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INJECTIVE DIMENSIONS OF LOCAL COHOMOLOGY MODULES

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Abstract. Assume that R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R, X is an R–module, and t is a non-negative integer. In this paper, we present upper bounds for the injective dimension of X in terms of the injective dimensions of its local cohomology modules and an upper bound for the injective dimension of $\mathrm{H}^t_{\mathfrak{a}}(X)$ in terms of the injective dimensions of the modules $\mathrm{H}^i_{\mathfrak{a}}(X)$, $i \neq t$, and that of X. As a consequence, we observe that R is Gorenstein whenever $\mathrm{H}^i_{\mathfrak{a}}(R)$ is of finite injective dimension for all i.

1. Introduction

Let R be a commutative Noetherian ring with non-zero identity. We use symbols $\mathfrak{a}$, M, X, and t as an ideal of R, a finite (i.e., finitely generated) R–module, an arbitrary R–module which is not necessarily finite, and a non-negative integer. We denote $\mathrm{id}_R(X)$ and $\mathrm{fd}_R(X)$ as the injective dimension and the flat dimension of X, respectively. We also write $\mathrm{H}^i_{\mathfrak{a}}(X)$ for the ith local cohomology module of X with respect to $\mathfrak{a}$. For basic results, notations, and terminologies not given in this paper, the reader is referred to [1], [2], and [7].

The main ideas of this paper come from the article [8] in which it is shown that there are some inequalities between the flat dimensions of a module and its local cohomology modules. Although one may expect some consistency for the injective dimensions, the similarities are far from obvious. For the local case, in [8, Corollary 4.1], it is shown that

$$\mathrm{fd}_R(M) \leq \sup\{\mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) - i : i \geq 0\}$$

while we show, in Theorem 2.4, that

$$\mathrm{id}_R(M) \leq \sup\ \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) + i : i \geq 0\}.$$

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Also, it is proven in [8, Corollary 4.3] that

$$\begin{aligned}\mathrm{fd}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq\ & \sup\ \{\mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + t - i + 1 : i < t\}\\ & \cup \{\mathrm{fd}_R(X) + t\} \cup \{\mathrm{fd}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + t - i - 1 : i > t\}.\end{aligned}$$

We prove, in Theorem 2.8, that

$$\begin{aligned}\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq\ & \sup\ \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i - 1 : i < t\}\\ & \cup \{\mathrm{id}_R(X) - t\} \cup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i + 1 : i > t\}.\end{aligned}$$

As applications, we show that R is Gorenstein whenever $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(R)) < \infty$ for all i, and we find that if $\mathrm{H}^i_{\mathfrak{a}}(M)$ is injective for all i, then $\mathrm{cd}_{R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = \dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Supp}_R(M)$. Recall that cohomological dimension of X with respect to $\mathfrak{a}$, denoted by $\mathrm{cd}_R(\mathfrak{a}, X)$, is the largest integer i in which $\mathrm{H}^i_{\mathfrak{a}}(X)$ is not zero [4]. We also prove that if $\mathrm{H}^i_{\mathfrak{a}}(X)$ is injective (resp. $\mathrm{H}^i_{\mathfrak{a}}(X) = 0$) for all $i \neq t$, then $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \mathrm{id}_R(X) - t + 1$ (resp. $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \mathrm{id}_R(X) - t$). When R is local and $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) < \infty$ (resp. $\mathrm{H}^i_{\mathfrak{a}}(M) = 0$) for all $i \neq t$, we observe that $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) < \infty$ if and only if $\mathrm{id}_R(M) < \infty$ (resp. $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) = \mathrm{id}_R(M) - t$).

2. Main results

Let Y be an arbitrary R–module. We denote

$$\mathrm{H}^i_{\mathfrak{a}}(Y, X) = \varinjlim_{n\in\mathbb{N}} \mathrm{Ext}^i_R(Y/\mathfrak{a}^n Y, X),$$

as the ith generalized local cohomology module of Y and X with respect to $\mathfrak{a}$ [5]. We have the isomorphisms $\mathrm{H}^i_{\mathfrak{a}}(M, X) \cong \mathrm{H}^i(\mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(E^{\bullet X})))$, where $E^{\bullet X}$ is a deleted injective resolution of X [3, Lemma 2.1(i)]. Thus $\Gamma_{\mathfrak{a}}(M, X) \cong \mathrm{Hom}_R(M, \Gamma_{\mathfrak{a}}(X))$ and if $\mathrm{Supp}_R(M) \cap \mathrm{Supp}_R(X) \subseteq \mathrm{Var}(\mathfrak{a})$, then $\mathrm{H}^i_{\mathfrak{a}}(M, X) \cong \mathrm{Ext}^i_R(M, X)$. Here, we denote $\mathrm{Var}(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathfrak{a} \subseteq \mathfrak{p}\}$.

Lemma 2.1. *Suppose that X is an arbitrary R–module and that N is a finite R–module. Assume also that $\overline{X} = X/\Gamma_{\mathfrak{a}}(X)$ and $Q = E_R(\overline{X})/\overline{X}$, where $E_R(\overline{X})$ is an injective hull of $\overline{X}$. Then the following statements hold true.*

(i) $\mathrm{H}^i_{\mathfrak{a}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(X)$ *for all* $i \geq 0$.
(ii) $\mathrm{H}^i_{\mathfrak{a}}(N, Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(N, \overline{X})$ *for all* $i \geq 0$.

Proof. Since $\Gamma_{\mathfrak{a}}(\overline{X}) = 0$, $\Gamma_{\mathfrak{a}}(E_R(\overline{X})) = 0$ and so

$$\Gamma_{\mathfrak{a}}(N, E_R(\overline{X})) \cong \mathrm{Hom}_R(N, \Gamma_{\mathfrak{a}}(E_R(\overline{X}))) = 0.$$

Applying the derived functors of $\Gamma_{\mathfrak{a}}(-)$ and $\Gamma_{\mathfrak{a}}(N, -)$ to the short exact sequence

$$0 \longrightarrow \overline{X} \longrightarrow E_R(\overline{X}) \longrightarrow Q \longrightarrow 0,$$

we obtain the isomorphisms

$$\mathrm{H}^i_{\mathfrak{a}}(Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(\overline{X}) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(X)$$

and
$$\mathrm{H}^i_{\mathfrak{a}}(N,Q) \cong \mathrm{H}^{i+1}_{\mathfrak{a}}(N,\overline{X})$$
for all $i \geq 0$. □

The following lemma is crucial to prove the main results of this paper.

Lemma 2.2. *Let s be a non-negative integer, X an arbitrary R–module such that*
$$\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + i : i \geq 0\} < s,$$
and N a finite R–module. Then $\mathrm{H}^s_{\mathfrak{a}}(N,X) = 0$.

Proof. We prove by using induction on s. The case $s = 0$ is clear. Suppose that $s > 0$ and that $s-1$ is settled. Let $\overline{X} = X/\Gamma_{\mathfrak{a}}(X)$ and $Q = E_R(\overline{X})/\overline{X}$. Since $\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + i : i \geq 0\} < s$, $\mathrm{id}_R(\Gamma_{\mathfrak{a}}(X)) < s$ and $\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + i : i \geq 1\} < s$. Thus $\mathrm{H}^s_{\mathfrak{a}}(N,\Gamma_{\mathfrak{a}}(X)) = 0$ and $\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(Q)) + i : i \geq 0\} < s-1$ from Lemma 2.1(i). Hence, by the induction hypothesis on Q, we have $\mathrm{H}^{s-1}_{\mathfrak{a}}(N,Q) = 0$ and so, by Lemma 2.1(ii), $\mathrm{H}^s_{\mathfrak{a}}(N,\overline{X}) = 0$. Now, by the short exact sequence
$$0 \longrightarrow \Gamma_{\mathfrak{a}}(X) \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0,$$
we get the long exact sequence
$$\cdots \longrightarrow \mathrm{H}^s_{\mathfrak{a}}(N,\Gamma_{\mathfrak{a}}(X)) \longrightarrow \mathrm{H}^s_{\mathfrak{a}}(N,X) \longrightarrow \mathrm{H}^s_{\mathfrak{a}}(N,\overline{X}) \longrightarrow \cdots$$
which shows that $\mathrm{H}^s_{\mathfrak{a}}(N,X) = 0$. □

For an R–module X, we say that $\mathrm{id}_R(\mathfrak{a},X) \leq n$ if $\mathrm{Ext}^s_R(R/\mathfrak{p},X) = 0$ for all $\mathfrak{p} \in \mathrm{Var}(\mathfrak{a})$ and all $s > n$. We call $\mathrm{id}_R(\mathfrak{a},X)$ the $\mathfrak{a}$–relative injective dimension of X [6].

Corollary 2.3. *Let X be an arbitrary R–module and $\mathfrak{b}$ an ideal of R such that* $\mathrm{Var}(\mathfrak{b}) \subseteq \mathrm{Var}(\mathfrak{a})$. *Then*
$$\mathrm{id}_R(\mathfrak{b},X) \leq \sup\ \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + i : i \geq 0\}.$$
In particular,
$$\mathrm{id}_R(\mathfrak{a},X) \leq \sup\ \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + i : i \geq 0\}.$$

Proof. Assume that the right-hand side of the inequality is a finite number, say n. Assume also that $\mathfrak{p} \in \mathrm{Var}(\mathfrak{b})$ and $s > n$. Take $N = R/\mathfrak{p}$ in Lemma 2.2 to get $\mathrm{H}^s_{\mathfrak{a}}(R/\mathfrak{p},X) = 0$. The assertion follows because we have the isomorphism $\mathrm{Ext}^s_R(R/\mathfrak{p},X) \cong \mathrm{H}^s_{\mathfrak{a}}(R/\mathfrak{p},X)$. □

The following theorem is the first main result of this paper and, among other things, shows that, in the local case, if $\mathrm{id}_R(M) = \infty$, then for every ideal $\mathfrak{a}$ of R there exists an integer $n_{\mathfrak{a}}$ such that $\mathrm{id}_R(\mathrm{H}^{n_{\mathfrak{a}}}_{\mathfrak{a}}(M)) = \infty$.

Theorem 2.4. *Let $(R,\mathfrak{m})$ be a local ring and M a finite R–module. Then*
$$\mathrm{id}_R(M) \leq \sup\ \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) + i : i \geq 0\}.$$
In particular, if $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) < \infty$ *for all i, then* $\mathrm{id}_R(M) < \infty$.

Proof. Since $\mathrm{id}_R(M) = \mathrm{id}_R(\mathfrak{m}, M)$, the assertion follows from Corollary 2.3. For the last part, note that $\mathrm{cd}_R(\mathfrak{a}, M)$ is finite from [1, Theorem 3.3.1 or Theorem 6.1.2] and we have

$$\sup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) + i : i \geq 0\} = \sup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) + i : 0 \leq i \leq \mathrm{cd}_R(\mathfrak{a}, M)\}$$

because for all $i > \mathrm{cd}_R(\mathfrak{a}, M)$, $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) + i = -\infty + i = -\infty$ (see [1, Conventions 11.1.8]). □

It is well known that if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring and

$$\mathrm{H}^{\dim(R)}_{\mathfrak{m}}(R) \cong E_R(R/\mathfrak{m}),$$

then R is Gorenstein. That means R is Gorenstein whenever $\mathrm{H}^{\dim(R)}_{\mathfrak{m}}(R) \cong E_R(R/\mathfrak{m})$ and $\mathrm{H}^i_{\mathfrak{m}}(R) = 0$ for all $i \neq \dim(R)$. In the following corollary, we prove that R is Gorenstein if $\mathrm{H}^i_{\mathfrak{a}}(R)$ has finite injective dimension for all i, where R is not necessarily local and $\mathfrak{a}$ is an arbitrary ideal of R.

Corollary 2.5. *Suppose that* $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(R)) < \infty$ *for all* i. *Then* R *is Gorenstein.*

Proof. Let $\mathfrak{p}$ be a prime ideal of R. By assumption, we have $\mathrm{id}_{R_{\mathfrak{p}}}(\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(R_{\mathfrak{p}})) < \infty$ for all i. Thus we get $\mathrm{id}_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}) < \infty$ from Theorem 2.4. Therefore $R_{\mathfrak{p}}$ is a Gorenstein local ring. Hence R is Gorenstein, as we desired. □

Corollary 2.6. *Let* M *be a finite* R*–module such that* $\mathrm{H}^i_{\mathfrak{a}}(M)$ *is injective for all* i. *Then*

$$\mathrm{cd}_{R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) = \dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}})$$

for all $\mathfrak{p} \in \mathrm{Supp}_R(M)$.

Proof. Let $\mathfrak{p} \in \mathrm{Supp}_R(M)$. For all i, $\mathrm{H}^i_{\mathfrak{a}R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ is injective from assumption. Hence, by Theorem 2.4, we get $\mathrm{id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{cd}_{R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}})$. Thus $\mathrm{id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mathrm{depth}(R_{\mathfrak{p}})$ and

$$\mathrm{id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{cd}_{R_{\mathfrak{p}}}(\mathfrak{a}R_{\mathfrak{p}}, M_{\mathfrak{p}}) \leq \dim_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) \leq \mathrm{id}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}).$$

The assertion follows. □

We need the following lemma to prove the second main result of this paper.

Lemma 2.7. *Let* s, t *be non-negative integers,* X *an arbitrary* R*–module such that*

$$\begin{aligned} &\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i - 1 : i < t\} \\ &\cup \{\mathrm{id}_R(X) - t\} \cup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i + 1 : i > t\} < s, \end{aligned}$$

and N *a finite* R*–module. Then* $\mathrm{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)) = 0$.

Proof. Let $\overline{X} = X/\Gamma_{\mathfrak{a}}(X)$ and $Q = E_R(\overline{X})/\overline{X}$. We prove by using induction on t. Let $t = 0$. Since $\mathrm{id}_R(X) < s$ and $\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) + i + 1 : i > 0\} < s$, $\mathrm{H}^s_{\mathfrak{a}}(N, X) = 0$ and $\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(\overline{X})) + i : i \geq 0\} < s - 1$. Thus $\mathrm{H}^{s-1}_{\mathfrak{a}}(N, \overline{X}) = 0$ from Lemma 2.2. Hence by the long exact sequence

$$\cdots \longrightarrow \mathrm{H}^{s-1}_{\mathfrak{a}}(N, \overline{X}) \longrightarrow \mathrm{H}^s_{\mathfrak{a}}(N, \Gamma_{\mathfrak{a}}(X)) \longrightarrow \mathrm{H}^s_{\mathfrak{a}}(N, X) \longrightarrow \cdots,$$

we get $\mathrm{Ext}^s_R(N, \Gamma_{\mathfrak{a}}(X)) \cong \mathrm{H}^s_{\mathfrak{a}}(N, \Gamma_{\mathfrak{a}}(X)) = 0$.

Suppose that $t > 0$ and that $t - 1$ is settled. From the exact sequences

$$0 \longrightarrow \Gamma_{\mathfrak{a}}(X) \longrightarrow X \longrightarrow \overline{X} \longrightarrow 0$$

and

$$0 \longrightarrow \overline{X} \longrightarrow E_R(\overline{X}) \longrightarrow Q \longrightarrow 0,$$

we get $\mathrm{id}_R(Q) < s + t - 1$ because we have $\mathrm{id}_R(\Gamma_{\mathfrak{a}}(X)) < s + t + 1$ and $\mathrm{id}_R(X) < s + t$ by assumptions. Thus

$$\begin{aligned}&\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i - 1 : 0 < i < t\}\\ &\cup \{\mathrm{id}_R(Q) - (t-1)\} \cup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i + 1 : i > t\} < s\end{aligned}$$

and so by Lemma 2.1(i),

$$\begin{aligned}&\sup\{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(Q)) - (t-1) + i - 1 : i < t - 1\}\\ &\cup \{\mathrm{id}_R(Q) - (t-1)\} \cup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(Q)) - (t-1) + i + 1 : i > t - 1\} < s.\end{aligned}$$

Now, from the induction hypothesis on Q, we have $\mathrm{Ext}^s_R(N, \mathrm{H}^{t-1}_{\mathfrak{a}}(Q)) = 0$. Therefore, again by Lemma 2.1(i), $\mathrm{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)) = 0$. □

Now, we can state our second main result.

Theorem 2.8. *Let X be an arbitrary R–module and t a non-negative integer. Then*

$$\begin{aligned}\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq\ &\sup\ \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i - 1 : i < t\}\\ &\cup \{\mathrm{id}_R(X) - t\} \cup \{\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) - t + i + 1 : i > t\}.\end{aligned}$$

Proof. Assume that the right-hand side of the inequality is a finite number, say n. Assume also that $s > n$ and N is a finite R–module. By Lemma 2.7, $\mathrm{Ext}^s_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)) = 0$. Thus the assertion follows. □

The following results are immediate applications of the above theorems.

Corollary 2.9. *Let R be a local ring, M a finite R–module, and t a non-negative integer such that $\mathrm{id}_R(\mathrm{H}^i_{\mathfrak{a}}(M)) < \infty$ for all $i \neq t$. Then $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(M)) < \infty$ if and only if $\mathrm{id}_R(M) < \infty$.*

Proof. This follows from Theorems 2.4 and 2.8. □

Corollary 2.10. *Let X be an arbitrary R–module and t a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is injective for all $i \neq t$. Then*

$$\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \mathrm{id}_R(X) - t + 1.$$

Proof. By Theorem 2.8, we have $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \sup\{\mathrm{id}_R(X) - t, \mathrm{cd}_R(\mathfrak{a}, X) - t + 1\}$. Thus the assertion follows because $\mathrm{cd}_R(\mathfrak{a}, X) \leq \mathrm{id}_R(X)$. □

Corollary 2.11. *Let R be a local ring, M a finite R–module, and t a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(M)$ is injective for all $i < \mathrm{cd}_R(\mathfrak{a}, M)$. Then*

$$\mathrm{id}_R(M) - \mathrm{cd}_R(\mathfrak{a}, M) \leq \mathrm{id}_R(\mathrm{H}^{\mathrm{cd}_R(\mathfrak{a},M)}_{\mathfrak{a}}(M)) \leq \mathrm{id}_R(M) - \mathrm{cd}_R(\mathfrak{a}, M) + 1.$$

Proof. It follows from Theorem 2.4 and Corollary 2.10. □

Let R be a local ring and t a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(X) = 0$ for all $i \neq t$. Zargar and Zakeri, in [9, Theorem 2.5], proved that if $\mathrm{id}_R(X) < \infty$, then $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) < \infty$. In the first part of the following corollary, we generalize this result for a general ring R as an application of Theorem 2.8.

Corollary 2.12. *Let X be an arbitrary R–module and t a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(X) = 0$ for all $i \neq t$. Then the following statements hold true*:

(i) $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) \leq \mathrm{id}_R(X) - t$.

(ii) $\mathrm{id}_R(\mathrm{H}^t_{\mathfrak{a}}(X)) = \mathrm{id}_R(X) - t$ *whenever R is local and X is finite.*

Proof. (i) This follows from Theorem 2.8.

(ii) It follows from Theorem 2.4 and the first part. □

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References

[1] M. P. Brodmann and R. Y. Sharp, *Local Cohomology: An Algebraic Introduction with Geometric Applications*, Cambridge, Cambridge University Press, 1998.

[2] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge, Cambridge University Press, 1998.

[3] K. Divaani-Aazar, R. Sazeedeh, and M. Tousi, *On vanishing of generalized local cohomology modules*, Algebra Colloq. **12** (2005), no. 2, 213–218.

[4] R. Hartshorne, *Cohomological dimension of algebraic varieties*, Ann. of Math. (2) **88** (1968), 403–450.

[5] J. Herzog, *Komplexe, auflösungen und dualität in der lokalen algebra*, Habilitationsschrift, Universitat Regensburg, 1970.

[6] A. R. Naghipour, *Some results on cofinite modules*, Int. Electron. J. Algebra **11** (2012), 82–95.

[7] J. J. Rotman, *An Introduction to Homological Algebra*, New York, Springer-Verlag, 2009.

[8] A. Vahidi, *Betti numbers and flat dimensions of local cohomology modules*, Canad. Math. Bull. **58** (2015), no. 3, 664–672.

[9] M. R. Zargar and H. Zakeri, *On injective and Gorenstein injective dimensions of local cohomology modules*, Algebra Colloq. **22** (2015), no. 1, 935–946.

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