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BLOCH-TYPE SPACES ON THE UPPER HALF-PLANE

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Abstract. We define Bloch-type spaces of $\mathcal{C}^1(\mathbb{H})$ on the upper half plane $\mathbb{H}$ and characterize them in terms of weighted Lipschitz functions. We also discuss the boundedness of a composition operator C_ϕ acting between two Bloch spaces. These obtained results generalize the corresponding known ones to the setting of upper half plane.

1. Introduction

Let $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}z > 0\}$ be the upper half-plane, and $\mathcal{C}^1(\mathbb{H})$ be the set of all complex-valued functions having continuous partial derivatives on $\mathbb{H}$. For $\alpha > 0$, the α-Bloch space on $\mathbb{H}$, denoted by $\mathcal{B}^\alpha$, is defined to be the space of all functions $f \in \mathcal{C}^1(\mathbb{H})$ such that

$$\|f\|_\alpha = \sup_{z\in\mathbb{H}} (\mathrm{Im}z)^\alpha (|f_z(z)| + |f_{\overline{z}}(z)|) < \infty.$$

It is easy to check that the space $\mathcal{B}^\alpha$ is a Banach space with the norm

$$\|f\|_{\mathcal{B}^\alpha} = |f(i)| + \|f\|_\alpha.$$

Let $\omega : [0, +\infty) \to [0, +\infty)$ be an increasing function with $\omega(0) = 0$, we say that ω is a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$ (cf. [6]). Following [2], given a majorant ω and $\alpha > 0$, the *ω-α-Bloch space* $\mathcal{B}^\alpha_\omega$ consists of all functions $f \in \mathcal{C}^1(\mathbb{H})$ such that

$$\|f\|_{\omega,\alpha} = \sup_{z\in\mathbb{H}} \omega\big((\mathrm{Im}z)^\alpha\big)(|f_z(z)| + |f_{\overline{z}}(z)|) < \infty, \tag{1}$$

and the *little ω-α-Bloch space* $\mathcal{B}^\alpha_{\omega,0}$ consists of the functions $f \in \mathcal{B}^\alpha_\omega$ such that

$$\lim_{z\to\partial^\infty\mathbb{H}} \omega\big((\mathrm{Im}z)^\alpha\big)(|f_z(z)| + |f_{\overline{z}}(z)|) = 0, \tag{2}$$

where $\partial^\infty\mathbb{H}$ denotes the union of $\partial\mathbb{H}$ and $\{\infty\}$.

If we denote by $\widetilde{\mathcal{B}}^\alpha_\omega$ the set of all functions $f \in \mathcal{B}^\alpha_\omega$ such that $\omega\big((\mathrm{Im}z)^\alpha\big)(|f_z(z)| +|f_{\overline{z}}(z)|)$ vanishing at ∞, then the condition (2) is equivalent to the condition

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that the function $f \in \widetilde{\mathcal{B}}_{\omega}^{\alpha}$ satisfies

$$\lim_{Imz\to 0} \omega\big((\mathrm{Im}z)^{\alpha}\big)(|f_z(z)| + |f_{\overline{z}}(z)|) = 0. \tag{3}$$

In particular, when $\omega(t) = t$, we remark that the space $\mathcal{B}_{\omega}^{\alpha}$ is the *α-Bloch space* $\mathcal{B}^{\alpha}$.

For $0 < \alpha \leq 1$, the weighted Poincaré metric ds_{α} of $\mathbb{H}$, introduced in [1] is defined as

$$ds_{\alpha}^2 = \frac{|dz|^2}{(\mathrm{Im}z)^{2\alpha}}.$$

Suppose that $\gamma(t)(0 \leq t \leq 1)$ is a continuous and piecewise smooth curve in $\mathbb{H}$. Then the length of $\gamma(t)$ with respect to the weighted Poincaré metric ds_{α} is equal to

$$L_{p_{\alpha}}(\gamma) = \int_{\gamma} ds_{\alpha} = \int_0^1 \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^{\alpha}} dt.$$

Consequently, the associated distance between z and w in $\mathbb{H}$ is

$$p_{\alpha}(z, w) = \inf\{L_{p_{\alpha}}(\gamma) : \gamma(0) = z, \gamma(1) = w\},$$

where γ is a continuous and piecewise smooth curve in $\mathbb{H}$. Note that p_1 ($\alpha = 1$) is the classical Poincaré distance in $\mathbb{H}$.

Let $\mu, \nu \geq 0$ and f be a continuous function in $\mathbb{H}$. If there exists a constant C such that

$$(\mathrm{Im}z)^{\mu}(\mathrm{Im}w)^{\nu}|f(z) - f(w)| \leq C|z - w| \quad (\text{resp. } \leq Cp_{\alpha}(z, w))$$

for any $z, w \in \mathbb{H}$, then we say that f is a *weighted Euclidian* (*resp. hyperbolic*) *Lipschitz function* of indices (μ, ν). In particular, when $\mu = \nu = 0$, we say that f is a *Euclidian* (*resp. hyperbolic*) *Lipschitz function* (cf. [14]).

In the theory of function spaces, the relationship between Bloch spaces and (weighted) Lipschitz functions has attracted much attention. In 1986, Holland and Walsh ([8]) established a classical criterion for analytic Bloch space in the unit disc $\mathbb{D}$ in terms of weighted Euclidian Lipschitz functions of indices $(\frac{1}{2}, \frac{1}{2})$. Since then, a series of work has been carried out to characterize Bloch, α-Bloch, little α-Bloch and Besov spaces of holomorphic and harmonic functions along this line. For instance, Ren and Tu [15] extended Holland and Walsh's criterion to the Bloch space in the unit ball of $\mathbb{C}^n$, Li and Wulan [9], Zhao [18] characterized holomorphic α-Bloch space in terms of $(1 - |z|^2)^{\beta}(1 - |w|^2)^{\alpha-\beta}|f(z) - f(w)|/|z - w|$. In [19, 20], Zhu investigated the relationship between Bloch spaces and hyperbolic Lipschitz functions and proved that a holomorphic function belongs to Bloch space if and only if it is hyperbolic Lipschitz. For the related results of harmonic functions, we refer to [2, 3, 4, 7, 14] and the references therein.

Motivated by the known results mentioned above, we consider the corresponding problems in the setting of $\mathcal{C}^1(\mathbb{H})$ in this paper. In Section 2, we collect some known results that will be needed in the sequel. The main results and their proofs are presented in Sections 3 and 4.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \leq A \leq CB$.

2. Several lemmas

In this section, we introduce some notations and collect some preliminary results that we need later.

Let $z, w \in \mathbb{H}$, the *pseudo-hyperbolic distance* ρ is defined as

$$\rho(z,w) = \Big|\frac{z-w}{z-\overline{w}}\Big|.$$

It is easy to see that ρ on $\mathbb{H}$ is a distance function and horizontal translation and dilatation invariant (cf. [11]). For $z \in \mathbb{H}$ and $r \in (0,1)$, the *pseudo-hyperbolic ball* with center z and radius r is denoted by

$$E(z,r) = \{w \in \mathbb{H} : \rho(z,w) < r\}.$$

A straightforward calculation shows that $E(z,r)$ is a Euclidean ball $\mathbb{B}(x^*, r_0)$ where

$$x^* = (x, \frac{1+r^2}{1-r^2}y)\ ,\ r_0 = \frac{2ry}{1-r^2},\ \text{and}\ z = x + yi.$$

The following lemma is proved in [11, Lemma 2.1].

Lemma 2.1. *The inequality*

$$\frac{1-\rho(z,w)}{1+\rho(z,w)} \leq \Big|\frac{z-\overline{u}}{w-\overline{u}}\Big| \leq \frac{1+\rho(z,w)}{1-\rho(z,w)}$$

holds for all $z, w, u \in \mathbb{H}$.

As an application of Lemma 2.1, we easily get the following (see [4]).

Corollary 2.1. *Let* $r \in (0,1)$, $u, v \in E(z,r)$. *Then we have*

$$Im\ u \asymp Im\ v \asymp |z-\overline{u}| \asymp |E(z,r)|^{\frac{1}{2}},$$

where $|E(z,r)|$ *denotes the area of* $E(z,r)$.

We end this section with two inequalities which will be used in the sequel.

Lemma 2.2 ([2, Lemma 6]). *Let* $\omega(t)$ *be a majorant and* $u \in (0,1]$, $v \in (1,\infty)$. *Then for* $t \in (0,\infty)$,

$$\omega(ut) \geq u\omega(t), \quad \omega(vt) \leq v\omega(t).$$

Lemma 2.3. *Let* $a, b > 0$, $0 < s < 1$. *Then* $sa + (1-s)b \geq a^s b^{1-s}$.

3. Bloch spaces

Let f be a harmonic Bloch mapping in the unit disc $\mathbb{D}$. In [5], Colonna proved that the Bloch constant B_f of f is equals to its Bloch semi-norm, i.e.,

$$B_f = \sup_{z,w\in\mathbb{D}, z\neq w} \frac{|f(z)-f(w)|}{h(z,w)} = \sup_{z\in\mathbb{D}} (1-|z|^2)(|f_z(z)|+|f_{\overline{z}}(z)|),$$

where h is the hyperbolic distance in $\mathbb{D}$.

Firstly, we characterize the space $\mathcal{B}^\alpha$ in terms of hyperbolic Lipschitz functions and generalize Colonna's result to the setting of $\mathcal{C}^1(\mathbb{H})$.

Theorem 3.1. *Let $f \in \mathcal{C}^1(\mathbb{H})$ and $0 < \alpha \leq 1$. Then $f \in \mathcal{B}^\alpha$ if and only if there is a constant $C > 0$ such that*

$$|f(z)-f(w)| \leq Cp_\alpha(z,w), \qquad z,w \in \mathbb{H}.$$

Moreover, we have

$$\|f\|_\alpha = \sup_{z,w\in\mathbb{H}, z\neq w} \frac{|f(z)-f(w)|}{p_\alpha(z,w)}$$

for all $f \in \mathcal{B}^\alpha$.

Proof. We first prove the sufficiency. For any $z, w \in \mathbb{H}$, by the definition of $p_\alpha(z,w)$, we assume that $\gamma(s)$ is the geodesic between z and w (parametrized by arc-length) with respect to p_α. Since $p_\alpha(\gamma(0),\gamma(s)) = s$, we have

$$|f(z)-f(w)| \leq Cs.$$

Dividing both sides by s and then letting $s \to 0$ in the above inequality gives

$$(|f_z(z)|+|f_{\overline{z}}(z)|)|\gamma'(0)| \leq C.$$

From the minimal length property of geodesics,

$$p_\alpha(\gamma(0),\gamma(s)) = \int_0^s \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha} dt = s,\ \ 0 < s < \epsilon,$$

we obtain that

$$\lim_{s\to 0} \frac{1}{s} \int_0^s \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha} dt = \frac{|\gamma'(0)|}{(\mathrm{Im}z)^\alpha} = 1.$$

It follows that $(\mathrm{Im}z)^\alpha(|f_z(z)|+|f_{\overline{z}}(z)|) \leq C$ and hence $f \in \mathcal{B}^\alpha$ with

$$(\mathrm{Im}z)^\alpha(|f_z(z)|+|f_{\overline{z}}(z)|) : z \in \mathbb{H}\} \leq \sup\Big\{\frac{|f(z)-f(w)|}{p_\alpha(z,w)} : z \neq w\Big\}.$$

Conversely, we assume that $f \in \mathcal{B}^\alpha$. Let $z, w \in \mathbb{H}$ and $\gamma(t)(0 \leq t \leq 1)$ be a smooth curve from z to w. Then we have

$$\begin{aligned} |f(z)-f(w)| &= \Big|\int_0^1 \frac{df}{dt}(\gamma(t))dt\Big| \\ &\leq \int_0^1 (|f_z(\gamma(t))|+|f_{\overline{z}}(\gamma(t))|)|\gamma'(t)|dt \end{aligned}$$

$$\leq \|f\|_\alpha \int_0^1 \frac{|\gamma'(t)|}{[\mathrm{Im}\gamma(t)]^\alpha} dt$$
$$\leq \|f\|_\alpha p_\alpha(\gamma(t)).$$

Taking the infimum over all piecewise continuous curves connecting z and w, we conclude that

$$|f(z) - f(w)| \leq \|f\|_\alpha p_\alpha(z, w)$$

for all $z, w \in \mathbb{H}$. This completes the proof. □

In the following, we characterize the spaces $\mathcal{B}^\alpha_\omega$, $\mathcal{B}^\alpha_{\omega,0}$ in terms of Euclidean weighted Lipschitz functions.

Theorem 3.2. *Let $r \in (0, 1)$, $f \in \mathcal{C}^1(\mathbb{H})$, $0 < \beta \leq \alpha$. Then $f \in \mathcal{B}^\alpha_\omega$ if and only if*

$$K = \sup_{w \in E(z,r), z \neq w} \omega((\mathrm{Im}z)^\beta (\mathrm{Im}w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} < \infty.$$

Proof. We first prove the sufficiency. Let $f \in \mathcal{C}^1(\mathbb{H})$. For each $z \in \mathbb{H}$, we have

$$\sup_{w \in E(z,r), z \neq w} \frac{|f(z) - f(w)|}{|z - w|} \leq \frac{K}{\omega((\mathrm{Im}z)^\beta (\mathrm{Im}w)^{\alpha-\beta})}.$$

By letting $w \to z$, we obtain that

$$|f_z(z)| + |f_{\overline{z}}(z)| \leq \frac{K}{\omega((\mathrm{Im}z)^\alpha)},$$

from which we conclude that $f \in \mathcal{B}^\alpha_\omega$.

Conversely, let $f \in \mathcal{B}^\alpha_\omega$ and for any $w \in E(z, r), z \neq w$,

$$|f(z) - f(w)| = \Big| \int_0^1 \frac{df}{ds}(sz + (1 - s)w) ds \Big|$$
$$\leq |z - w| \int_0^1 \Big(\Big|\frac{\partial f}{\partial \zeta}(sz + (1 - s)w)\Big| + \Big|\frac{\partial f}{\partial \overline{\zeta}}(sz + (1 - s)w)\Big| \Big) ds$$
$$\leq C|z - w| \|f\|_{\omega,\alpha} \int_0^1 \frac{ds}{\omega((\mathrm{Im}(sz + (1 - s)w))^\alpha)}$$
$$\leq C|z - w| \int_0^1 \frac{ds}{\omega((\mathrm{Im}z)^{\alpha s} (\mathrm{Im}w)^{\alpha-\alpha s})},$$

where the last inequality follows from Lemma 2.3.

Since for each $w \in E(z, r), z \neq w$, $\mathrm{Im}z \asymp \mathrm{Im}w$, we can find a $\lambda \in (0, 1)$ such that $\mathrm{Im}z \geq \lambda(\mathrm{Im}w)$ and $\mathrm{Im}w \geq \lambda(\mathrm{Im}z)$. Then we infer that

$$\frac{|f(z) - f(w)|}{|z - w|} \leq C \int_0^1 \frac{ds}{\omega((\mathrm{Im}z)^{\alpha s} (\mathrm{Im}w)^{\alpha-\alpha s})}$$
$$\leq C \int_0^1 \frac{ds}{\omega((\mathrm{Im}z)^\alpha \lambda^{\alpha-\alpha s})}$$

$$\leq \frac{C}{\omega((\mathrm{Im}z)^{\alpha})}\int_0^1 \frac{ds}{\lambda^{\alpha-\alpha s}}$$
$$\leq \frac{C}{\omega((\mathrm{Im}z)^{\alpha})}.$$

Thus,

$$\sup_{w\in E(z,r),z\neq w} \omega((\mathrm{Im}z)^{\alpha})\frac{|f(z)-f(w)|}{|z-w|} < \infty.$$

By Lemma 2.2 again, we deduce that

$$\omega((\mathrm{Im}z)^{\alpha}) \geq \lambda^{\alpha-\beta}\omega(\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta}),$$

from which we see that $K < \infty$. The proof of Theorem 3.2 is completed. □

Theorem 3.3. *Let $r \in (0,1)$, $f \in \widetilde{\mathcal{B}}^{\alpha}_{\omega}$, $0 < \beta \leq \alpha$. Then $f \in \mathcal{B}^{\alpha}_{\omega,0}$ if and only if*

$$\lim_{\mathrm{Im}z\to 0}\sup_{w\in E(z,r),z\neq w} \omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f(z)-f(w)|}{|z-w|} = 0. \tag{4}$$

Proof. Sufficiency. Assume that (4) holds. Then for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup_{w\in E(z,r),z\neq w} \omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f(z)-f(w)|}{|z-w|} < \epsilon$$

whenever $0 < \mathrm{Im}z < \delta$. It follows by an argument similar to that in the proof of Theorem 3.2, we have

$$\omega((\mathrm{Im}z)^{\alpha})(|f_z(z)|+|f_{\overline{z}}(z)|) < C\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f(z)-f(w)|}{|z-w|} < C\epsilon,$$

whenever $0 < \mathrm{Im}z < \delta$, $w \in E(z,r)$. Hence

$$\lim_{Imz\to 0}\omega((\mathrm{Im}z)^{\alpha})(|f_z(z)|+|f_{\overline{z}}(z)|) = 0.$$

Necessity. Now we assume that $f \in \mathcal{B}^{\alpha}_{\omega,0}$. For $t \in (0,+\infty)$, let $f_t(z) = f(z+ti)$. By the proof of Theorem 3.2, for $w \in E(z,r)$, we have

$$\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|(f-f_t)(z)-(f-f_t)(w)|}{|z-w|} \leq C\|f-f_t\|_{\omega,\alpha}$$

and

$$\begin{aligned}
&\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f_t(z)-f_t(w)|}{|z-w|}\\
&= \frac{\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})}{\omega((\mathrm{Im}z+t)^{\beta}(\mathrm{Im}w+t)^{\alpha-\beta})}\frac{\omega((\mathrm{Im}z+t)^{\beta}(\mathrm{Im}w+t)^{\alpha-\beta})|f(z+ti)-f(w+ti)|}{|z-w|}\\
&\leq \frac{C\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})}{\omega((\mathrm{Im}z+t)^{\beta}(\mathrm{Im}w+t)^{\alpha-\beta})}\|f\|_{\omega,\alpha}.
\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}&\sup_{w\in E(z,r),z\neq w}\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f(z)-f(w)|}{|z-w|}\\&\leq C\|f-f_t\|_{\omega,\alpha}+\frac{C\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})}{\omega((\mathrm{Im}z+t)^{\beta}(\mathrm{Im}w+t)^{\alpha-\beta})}\|f\|_{\omega,\alpha}.\end{aligned}$$

In the above inequality, first by letting $\mathrm{Im}z\to 0$ and then letting $t\to 0$, we obtain the desired result. □

Remark 3.1. When $\omega(t)=t$, Li and Wulan [9] obtained the analogues of Theorems 3.2 and 3.3 for holomorphic Bloch space on the unit ball of $\mathbb{C}^n$.

In the following, we remove the restriction $w\in E(z,r)$ in Theorems 3.2 and 3.3 and obtain the following results which can be viewed as generalizations of [2, Theorems 3, 5] to the case of $\mathcal{C}^1(\mathbb{H})$.

Theorem 3.4. *Let $f\in\mathcal{C}^1(\mathbb{H})$, $0<\beta<1,\beta\leq\alpha<1+\beta$. Then $f\in\mathcal{B}^{\alpha}_{\omega}$ if and only if*

$$\sup_{z,w\in\mathbb{H},z\neq w}\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f(z)-f(w)|}{|z-w|}<\infty. \tag{5}$$

Proof. Assume that (5) holds. Fix $r\in(0,1)$ and $x\in\mathbb{H}$, it follows from the proof of Theorem 3.2, we can easily prove that $f\in\mathcal{B}^{\alpha}_{\omega}$. For the converse, we assume that $f\in\mathcal{B}^{\alpha}_{\omega}$. Then for $z,w\in\mathbb{H}$,

$$|f(z)-f(w)|\leq C|z-w|\int_0^1\frac{ds}{\omega((\mathrm{Im}(sz+(1-s)w))^{\alpha})}.$$

Since for $z,w\in\mathbb{H}$ and $s\in[0,1]$,

$$(s\mathrm{Im}z+(1-s)\mathrm{Im}w)^{\alpha}\geq(s\mathrm{Im}z)^{\beta}((1-s)\mathrm{Im}w)^{\alpha-\beta},$$

we get that

$$\begin{aligned}|f(z)-f(w)|&\leq C|z-w|\int_0^1\frac{ds}{\omega\big((s\mathrm{Im}z+(1-s)\mathrm{Im}w)^{\alpha}\big)}\\&\leq C|z-w|\int_0^1\frac{ds}{\omega(s^{\beta}(1-s)^{\alpha-\beta}(\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})}\\&\leq\frac{C|z-w|}{\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})}\int_0^1\frac{ds}{s^{\beta}(1-s)^{\alpha-\beta}}\\&\leq\frac{C|z-w|}{\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})},\end{aligned}$$

where the last integral converges since $\alpha<1+\beta$. Thus

$$\sup_{z,w\in\mathbb{H},z\neq w}\omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta})\frac{|f(z)-f(w)|}{|z-w|}<\infty.$$

This completes the proof of Theorem 3.4. □

Similarly, we can prove the following.

Theorem 3.5. *Let* $f \in \widetilde{\mathcal{B}}^{\alpha}_{\omega}$, $0 < \beta < 1$, $\beta \leq \alpha < 1 + \beta$. *Then* $f \in \mathcal{B}^{\alpha}_{\omega,0}$ *if and only if*

$$\lim_{\mathrm{Im}z \to 0} \sup_{z,w \in \mathbb{H}, z \neq w} \omega((\mathrm{Im}z)^{\beta}(\mathrm{Im}w)^{\alpha-\beta}) \frac{|f(z) - f(w)|}{|z - w|} = 0.$$

4. Composition operators

Let ϕ be a holomorphic self-mapping of $\mathbb{H}$. The composition operator C_{ϕ}, induced by ϕ is defined by $C_{\phi}(f) = f \circ \phi$ for $f \in \mathcal{C}^1(\mathbb{H})$. During the past few years, composition operators have been studied extensively on spaces of holomorphic functions on various domains in $\mathbb{C}$ and $\mathbb{C}^n$, see e.g., [13, 10, 16, 21]. In this section, we discuss the boundedness of composition operators between Bloch spaces on the upper half plane $\mathbb{H}$.

Theorem 4.1. *Let* $\alpha, \beta > 0$ *and* ϕ *be a holomorphic self-mapping of* $\mathbb{H}$. *Then* $C_{\phi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ *is bounded if and only if*

$$\sup_{z \in \mathbb{H}} \frac{(Imz)^{\beta}|\phi'(z)|}{(Im\phi(z))^{\alpha}} < \infty. \tag{6}$$

Proof. First suppose that

$$M = \sup_{z \in \mathbb{H}} \frac{(\mathrm{Im}z)^{\beta}|\phi'(z)|}{(\mathrm{Im}\phi(z))^{\alpha}} < \infty.$$

For $f \in \mathcal{B}^{\alpha}$ and $z \in \mathbb{H}$, we have

$$\begin{aligned}(\mathrm{Im}z)^{\beta}(|C_{\phi}(f)_z| + |C_{\phi}(f)_{\overline{z}}|) &= (\mathrm{Im}z)^{\beta}(|(f \circ \phi)_z(z)| + |(f \circ \phi)_{\overline{z}}(z)|) \\ &= (\mathrm{Im}z)^{\beta}|\phi'(z)|(|f_z(\phi(z))| + |f_{\overline{z}}(\phi(z))|) \\ &\leq M(\mathrm{Im}\phi(z))^{\alpha}|(|f_z(\phi(z))| + |f_{\overline{z}}(\phi(z))|) \\ &\leq C\|f\|_{\alpha}\end{aligned}$$

and

$$|f(\phi(i))| \leq C\|f\|_{\alpha}.$$

Hence $C_{\phi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is bounded.

For the converse, assume that $C_{\phi} : \mathcal{B}^{\alpha} \to \mathcal{B}^{\beta}$ is a bounded operator with

$$\|C_{\phi}(f)\|_{\beta} \leq C\|f\|_{\alpha}$$

for all $f \in \mathcal{B}^{\alpha}$. Fix a point $z_0 \in \mathbb{H}$ and let $w = \phi(z_0)$. If $\alpha \neq 1$, consider the function $f_w(z) = (z - \overline{w})^{1-\alpha}$. Then it is easy to check that $f_w \in \mathcal{B}^{\alpha}$. The boundedness of C_{ϕ} implies that

$$\frac{(\mathrm{Im}z)^{\beta}|\phi'(z))|}{|\phi(z) - \overline{w}|^{\alpha}} \leq C.$$

In particular, take $z = z_0$, we get

$$\frac{(\mathrm{Im}z_0)^\beta |\phi'(z_0)|}{(\mathrm{Im}\phi(z_0))^\alpha} \leq C.$$

Since z_0 is arbitrary, the result follows.

If $\alpha = 1$, we only need to consider the function $f_w(z) = \ln(z - \overline{w})$. Following a discussion similar to the above, it can be proved that (6) holds. The proof of Theorem 4.1 is completed. □

Recall that the classical Schwarz-Pick Lemma in the upper half-plane gives that for a holomorphic self-mapping ϕ of $\mathbb{H}$, $(\mathrm{Im}z)|\phi'(z)| \leq \mathrm{Im}\phi(z)$ holds for all $z \in \mathbb{H}$. As an application of this result, it is easy to derive the following corollary.

Corollary 4.1. *Let ϕ be a holomorphic self-mapping of $\mathbb{H}$. Then $C_\phi : \mathcal{B}^1 \to \mathcal{B}^1$ is bounded.*

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