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# ON MULTISECANT PLANES OF LOCALLY NON-COHEN-MACAULAY SURFACES

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Abstract. For a nondegenerate projective irreducible variety $X \subset \mathbb{P}^r$, it is a natural problem to find an upper bound for the value of

$$\ell_\beta(X) = \max\{\mathrm{length}(X \cap L) \mid L = \mathbb{P}^\beta \subset \mathbb{P}^r,\ \dim\ (X \cap L) = 0\}$$

for each $1 \le \beta \le e$. When $X$ is locally Cohen-Macaulay, A. Noma in [10] proves that $\ell_\beta(X)$ is at most $d - e + \beta$ where $d$ and $e$ are respectively the degree and the codimension of $X$. In this paper, we construct some surfaces $S \subset \mathbb{P}^5$ of degree $d \in \{7, \ldots, 12\}$ which satisfies the inequality

$$\ell_2(S) \ge d - 3 + \lfloor \frac{d}{2} \rfloor.$$

This shows that Noma's bound is no more valid for locally non-Cohen-Macaulay varieties.

## 1. Introduction

Throughout this paper, we work over an algebraically closed field $\Bbbk$ of arbitrary characteristic.

Let $X \subset \mathbb{P}^r$ be an $n$-dimensional nondegenerate projective irreducible variety of degree $d$ and codimension $e = r - n$. A linear subspace $L = \mathbb{P}^\beta \subset \mathbb{P}^r$ is said to be *$k$-secant to $X$* when the integer

$$\mathrm{length}(X \cap L) := \dim_{\Bbbk}(\mathcal{O}_{\mathbb{P}^r}/\mathcal{I}_X + \mathcal{I}_L)$$

is bigger than or equal to $k$. For each $1 \le \beta \le e$, we define $\ell_\beta(X)$ by

$$\ell_\beta(X) = \max\{\mathrm{length}(X \cap L) \mid L = \mathbb{P}^\beta \subset \mathbb{P}^r,\ \dim\ (X \cap L) = 0\}.$$

It is a natural problem in projective algebraic geometry to find upper bounds for the values of $\ell_\beta(X)$ in terms of basic invariants of $X$. To put things in perspective, we would like to provide a historical review about the integers $\ell_1(X), \ldots, \ell_e(X)$.

**1.1. The case of $\ell_e(X)$:** It is an elementary fact that $\ell_e(X)$ is at least $d$. Moreover, the equality $\ell_e(X) = d$ occurs if and only if $X$ is locally Cohen-Macaulay. In consequence, if $X$ is not locally Cohen-Macaulay, then there

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should be an $e$-dimensional subspace $L$ such that dim $(X \cap L) = 0$ and $\ell_e(X) > d$ (cf. Exercise 18.17 in [6]). In [10, Example 1.2], the author constructs such an example explicitly. His example is a locally non-Cohen-Macaulay surface $S$ in $\mathbb{P}^4$ of degree 4 which admits a 5-secant plane.

**1.2. The case of $\ell_1(X)$:** We say that $X$ is $m$-regular if

$$H^i(\mathbb{P}^r, \mathcal{I}_X(j)) = 0 \quad \text{for all} \quad j \geq m - i.$$

The *regularity of* $X$, denoted by reg$(X)$, is defined to be the least $m$ such that $X$ is $m$-regular. The interest in this concept stems partly from the fact that if $X$ is $m$-regular, then it is cut out ideal-theoretically by forms of degree at most $m$. This algebraic property of the $m$-regularity has the elementary geometric consequence that

$$\ell_1(X) \leq \text{reg}(X). \tag{1.1}$$

The famous Eisenbud-Goto's Regularity Conjecture addresses that

**Eisenbud-Goto's Regularity Conjecture:** reg$(X) \leq d - e + 1$.

By the inequality (1.1), this conjecture implies the following

**Multisecant Line Conjecture:** $\ell_1(X) \leq d - e + 1$.

The first conjecture is known to be true for curves by [7] and for smooth complex surfaces by [11] and [9]. The second conjecture is shown by A. Noma in [10] when $X$ is locally Cohen-Macaulay. In the classical paper [7], the authors also classify the boundary case for curves. Namely, reg$(X) = d - r + 2$ if and only if either $d \leq r+1$ or else $X$ is a smooth rational curve such that $\ell_1(X) = d - r + 2$. This beautiful result has been generalized to several directions. In [1], A. Bertin shows that if $X$ is a smooth variety and $\ell_1(X) \geq d - e + 1$, then $X$ is a rational scroll and $\ell_1(X) = \text{reg}(X) = d - e + 1$.

**1.3. The case of $\ell_\beta(X)$ for $1 < \beta < e$:** A. Bertin's view point leads the mathematicians to study the upper bound on $\ell_\beta(X)$ for all $1 < \beta < e$. In [8], S. Kwak proves the inequality

$$\text{length}(X \cap L) \leq d - e + \beta$$

when $X$ is smooth and $L = \mathbb{P}^\beta$ is a *curvilinear* multi-secant space to $X$ in the sense that $X \cap L$ lies on a smooth curve. Then he classifies all varieties having a $k$-secant curvilinear subspace of dimension $\beta$ for the extremal case where $k = d - e + \beta$ and next to the extremal case where $k = d - e + \beta - 1$. In [10], A. Noma proves the following.

**Theorem 1.1** (Theorem 1.1 in [10])**.** *Let $X \subset \mathbb{P}^r$ be a nondegenerate projective irreducible variety of degree $d$ and codimension $e$. If $X$ is locally Cohen-Macaulay, then*

$$\ell_\beta(X) \leq d - e + \beta \quad \textit{for all } 1 \leq \beta \leq e. \tag{1.2}$$

In summary, by Eisenbud-Goto's regularity conjecture it is strongly believed that $\ell_1(X)$ is at most $d-e+1$ for all projective irreducible varieties. On the other hand, $\ell_e(X)=d$ if $X$ is locally Cohen-Macaulay and $\ell_e(X)>d$ otherwise. So, the value of $\ell_e(X)$ is closely related to whether $X$ is locally Cohen-Macaulay or not. These facts lead us naturally to ask the following.

**Question.** Let $X \subset \mathbb{P}^r$ be a nondegenerate projective irreducible variety of degree $d$ and codimension $e$, which is not locally Cohen-Macaulay. For each integer $\beta < e$, is the value of $\ell_\beta(X)$ bounded by $d-e+\beta$?

Concerning this question, our aim in this short note is to show that the answer is "NO" by constructing an example of a variety $X_d \subset \mathbb{P}^{n+3}$ of dimension $n$, codimension $e=3$ and degree $d \in \{7,8,9,10,11,12\}$ such that

$$\ell_2(X) \geq d-3+\lfloor\frac{d}{2}\rfloor > d-1.$$

More precisely, our $X_d$ has a locally non-Cohen-Macaulay point, say $P$. We find explicitly a plane $L$ which passes through $P$ and satisfies the two conditions

$$\dim(X_d \cap L)=0 \quad \text{and} \quad \text{length}(X_d \cap L) \geq d-3+\lfloor\frac{d}{2}\rfloor.$$

For details, see Theorem 2.3 and Corollary 2.4.

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## 2. Construction of examples

For each $k \geq 5$, let $S_k \subset \mathbb{P}^5$ be a subset parameterized by

$$S_k := \{[us^2 : ust : ut^2 : vs^k : vst^{k-1} : vt^k] \mid (s,t),(u,v) \in \Bbbk^2 \setminus \{(0,0)\}\}.$$

**Proposition 2.1.** *Let $S_k$ be as above. Then*

(1) *$S_k \subset \mathbb{P}^5$ is a nondegenerate projective surface of degree $k+2$.*
(2) *$\text{reg}(S_k)=\ell_1(S_k)=k$.*

*Proof.* (1) Let $\widetilde{S_k} := S(2,k) \subset \mathbb{P}^{k+3}$, $k \geq 5$, be the standard rational normal surface scroll of degree $k+2$ parameterized by

$$\widetilde{S_k} := \{[us^2 : ust : ut^2 : vs^k : vs^{k-1}t : \cdots : vst^{k-1} : vt^k] \mid (s,t),(u,v) \in K^2 \setminus \{(0,0)\}\}.$$

Let $\Lambda$ be the $(k-3)$-dimensional subspace spanned by the $(k-2)$ coordinate points $P_4, P_5, \ldots, P_{k+1}$ of $\mathbb{P}^{k+3}$. Then $\Lambda$ avoids $\widetilde{S_k}$ and $S_k$ is the image of $\widetilde{S_k}$ under the linear projection map

$$\pi_\Lambda : \mathbb{P}^{k+3} \setminus \Lambda \to \mathbb{P}^5.$$

Moreover, it maps $S(2) \subset \widetilde{S_k}$ to a plane conic isomorphically, the $k$-dimensional subspace $\langle S(k) \rangle$ onto a plane $\mathbb{P}^2$ and the rational normal curve $S(k) \subset \widetilde{S_k}$ birationally onto the plane curve $C_k$ of degree $k$ corresponding to the relation

$$\left(vst^{k-1}\right)^k = \left(vs^k\right) \times \left(vt^k\right)^{k-1}.$$

Therefore the restriction map $f : \widetilde{S_k} \to S_k$ of $\pi_\Lambda$ to $\widetilde{S_k}$ is finite and birational. In particular, we have

$$\deg(S_k) = \deg(\widetilde{S_k}) = k + 2.$$

(2) By the above description of $S_k$ as the image of a finite birational linear projection of $\widetilde{S_k}$, we know that $S_k$ is a *surface of maximal sectional regularity* in the sense that its general hyperplane section curve is of maximal regularity. For details, see [2, Theorem 6.3]. Indeed, the plane curve $C_k$ provides infinitely many $k$-secant lines to $S_k$. This implies that $\ell_1(S_k) \geq k$. Also it follows by [3, Theorem 1.1] that $\operatorname{reg}(S_k) = k$ and hence $\ell_1(S_k) \leq k$. Therefore we get the desired equalities. □

*Notation and Remark* 2.2. Here we consider the defining ideal and the Betti diagram of $S_k$. All computations are obtained by means of the Computer Algebra System "SINGULAR" [5].

(1) Let $R = \Bbbk[Z_0, Z_1, Z_2, Z_3, Z_4, Z_5]$ be the homogeneous coordinate ring of $\mathbb{P}^5$ and consider the plane curve $C_k$ of degree $k$ which is defined as the image of the rational normal curve $S(k) \subset \widetilde{S_k}$ under the projection map $f$. Note that $C_k$ is defined in $\mathbb{P}^5$ by $Z_0, Z_1, Z_2$ and

$$G_k := Z_3 Z_5^{k-1} - Z_4^k.$$

Also $P := [0 : 0 : 0 : 1 : 0 : 0]$ is the unique singular point of $C_k$.

(2) Let $Y = S(0, 1, 2)$ be the threefold scroll in $\mathbb{P}^5$ which is defined as the rank 1 locus of the matrix

$$\begin{pmatrix} Z_0 & Z_1 & Z_4 \\ Z_1 & Z_2 & Z_5 \end{pmatrix}.$$

Thus the homogeneous ideal $I_Y$ of $Y$ is

$$I_Y = \langle Z_0Z_5 - Z_1Z_4, Z_0Z_2 - Z_1^2, Z_1Z_5 - Z_2Z_4 \rangle.$$

Observe that $S_k$ is contained in $Y$ since the matrix

$$\begin{pmatrix} us^2 & ust & vst^{k-1} \\ ust & ut^2 & vt^k \end{pmatrix}$$

of rank 1 for all $(s, t), (u, v) \in \Bbbk^2 \setminus \{(0, 0)\}$.

(3) Consider the plane $L$ defined by $Z_0, Z_1, Z_5$. One can easily check that the intersection $S_k \cap L$ consists of exactly the two points $P$ and $Q := [0 : 0 : 1 : 0 : 0 : 0]$. Also, $Y \cap L$ is equal to $\operatorname{Proj}\big(\Bbbk[Z_2, Z_3, Z_4]/\langle Z_2Z_4\rangle\big)$, the union of two lines in $L$.

(4) For $5 \leq k \leq 10$, the homogeneous ideal of $S_k$ is as below. For the simplicity, put

$$H_{k,i} := Z_0^i Z_4^{k-2i} - Z_2^i Z_3 Z_5^{k-1-2i} \quad (1 \leq i \leq \lceil \frac{k}{2} \rceil - 2)$$

and

$$G_k := Z_3 Z_5^{k-1} - Z_4^k.$$

Then

$$\begin{cases} I_{S_5} = \langle G_5, H_{5,1}, Z_0^2Z_4 - Z_2^2Z_3\rangle + I_Y, \\ I_{S_6} = \langle G_6, H_{6,1}, Z_0^2Z_4^2 - Z_2^2Z_3Z_5, Z_0^2Z_1Z_4 - Z_2^3Z_3, Z_0^3Z_4 - Z_1Z_2^2Z_3\rangle \\ \qquad + I_Y, \\ I_{S_7} = \langle G_7, H_{7,1}, H_{7,2}, Z_0^3Z_4 - Z_2^3Z_3\rangle + I_Y, \\ I_{S_8} = \langle G_8, H_{8,1}, H_{8,2}, Z_0^3Z_4^2 - Z_2^3Z_3Z_5, Z_0^3Z_1Z_4 - Z_2^4Z_3, Z_0^4Z_4 - Z_1Z_2^3Z_3\rangle \\ \qquad + I_Y, \\ I_{S_9} = \langle G_9, H_{9,1}, H_{9,2}, H_{9,3}, Z_0^4Z_4 - Z_2^4Z_3\rangle + I_Y \quad \text{and} \\ I_{S_{10}} = \langle G_{10}, H_{10,1}, H_{10,2}, H_{10,3}, Z_0^4Z_4^2 - Z_2^4Z_3Z_5, Z_0^4Z_1Z_4 - Z_2^5Z_3, Z_0^5Z_4 \\ \qquad - Z_1Z_2^4Z_3\rangle + I_Y. \end{cases}$$

(5) For $5 \le k \le 10$, we could check that the singular locus of $S_k$ is exactly the set $\{P\}$ by applying "SINGULAR" to these defining equations of $S_k$. Note that the singular locus and the locally non-Cohen-Macaulay locus of $S_k$ are same by [4, Lemma 4.5]. Consequently, it is shown that $P$ is the unique locally non-Cohen-Macaulay point of $S_k$. In particular, the depth of the local ring $\mathcal{O}_{S_k,P}$ is equal to 1 while that of $\mathcal{O}_{S_k,Q}$ is equal to 2 for all $Q \in S_k \setminus \{P\}$.

(6) For $5 \le k \le 10$, the occurring Betti diagrams of $S_k$ are as below.

| | $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $S_5$ | $\beta_{i,1}$ | 3 | 2 | 0 | 0 |
| | $\beta_{i,2}$ | 1 | 0 | 0 | 0 |
| | $\beta_{i,3}$ | 1 | 6 | 5 | 1 |
| | $\beta_{i,4}$ | 1 | 3 | 3 | 1 |

| | $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $S_6$ | $\beta_{i,1}$ | 3 | 2 | 0 | 0 |
| | $\beta_{i,2}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,3}$ | 3 | 6 | 3 | 0 |
| | $\beta_{i,4}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,5}$ | 1 | 3 | 3 | 1 |

| | $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $S_7$ | $\beta_{i,1}$ | 3 | 2 | 0 | 0 |
| | $\beta_{i,2}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,3}$ | 1 | 0 | 0 | 0 |
| | $\beta_{i,4}$ | 1 | 6 | 5 | 1 |
| | $\beta_{i,5}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,6}$ | 1 | 3 | 3 | 1 |

**Theorem 2.3.** *For $5 \le k \le 10$, it holds that*

$$\ell_2(S_k) \ge k + \lfloor \frac{k}{2} \rfloor. \tag{2.1}$$

*Proof.* Consider the intersection $\Gamma_k := S_k \cap L$ which is defined by the homogeneous ideal

$$I_{\Gamma_k} := I_{S_k} + \langle Z_0, Z_1, Z_5\rangle.$$

Since $\Gamma_k$ is a finite scheme, there is a $\Bbbk$-algebra $A_k$ such that $\Gamma_k \cong \mathrm{Spec}(A_k)$. In particular, it holds that

$$\mathrm{length}(\Gamma_k) = \dim_{\Bbbk} A_k.$$

| | $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $S_8$ | $\beta_{i,1}$ | 3 | 2 | 0 | 0 |
| | $\beta_{i,2}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,3}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,4}$ | 3 | 6 | 3 | 0 |
| | $\beta_{i,5}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,6}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,7}$ | 1 | 3 | 3 | 1 |

| | $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $S_9$ | $\beta_{i,1}$ | 3 | 2 | 0 | 0 |
| | $\beta_{i,2}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,3}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,4}$ | 1 | 0 | 0 | 0 |
| | $\beta_{i,5}$ | 1 | 6 | 5 | 1 |
| | $\beta_{i,6}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,7}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,8}$ | 1 | 3 | 3 | 1 |

| | $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| $S_{10}$ | $\beta_{i,1}$ | 3 | 2 | 0 | 0 |
| | $\beta_{i,2}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,3}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,4}$ | 0 | 0 | 0 | 0 |
| | $\beta_{i,5}$ | 3 | 6 | 3 | 0 |
| | $\beta_{i,6}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,7}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,8}$ | 1 | 3 | 3 | 1 |
| | $\beta_{i,9}$ | 1 | 3 | 3 | 1 |

From now on, we suppose that $5 \leq k \leq 10$. By using the computational result in Notation and Remark 2.2(4), we have

$$I_{\Gamma_k} := \langle Z_0, Z_1, Z_5, Z_2^{\lfloor \frac{k}{2} \rfloor} Z_3, Z_2 Z_4, Z_4^k \rangle.$$

Observe that $\Gamma_k$ is contained in the complement of the hyperplane $Z_2 + Z_3$. Obviously this implies that $A_k$ is isomorphic to the dehomogenization of the graded ring $R/I_{\Gamma_k}$ with respect to the linear form $Z_2 + Z_3$. Thus we have

$$A_k \cong K[\alpha, \beta]/\langle \alpha^{\lfloor \frac{k}{2} \rfloor}(1-\alpha), \alpha\beta, \beta^k \rangle, \tag{2.2}$$

where $\alpha = \frac{Z_2}{Z_2+Z_3}$ and $\beta = \frac{Z_4}{Z_2+Z_3}$. In particular, the set

$$\{1, \alpha, \ldots, \alpha^{\lfloor \frac{k}{2} \rfloor}, \beta, \beta^2, \ldots, \beta^{k-1}\}$$

is a basis of $A_k$ as a $\Bbbk$-vector space. It follows that

$$\dim_{\Bbbk} A_k = k + \lfloor \frac{k}{2} \rfloor \quad \text{for } 5 \leq k \leq 10. \tag{2.3}$$

Thus we get the inequality

$$\ell_2(S_k) \geq \text{length}(\Gamma_k) = \dim_{\Bbbk} A_k = k + \lfloor \frac{k}{2} \rfloor,$$

which completes the proof of (2.1). □

**Corollary 2.4.** *For each $n \geq 2$ and $7 \leq d \leq 12$, there exists an $n$-dimensional projective irreducible variety $X \subset \mathbb{P}^{n+3}$ of degree $d$ and codimension 3 which admits a proper $(d-3+\lfloor \frac{d}{2} \rfloor)$-secant plane.*

*Proof.* For $5 \leq k \leq 10$, let $X \subset \mathbb{P}^{n+3}$ be a cone over $S_k$. Then the assertion comes immediately from Theorem 2.3. □

Our computational result in Theorem 2.3 leads us to pose the following.

**Conjecture.** For all $k \geq 5$, it holds that

$$| S_k \cap L | = k + \lfloor \frac{k}{2} \rfloor \quad \text{and hence} \quad \ell_2(S_k) \geq k + \lfloor \frac{k}{2} \rfloor.$$

In the proof of Theorem 2.3 we use a specific set of generators of the homogeneous ideal of $S_k$. Indeed, this conjecture can be checked for more $k$'s by the same play once we can solve the problem to describe the homogeneous ideal of $S_k$ precisely. But this problem seems to be difficult for arbitrary $k$.

The above conjecture is interesting since it says that the value of $\ell_2(X)$ for locally non-Cohen-Macaulay varieties can be enlargeable arbitrarily comparing with Noma's bound $d - c + 2$ in Theorem 1.1 for the value of $\ell_2(X)$ for locally Cohen-Macaulay varieties.

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