{dr_i}{dt} = K_{av} r_i - K_i r_i$. Take $(\alpha, \beta) = (0, 1)$, and $n = 2$, then the unified (α, β)-flow (2.2) becomes Chow-Luo's flow. The unified (α, β)-flow (2.2) modeled its form after that of the well known smooth Yamabe flow $\frac{\partial g}{\partial t} = (s - R)g$, where g is a smooth Riemannian metric tensor, R is the scalar curvature and s is the average curvature. If we express the unified (α, β)-flow (2.2) as $\frac{dr_i}{dt} = (s_\alpha - \frac{K_i}{r_i^\alpha}) r_i^\beta$ and further take $\beta = 1$, then it's easy to see that the term $\frac{K_i}{r_i^\alpha}$ plays similar role as the term R plays in the smooth Yamabe flow. To some extent, $\frac{K_i}{r_i^\alpha}$ is the discrete analogy of the smooth scalar curvature R.

Now it's the time to say a bit more about the motivations to introduce the unified (α, β)-flow. The first motivation is to unify the various discrete curvature flows previously defined in the literature, see Section 3 for more details. The second motivation is to approach the following combinatorial α-Yamabe problem, which is modeled after the smooth Yamabe problem and was previously raised by the first of the authors and Xu [9].

The combinatorial α-Yamabe problem. Given a 2-dimensional (or 3-dimensional) manifold M^2 (or M^3) with triangulation $\mathcal{T}$, find a circle (or ball) packing metric with constant combinatorial α-curvature in the combinatorial conformal class $\mathbb{R}^N_{>0}$ (or $\mathfrak{M}_{\mathcal{T}}$).

The unified (α, β)-flow provides a natural way to approach the combinatorial α-Yamabe problem. We want to deform an arbitrary metric $r(0)$ to a discrete α-quasi-Einstein metric r^*, one effective way is to evolve it according to an ODE system $r_i'(t) = f_i(r(t))$ with r^* as its critical point. Thus $f_i(r^*)$ equals to zero. The easiest way is to choose $f_i(r) = s_\alpha - \frac{K_i}{r_i^\alpha}$. In fact, we found this selection of $f_i(r)$ is too restrictive, this fact motivates us to relax it as $f_i(r) = s_\alpha r_i^\alpha - K_i r_i^{\beta-\alpha}$. In the following Theorem 4.2, we will show that if the solution to the (α, β)-flow converges, then the combinatorial α-Yamabe problem is solvable.

We shall show that the unified (α, β) flow (2.2) is generally not a "normalization" of the following flow

$$\frac{dr_i}{dt} = -K_i r_i^{\beta-\alpha} \tag{2.3}$$

except for $(\alpha, \beta) = (0, 1)$, where the word "normalization" means that the solutions to this two flows are related by a change of the scale in space and a change of the parametrization in time. Assume the solution of the unified (α, β) flow (2.2) differs from the solution of the flow (2.3) only by a change of the scale in space and a change of the parametrization in time. Let t, r, K, s_α denote the variables for the flow (2.3), and $\tilde{t}, \tilde{r}, \tilde{K}, \tilde{s}_\alpha$ for the flow (2.2). Let $r(t), t \in [0, T)$ be a solution of the flow (2.3), while $\tilde{r}(\tilde{t}), \tilde{t} \in [0, \tilde{T})$ be a solution of the flow (2.2). Set $\tilde{r}(\tilde{t}) = \varphi(t) r(t)$, where $\varphi(t) > 0$ is a scaling factor and is independent of the vertices. Obviously, $\tilde{K}(\tilde{t}) = K(t)$, $\tilde{s}_\alpha(\tilde{t}) = s_\alpha(t)\varphi^{-\alpha}(t)$, and $\tilde{s}_\alpha \tilde{r}_i^\alpha = s_\alpha r_i^\alpha$. Hence

$$\frac{d\tilde{r}_i}{d\tilde{t}} = \frac{d(\varphi r_i)}{dt}\frac{dt}{d\tilde{t}} = \Big(r_i \frac{d\varphi}{dt} - \varphi \tilde{K}_i r_i^{\beta-\alpha}\Big)\frac{dt}{d\tilde{t}}.$$

On the other hand, $\tilde{r}$ satisfies the equation (2.2), i.e.,

$$\frac{d\tilde{r}_i}{d\tilde{t}} = \tilde{s}_\alpha \tilde{r}_i^\beta - \tilde{K}_i \tilde{r}_i^{\beta-\alpha} = s_\alpha r_i^\beta \varphi^{\beta-\alpha} - K_i r_i^{\beta-\alpha}\varphi^{\beta-\alpha}.$$

Comparing the above two expressions about $d\tilde{r}_i/d\tilde{t}$, we obtain $\beta - \alpha = 1$, $dt/d\tilde{t} = 1$ and $\frac{1}{\varphi}\frac{d\varphi}{dt} = s_\alpha r_i^\alpha$ for every $i \in V$. Since φ is independent of $i \in V$, we further get $\alpha = 0$ and $\beta = 1$. Hence the unified (α, β) flow (2.2) is a normalization of the flow (2.3) if and only if $(\alpha, \beta) = (0, 1)$.

3. Some examples

There are several discrete curvature flows on two and three dimensional triangulated manifolds. We list some of them below. Firstly let us take a look at the 2-dimensional examples.

Example 1. Given a triangulated surface $(M^2, \mathcal{T})$, consider Thurston's circle packing metric r with fixed weight Φ. Denote the discrete α-curvature as $R_{\alpha,i} = K_i/r_i^\alpha$. Notice that $s_\alpha = 2\pi\chi(M)/\|r\|_\alpha^\alpha$ by discrete Gauss-Bonnet formula. Then we have the following six different discrete flows on triangulated surfaces:

(1) $\dot{u}_i = K_{av} - K_i$, where $u_i = \ln r_i$, $K_{av} = 2\pi\chi(M)/N$, see [1];
(2) $\dot{u}_i = s_\alpha r_i^\alpha - K_i$, where $u_i = \ln r_i$, $\alpha \in \mathbb{R}$, see [9];
(2)′ $\dot{u}_i = s_\alpha r_i^\alpha - K_i$, where $u_i = \ln r_i^\alpha$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, differs from (2) by a constant α;
(3) $\dot{g}_i = (R_{av} - R_i)g_i$, where $g_i = r_i^2$, $R_{av} = s_2$ and $R_i = R_{2,i}$, see [8];
(4) $\dot{u}_i = s_\alpha - R_{\alpha,i}$, where $u_i = \ln r_i$, $\alpha \in \mathbb{R}$, see [8];
(4)′ $\dot{u}_i = s_\alpha - R_{\alpha,i}$, where $u_i = \ln r_i^\alpha$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, differs from (4) by a constant α.

Next we see the cases with dimension three.

Example 2. Given a triangulated 3-manifold $(M^3, \mathcal{T})$, consider Cooper and Rivin's sphere packing metric r and α-curvature $R_{\alpha,i} = K_i/r_i^2$. Then we have the following nine different discrete flows on M^3:

(1) $\dot{u}_i = -K_i$, where $u_i = \ln r_i$, see [10];
(2) $\dot{r}_i = (s_0 - K_i)r_i$, a normalization of the flow in (1), see [4];
(3) $\dot{r}_i = \lambda r_i - K_i$, with $\lambda = s_1$, see [8];
(4) $\dot{r}_i = s_\alpha r_i^\alpha - K_i$, $\alpha \in \mathbb{R}$, see [9];
(5) $\dot{u}_i = s_\alpha - R_{\alpha,i}$ (or $\dot{r}_i = s_\alpha r_i - \frac{K_i}{r_i^{\alpha-1}}$), where $u_i = \ln r_i$, $\alpha \in \mathbb{R}$, see [5];
(5)′ $\dot{u}_i = s_\alpha r_i^\alpha - K_i$, where $u_i = r_i^\alpha$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, differs from (5) by a constant α;
(6) $\dot{g}_i = (R_{av} - R_i)g_i$, where $g_i = r_i^2$, $R_{av} = s_2$, $R_i = K_i/r_i^2$, see [8];
(7) $\dot{u}_i = s_\alpha r_i^\alpha - K_i$, where $u_i = \ln r_i$, $\alpha \in \mathbb{R}$, see [6];
(7)′ $\dot{u}_i = s_\alpha r_i^\alpha - K_i$, where $u_i = \ln r_i^\alpha$, $\alpha \in \mathbb{R}$ and $\alpha \neq 0$, differs from (7) by a constant α.

It is remarkable that all the discrete flows presented in Examples 1 and 2 can be unified by the (α, β)-flow (2.2):

- For the cases in Example 1. If $\alpha = 0$ and $\beta = 1$, then the $(0, 1)$-flow is just the flow (1). If $\beta = \alpha + 1$, then the $(\alpha, \alpha + 1)$-flow is just the flow (2). If $\beta = 1$, then the $(\alpha, 1)$-flow is just the flow (5). Notice that the flow (3) differs from the $(\alpha, \alpha + 1)$-flow by a constant α. Moreover, the flow (4) is a special case of the flow (6) with $\alpha = 2$. The flow (6) differs from the $(\alpha, 1)$-flow by a constant α.

- For the cases in Example 2. If $\beta=\alpha=1$, then the $(1,1)$-flow is just the flow (2). If $\beta=\alpha$, then the (α,α)-flow is just the flow (3). If $\beta=1$, then the $(\alpha,1)$-flow is just the flow (4). If $\beta=\alpha+1$, then the $(\alpha,\alpha+1)$-flow is just the flow (7). Notice that the flow (1) is in fact a type of $(\alpha,\alpha+1)$-flow without normalization. The flow (5) differs from the $(\alpha,1)$-flow by a constant α. The flow (6) is a special case of the flow (5) with $\alpha=2$. The last flow $(7)'$ differs from the $(\alpha,\alpha+1)$-flow (7) by a constant α.

4. Basic properties of (α,β)-flow

Since the two most representative flows (2) and (4) in Example 1 are have intensively been studied in [8,9], we mainly study 3-dimensional unified (α,β)-flow in the remaining of this section. It's obviously to prove:

Proposition 4.1. *Let $\delta=\alpha-\beta+n-1$. If $\delta\neq 0$, then $\sum_{i=1}^N r_i^\delta(t)$ is invariant along the (α,β)-flow. If $\delta=0$, then $\prod_{i=1}^N r_i(t)$ is invariant along the flow.*

Theorem 4.2. *The critical points of the (α,β)-flow (2.2) is a constant α-curvature metric, and the solution $r(t)$ to this flow always exists locally. Moreover, if the solution $r(t)$ converges, then the combinatorial α-Yamabe problem is solvable, that is, there exists a constant α-curvature metric in the combinatorial conformal class.*

Proof. Note that, in $\mathfrak{M}_{\mathcal{T}}$, K_i as a function of $r=(r_1,\dots,r_N)^T$ is smooth and hence locally Lipschitz continuous. By Picard theorem in classical ODE theory, flow (2.2) has a unique solution $r(t)$, $t\in[0,\epsilon)$ for some $\epsilon>0$. The convergence of $r(t)$ means that there exists a metric $r^*\in\mathfrak{M}_{\mathcal{T}}$, such that $r(t)\to r^*$ according to the Euclidean topology. By the classical ODE theory, r^* should be the critical point of flow (2.2), which implies the conclusion above. □

It follows natural to know when the unified (α,β)-flow converges. The following lemma is very useful:

Lemma 4.3 ([2, 11, 14])**.** *Suppose $(M,\mathcal{T})$ is a triangulated 3-manifold with sphere packing metric r, $\mathcal{S}=\sum K_ir_i$ is the Einstein-Hilbert-Regge functional. Then we have*

$$\nabla_r\mathcal{S}=K. \tag{4.1}$$

If we set

$$\Lambda=Hess_r\mathcal{S}=\frac{\partial(K_1,\dots,K_N)}{\partial(r_1,\dots,r_N)}=\begin{pmatrix}\frac{\partial K_1}{\partial r_1} & \cdot & \cdot & \cdot & \frac{\partial K_1}{\partial r_N}\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \cdot & \cdot & \cdot & \cdot & \cdot\\ \frac{\partial K_N}{\partial r_1} & \cdot & \cdot & \cdot & \frac{\partial K_N}{\partial r_N}\end{pmatrix},$$

then Λ is positive semi-definite with rank $N-1$ and the kernel of Λ is the linear space spanned by the vector r.

We recall the definition of α-order combinatorial Laplacian.

Definition 4.4 ([8, 9])**.** Given a triangulated n-manifold $(M,\mathcal{T})$ with $n=2$ or 3. For any $\alpha\in\mathbb{R}$, the α-order combinatorial Laplacian ("α-Laplacian" for short) $\Delta_\alpha : C(\mathcal{T}_0)\to C(\mathcal{T}_0)$ is defined as

$$\Delta_\alpha f_i = \frac{1}{r_i^\alpha}\sum_{j\sim i}(-\frac{\partial K_i}{\partial r_j}r_j)(f_j-f_i) \tag{4.2}$$

for $f\in C(\mathcal{T}_0)$.

The α-Laplacian (4.2) can also be written in a matrix form

$$\Delta_\alpha = -\Sigma^{-\alpha}\Lambda\Sigma \tag{4.3}$$

with $\Delta_\alpha f = -\Sigma^{-\alpha}\Lambda\Sigma f$ for each $f\in C(\mathcal{T}_0)$, where $\Sigma = diag\{r_1,\ldots,r_N\}$.

Theorem 4.5. *Given a triangulated manifold $(M^3,\mathcal{T})$. Suppose $r^*\in\mathbb{S}^3$ is a constant α-curvature metric satisfying $\lambda_1(-\Delta_\alpha)>\alpha s_\alpha^*$, or more specifically, $r^*\in\mathbb{S}^3$ is a constant α-curvature metric with $\alpha s_\alpha^*\le 0$. If $\|r(0)-r^*\|$ is small enough, then the solution of the unified (α,β)-flow* (2.2) *exists for all time and converges to r^*.*

Proof. Denote $\Gamma_i(r) = s_\alpha r_i^\beta - K_i r_i^{\beta-\alpha}$, $1\le i\le N$, then the (α,β)-flow (2.2) can be written as $\dot r = \Gamma(r)$, which is an autonomous ODE system. Differentiating Γ with respect to r, we can get

$$D_r\Gamma(r) = \Sigma^{\beta-\alpha}\left(-\Lambda+\alpha s_\alpha\left(\Sigma^{\alpha-1}-\frac{r^\alpha(r^\alpha)^T}{\|r\|_{\alpha+1}^{\alpha+1}}\right)-H\right),$$

where

$$H = (\beta-\alpha)\Sigma^{-1}\begin{pmatrix} K_1 - s_\alpha r_1^\alpha & & \\ & \ddots & \\ & & K_N - s_\alpha r_N^\alpha\end{pmatrix} + \frac{r^\alpha\left(K-s_\alpha r^\alpha\right)^T}{\|r\|_{\alpha+1}^{\alpha+1}}.$$

At constant α-curvature metric points r^*, $H=0$, thus we have

$$D_r\Gamma|_{r^*} = \Sigma^{\beta-\alpha}\left(-\Lambda+\alpha s_\alpha\left(\Sigma^{\alpha-1}-\frac{r^\alpha(r^\alpha)^T}{\|r\|_{\alpha+1}^{\alpha+1}}\right)\right)_{r^*}. \tag{4.4}$$

We follow the tricks in [7, 9] to derive the conclusion. Denote $\widetilde{\Lambda} = \Sigma^{\frac{1-\alpha}{2}}\Lambda\Sigma^{\frac{1-\alpha}{2}}$. Then

$$-\Delta_\alpha = \Sigma^{-\alpha}\Lambda\Sigma = \Sigma^{-\frac{1+\alpha}{2}}\widetilde{\Lambda}\Sigma^{\frac{1+\alpha}{2}},$$

which implies that

$$\lambda_1(-\Delta_\alpha) = \lambda_1(\widetilde{\Lambda}).$$

Choose a matrix $P\in O(N)$ such that $P^T\widetilde{\Lambda}P = diag\{0,\lambda_1(\widetilde{\Lambda}),\ldots,\lambda_{N-1}(\widetilde{\Lambda})\}$. Suppose $P=(e_0,e_1,\ldots,e_{N-1})$, where e_i is the $(i+1)$-column of P. Then $\widetilde{\Lambda}e_0 = 0$ and $\widetilde{\Lambda}e_i = \lambda_i e_i$, $1\le i\le N-1$, which implies $e_0 = r^{\frac{\alpha+1}{2}}/\|r^{\frac{\alpha+1}{2}}\|$

and $r^{\frac{\alpha+1}{2}} \perp e_i$, $1 \le i \le N-1$. Hence $\left(I - \frac{r^{\frac{\alpha+1}{2}}(r^{\frac{\alpha+1}{2}})^T}{\|r\|_{\alpha+1}^{\alpha+1}}\right)e_0 = 0$ and $\left(I - \frac{r^{\frac{\alpha+1}{2}}(r^{\frac{\alpha+1}{2}})^T}{\|r\|_{\alpha+1}^{\alpha+1}}\right)e_i = e_i$, $1 \le i \le N-1$. Furthermore,

$$\begin{aligned}
& - D_r\Gamma|_{r^*} \\
&= \Sigma^{\beta-\alpha}\Sigma^{\frac{\alpha-1}{2}} P diag\left\{0, \lambda_1(\widetilde{\Lambda}) - \alpha s_\alpha^*, \ldots, \lambda_{N-1}(\widetilde{\Lambda}) - \alpha s_\alpha^*\right\} P^T \Sigma^{\frac{\alpha-1}{2}} \\
&= \Sigma^{\frac{\beta-\alpha}{2}}\Sigma^{\frac{\beta-1}{2}} P diag\left\{0, \lambda_1(\widetilde{\Lambda}) - \alpha s_\alpha^*, \ldots, \lambda_{N-1}(\widetilde{\Lambda}) - \alpha s_\alpha^*\right\} P^T \Sigma^{\frac{\beta-1}{2}}\Sigma^{-\frac{\beta-\alpha}{2}} \\
&\sim \left(\Sigma^{\frac{\beta-1}{2}}P\right) diag\left\{0, \lambda_1(\widetilde{\Lambda}) - \alpha s_\alpha^*, \ldots, \lambda_{N-1}(\widetilde{\Lambda}) - \alpha s_\alpha^*\right\} \left(\Sigma^{\frac{\beta-1}{2}}P\right)^T.
\end{aligned}$$

This shows that the eigenvalues of $D_r\Gamma|_{r^*}$ are all negative when restricted to the hypersurface $\sum_{i=1}^N r_i^\delta(t)$ when $\delta \ne 0$, or to the hypersurface $\prod_{i=1}^N r_i(t)$ when $\delta = 0$. Then the theorem is a consequence of the Lyapunov Stability Theorem in classical ODE theory. □

Remark 1. The α=0 case is of special interest. The unified $(0,\beta)$-flow always converges to a metric r^* with Cooper and Rivin's curvature $K_i \equiv$ constant, if $\|r(0) - r^*\|$ is small enough. Specifically, there exists $\epsilon > 0$ such that if the initial metric $r(0)$ satisfies $\|r(0) - r^*\| < \epsilon$, then the solution $r(t)$ to the unified $(0,\beta)$-flow exists for all time $t \ge 0$ and converges to r^* as $t \to +\infty$. This means that constant K-curvature metric r^* is locally stable. More specifically, it seems that the $(0,1)$-flow, which lies at the intersection of $(\alpha,1)$-flow and $(\alpha,\alpha+1)$-flow (see Figure 1), shows much better convergence properties than other types of unified (α,β)-flows. For more details, see [4–6].

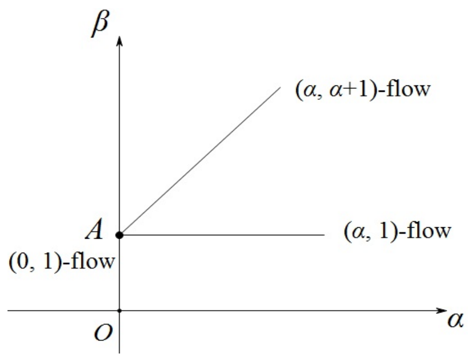

FIGURE 1. (α,β) flows

5. Calculating discrete constant α-curvature metrics

The (α,β)-flow provides an efficient way to find constant α-curvature metrics on triangulated manifolds. To see the power of this method, we take the 16-cell triangulation of $\mathbb{S}^3$ as an example, and see how the (α,β)-flow can be used in concrete calculations.

Consider a standard geometric triangulation $\mathcal{T}_s$ of $\mathbb{S}^3$. Let $A_1=(1,0,0,0)$, $A_2=(-1,0,0,0)$, $B_1=(0,1,0,0)$, $B_2=(0,-1,0,0)$, $C_1=(0,0,1,0)$, $C_2=(0,0,-1,0)$, $D_1=(0,0,0,1)$, $D_2=(0,0,0,-1)$ be the vertices of $\mathcal{T}_s$, $P_iQ_j(\{P,Q\}\in\{A,B,C,D\}, i,j=1,2)$ be the edges of $\mathcal{T}_s$, $P_iQ_jR_k$ ($\{P,Q,R\}\subset\{A,B,C,D\}, i,j,k=1,2)$ be the faces of $\mathcal{T}_s$, and the regular tetrahedrons $A_iB_jC_kD_l(i,j,k,l=1,2)$ be the tetrahedrons of $\mathcal{T}_s$.

We then consider a topological triangulation $\mathcal{T}$ of $\mathbb{S}^3$, which has the same combinatorial structure with $\mathcal{T}_s$. $\mathcal{T}$ is often called the 16-cell triangulation of $\mathbb{S}^3$ in previous literature. It is easy to see that $\mathcal{T}$ carries a trivial constant α-curvature metric for each α. In fact, let $r_i=1$, then r is exactly a constant α-curvature metric. We want to know whether there are other (up to scaling) constant α-curvature metrics? By evolving the unified (α,β)-flow (2.2) with appropriate initial value, we can easily get many constant α-curvature metrics (up to scaling). For example, when $\alpha=1$, we denote the metrics on A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2 by $r_1^+, r_1^-, r_2^+, r_2^-, r_3^+, r_3^-, r_4^+, r_4^-$ respectively. Then

$$r_1^{\pm}=r_2^{\pm}=\frac{1}{12},\ \ r_3^{\pm}=r_4^{\pm}=\frac{1}{6}$$

is exactly a constant 1-curvature metric. For this case,

$$\begin{aligned}K_1^{\pm}=K_2^{\pm}&=8\pi-16\arccos\frac{1}{\sqrt{10}},\\ K_3^{\pm}=K_4^{\pm}&=12\pi-8\arccos\frac{3}{5}-16\arccos\frac{1}{\sqrt{10}}.\end{aligned}$$

Furthermore, we have $\frac{K_i^{\pm}}{r_i^{\pm}}=12(8\pi-16\arccos\frac{1}{\sqrt{10}})\approx-61.8$ for each $1\le i\le 4$.

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