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STABILITY OF FRACTIONAL-ORDER NONLINEAR SYSTEMS DEPENDING ON A PARAMETER

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Abstract. In this paper, we present a practical Mittag Leffler stability for fractional-order nonlinear systems depending on a parameter. A sufficient condition on practical Mittag Leffler stability is given by using a Lyapunov function. In addition, we study the problem of stability and stabilization for some classes of fractional-order systems.

1. Introduction

The fractional order dynamical systems have attracted remarkable attention in the last decade. Many dynamic systems are better characterized by a dynamic model of fractional order, usually based on the concept of differentiation or integration of fractional order.

It is usually known that several physical systems are characterized by fractional-order state equations ([16]), such as, fractional Langevin equation ([3]), fractional Lotka-Volterra equation ([10]) in biological systems, fractional-order oscillator equation ([29]) in damping vibration, in anomalous diffusion and so on. Particularly, stability analysis is one of the most fundamental issues for systems. In this few years, there are many results about the stability and stabilization of fractional order systems ([4, 5, 6, 7, 11, 12, 13, 17, 18, 21, 23, 25, 27, 30]).

The problem of stability and stabilization for nonlinear integer-order dynamic systems is many studied in literature ([2, 9, 15, 20, 22, 28, 31]). One type of stability studied, deals with the so called practical stability, this notion was studied in ([1, 14]). In the present paper, we introduce the notion of practical stability of nonlinear fractional-order systems depending on a parameter, called ε^* practical Mittag Leffler stability. This stability ensures the Mittag Leffler stability of a ball containing the origin of the state space, the radius of the ball can be made arbitrarily small. Our goal is to investigate the practical Mittag Leffler stability of nonlinear fractional-order systems depending on

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a parameter by using the Lyapunov techniques. Precisely we give sufficient conditions that ensures the practical Mittag Leffler stability of such systems.

The paper is organised as follows. In Section 2, some necessary definitions and lemmas are presented. In Section 3 a sufficient condition on practical Mittag Leffler stability of nonlinear fractional differential equations is given. In addition, some other classes of perturbed fractional systems are studied in point of view stability and a continuous feedback controller is proposed to stabilize a large class of nonlinear fractional dynamical systems with uncertainties.

2. Preliminaries

In this section, some definitions, lemmas and theorems related to the fractional calculus are given.

In the literature, there are many definitions for fractional derivative ([17, 26]). In this paper we adopt the definition of Caputo fractional derivative.

Definition. The Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as,

$$I_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-\tau)^{\alpha-1} x(\tau) d\tau,$$

$$\Gamma(\alpha) = \int_{0}^{+\infty} e^{-t} t^{\alpha-1} dt,$$

where Γ is the Gamma function generalizing factorial for non-integer arguments.

Definition. The Caputo fractional derivative is defined as,

$${}^{C}D_{t_0,t}^{\alpha} x(t) = \frac{1}{\Gamma(m-\alpha)} \int_{t_0}^{t} (t-\tau)^{m-\alpha-1} \frac{d^m}{d\tau^m} x(\tau) d\tau, \ (m-1 < \alpha < m).$$

When $0 < \alpha < 1$, then the Caputo fractional derivative of order α of f reduces to

$$(2.1) \qquad {}^{C}D_{t_0,t}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int_{t_0}^{t} (t-\tau)^{-\alpha} \frac{d}{d\tau} x(\tau) d\tau.$$

On the other hand, there exists a frequently used function in the solution of fractional order systems named the Mittag Leffler function. Indeed, the proposed function is a generalization of the exponential function. In this context, the following definitions and lemmas are presented.

Definition. The Mittag-Leffler function with two parameters is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+\beta)},$$

where $\alpha > 0$, $\beta > 0$, $z \in \mathbb{C}$. When $\beta = 1$, one has $E_{\alpha}(z) = E_{\alpha,1}(z)$, furthermore, $E_{1,1}(z) = e^z$.

Lemma 2.1 ([24]). *For $0<\alpha<1$, we have $E_\alpha(-t)$ is nonincreasing in t.*

We consider the nonhomogeneous linear fractional differential equation with Caputo fractional derivative

$$
\begin{aligned}
{}^{C}D^{\alpha}_{t_0,t}x(t) &= \lambda x + h(t),\ t\geq t_0 \\
x(t_0) &= x_0,
\end{aligned}
\tag{2.2}
$$

The solution of (2.2) is given by

$$
x(t) = x_0 E_\alpha(\lambda(t-t_0)^\alpha) + \int_{t_0}^{t}(t-s)^{\alpha-1}E_{\alpha,\alpha}(\lambda(t-s)^\alpha)h(s)ds. \tag{2.3}
$$

Lemma 2.2 ([8]). *If one sets $h(t)=d$ in (2.2) with a constant d, then the solution of (2.3) reduces to*

$$
x(t) = x_0 E_\alpha(\lambda(t-t_0)^\alpha) + d(t-t_0)^\alpha E_{\alpha,\alpha+1}(\lambda(t-t_0)^\alpha). \tag{2.4}
$$

Lemma 2.3 ([8]). *Let $0<\alpha<1$ and $|arg(\lambda)|>\frac{\pi\alpha}{2}$. Then, one has*

$$
t^\alpha E_{\alpha,\alpha+1}(\lambda t^\alpha) = -\frac{1}{\lambda} - \frac{1}{\Gamma(1-\alpha)\lambda^2 t^\alpha} + O\big(\frac{1}{\lambda^3 t^{2\alpha}}\big) \quad as \quad t\to\infty.
$$

Lemma 2.4 ([8]). *Suppose that*

$$
{}^{C}D^{\alpha}_{t_0,t}m(t) \leq \lambda m(t) + d,\ m(t_0) = m_0,\ t\geq t_0\geq 0,
$$

where λ, $d\in\mathbb{R}$. Then, one has

$$
m(t) \leq m(t_0)E_\alpha\big(\lambda(t-t_0)^\alpha\big) + d(t-t_0)^\alpha E_{\alpha,\alpha+1}\big(\lambda(t-t_0)^q\big),\ t\geq t_0\geq 0.
$$

Moreover if $\lambda<0$, then

$$
m(t) \leq m(t_0)E_\alpha\big(\lambda(t-t_0)^\alpha\big) + M\ d,\ t\geq t_0\geq 0,
$$

where $M=\sup\limits_{s\geq 0}\Big(s^\alpha E_{\alpha,\alpha+1}\big(\lambda s^\alpha\big)\Big)$.

Remark 2.5. Authors in ([8]) studied the boundedness of solutions for fractional differential equations by using the previous lemma and the Lyapunov function.

Lemma 2.6 ([13]). *Let x be a vector of functions. Then, for any time instant $t\geq t_0$, the following relationship holds*

$$
\frac{1}{2}\ {}^{C}D^{\alpha}_{t_0,t}(x^T(t)Px(t)) \leq x^T(t)P\ {}^{C}D^{\alpha}_{t_0,t}x(t),\quad \forall\alpha\in(0,1),\ \forall t\geq t_0
$$

where $P\in\mathbb{R}^{n\times n}$ is a constant, square, symmetric and positive definite matrix.

Lemma 2.7. *For all $p\geq 1$ and $a,b\geq 0$, we have $(a+b)^p\leq 2^{p-1}(a^p+b^p)$ and $(a+b)^{\frac{1}{p}}\leq(a^{\frac{1}{p}}+b^{\frac{1}{p}})$.*

3. Stability of fractional differential equations depending on a parameter

Consider a parameterized family of fractional differential equations with a Caputo derivative for $0 < \alpha < 1$ having the following form:

$$^{C}D^{\alpha}_{t_0,t}x(t) = f(t,x,\varepsilon),\ t \geq t_0$$
$$x(t_0) = x_0,$$

where $t_0 \in \mathbb{R}_+$, $\varepsilon \in \mathbb{R}^*_+$, $x(t) \in \mathbb{R}^n$, $f(\cdot,\cdot,\varepsilon) : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is continuous and locally Lipschitz in x.

The goal of the paper is to study the ε^*-practical Mittag Leffler stability of certain classes of form of (3.1).

Definition. The system (3.1) is said to be

- ε^*-*uniformly practically Mittag Leffler stable* if for all $0 < \varepsilon \leq \varepsilon^*$ there exist positive scalars $K(\varepsilon)$, $\lambda(\varepsilon)$ and $\rho(\varepsilon)$ and such that the trajectory of (3.1) passing through any initial state $x_\varepsilon(t_0)$ at any initial time t_0 evaluated at time t satisfies:

$$\|x_\varepsilon(t)\| \leq \Big[K(\varepsilon)m\big(x_\varepsilon(t_0)\big)E_\alpha\big(-\lambda(\varepsilon)(t-t_0)^\alpha\big)\Big]^b + \rho(\varepsilon),\ \forall t \geq t_0 \geq 0, \tag{3.1}$$

  with $b > 0$, $\rho(\varepsilon) \longrightarrow 0$ as $\varepsilon \longrightarrow 0^+$ and there exist K, λ_1, $\lambda_2 > 0$ such that $\lambda_1 \leq \lambda(\varepsilon) \leq \lambda_2$, $0 < K(\varepsilon) \leq K$ for all $\varepsilon \in]0,\varepsilon^*]$, $m(0) = 0$, $m(x) \geq 0$ and m is locally Lipschitz.
- *Uniformly Mittag Leffler stable* if (3.1) is satisfied with $\rho = 0$.

Proposition 3.1. *Let $p \geq 1$ and $\varepsilon^* > 0$. Assume that for all $0 < \varepsilon \leq \varepsilon^*$, there exists a continuously differentiable function $V_\varepsilon : \mathbb{R}_+ \times \mathbb{R}^n \longrightarrow \mathbb{R}$, a continuous function $\mu : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ and positive constants scalar $a_1(\varepsilon)$, $a_2(\varepsilon)$, $a_3(\varepsilon)$, $r_1(\varepsilon)$ and $r_2(\varepsilon)$ such that*

(1)

$$a_1(\varepsilon)\|x\|^p \leq V_\varepsilon(t,x) \leq a_2(\varepsilon)\|x\|^p + r_1(\varepsilon),\ \forall t \geq 0,\ x \in \mathbb{R}^n, \tag{3.2}$$

(2)

$$^{C}D^{\alpha}_{t_0,t}V_\varepsilon(t,x_\varepsilon(t)) \leq -a_3(\varepsilon)\|x_\varepsilon(t)\|^p + \mu(t)r_2(\varepsilon),\ \forall t \geq t_0, \tag{3.3}$$

with

- $\forall\varepsilon \in]0,\varepsilon^*]$, $\frac{a_3(\varepsilon)}{a_2(\varepsilon)} \geq \lambda$ *and* $0 < \frac{a_2(\varepsilon)}{a_1(\varepsilon)} \leq K$, *with* λ, $K > 0$.
- $t \mapsto \displaystyle\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda(t-s)^\alpha\Big)\mu(s)ds$ *is a bounded function.*
- $c(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$ *where*

$$c(\varepsilon) = r_1(\varepsilon)\frac{(a_2(\varepsilon) + Ma_3(\varepsilon))}{a_1(\varepsilon)a_2(\varepsilon)} + r_2(\varepsilon)\frac{M}{a_1(\varepsilon)},$$

with, $M = M_1 + M_2$*, where*

$$M_1 = \sup_{s\geq 0}\Big(s^{\alpha}E_{\alpha,\alpha+1}\big(-\lambda s^{\alpha}\big)\Big)$$

and

$$M_2 = \sup_{t\geq 0}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda(t-s)^{\alpha}\Big)\mu(s)ds.$$

Then, the system (3.1) *is* ε^**-uniformly practically Mittag Leffler stable.*

Proof. Let us consider $t_0 \geq 0$. It follows from (3.2) and (3.3) that

$$\begin{aligned} {}^{C}D^{\alpha}_{t_0,t}V_{\varepsilon}(t,x_{\varepsilon}(t)) &\leq -\frac{a_3(\varepsilon)}{a_2(\varepsilon)}V_{\varepsilon}(t,x_{\varepsilon}(t)) + \rho(t)l(\varepsilon) \\ &\leq -\lambda V_{\varepsilon}(t,x_{\varepsilon}(t)) + \rho(t)l(\varepsilon),\ \forall t\geq t_0, \end{aligned} \tag{3.4}$$

where $\rho(t) = \big(1+\mu(t)\big)$ and $l(\varepsilon) = r_2(\varepsilon) + \frac{r_1(\varepsilon)a_3(\varepsilon)}{a_2(\varepsilon)}$.

There exists a nonnegative function $h(t)$ satisfying:

$${}^{C}D^{\alpha}_{t_0,t}V_{\varepsilon}(t,x_{\varepsilon}(t)) = -\lambda V_{\varepsilon}(t,x_{\varepsilon}(t)) + \rho(t)l(\varepsilon) - h(t). \tag{3.5}$$

It follows from (2.3) that

$$\begin{aligned} V_{\varepsilon}(t,x_{\varepsilon}(t)) &= E_{\alpha}\big(-\lambda(t-t_0)^{\alpha}\big)V_{\varepsilon}(t_0,x_{\varepsilon}(t_0)) \\ &\quad + \int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda(t-s)^{\alpha}\Big)\big(\rho(s)l(\varepsilon)-h(s)\big)ds. \end{aligned} \tag{3.6}$$

We have

$$\int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda(t-s)^{\alpha}\Big)h(s)ds \geq 0,$$

then,

$$\begin{aligned} V_{\varepsilon}(t,x_{\varepsilon}(t)) &\leq E_{\alpha}\big(-\lambda(t-t_0)^{\alpha}\big)V_{\varepsilon}(t_0,x_{\varepsilon}(t_0)) \\ &\quad + l(\varepsilon)\int_{t_0}^t (t-s)^{\alpha-1}E_{\alpha,\alpha}\Big(-\lambda(t-s)^{\alpha}\Big)\rho(s)ds. \end{aligned} \tag{3.7}$$

Hence,

$$V_{\varepsilon}(t,x_{\varepsilon}(t)) \leq E_{\alpha}\big(-\lambda(t-t_0)^{\alpha}\big)V_{\varepsilon}(t_0,x_{\varepsilon}(t_0)) + Ml(\varepsilon),\ \forall\ t\geq t_0. \tag{3.8}$$

By (3.2) we have:

$$\|x_{\varepsilon}(t)\|^p \leq \frac{1}{a_1(\varepsilon)}E_{\alpha}\big(-\lambda(t-t_0)^{\alpha}\big)\big(a_2(\varepsilon)\|x_{\varepsilon}(t_0)\|^p + r_1(\varepsilon)\big) + \frac{Ml(\varepsilon)}{a_1(\varepsilon)}\ ,\ t\geq t_0\geq 0. \tag{3.9}$$

It follows from Lemma 2.1 $E_{\alpha}\big(-\lambda s^{\alpha}\big) \leq 1,\ \forall s\geq 0$, so

$$\|x_{\varepsilon}(t)\|^p \leq \frac{a_2(\varepsilon)}{a_1(\varepsilon)}E_{\alpha}\big(-\lambda(t-t_0)^{\alpha}\big)\|x_{\varepsilon}(t_0)\|^p + c(\varepsilon)\ ,\ t\geq t_0\geq 0. \tag{3.10}$$

Hence, from Lemma 2.7 we obtain

$$\|x_\varepsilon(t)\| \le \Big[\frac{a_2(\varepsilon)}{a_1(\varepsilon)} E_\alpha\big(-\lambda(t-t_0)^\alpha\big)\|x_\varepsilon(t_0)\|^p\Big]^{\frac{1}{p}} + r(\varepsilon)\ ,\ t \ge t_0 \ge 0 \tag{3.11}$$

with $r(\varepsilon) = c(\varepsilon)^{\frac{1}{p}}$. Hence, the system (3.1) is ε^*-uniformly practically Mittag Leffler stable. □

3.1. Stability for a class of perturbed systems

In this section, we consider a perturbed system:

$${}^C D^\alpha_{t_0,t} x(t) = Ax(t) + g(t, x(t), \varepsilon),\ x(t_0) = x_0, \tag{3.12}$$

where, $0 < \alpha < 1$, $x(t) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$ is a constant matrix and $g(t, x(t), \varepsilon) \in \mathbb{R}^n$.

Consider the following assumption:
(H) The perturbation term $g(t, x(t), \varepsilon)$ satisfies, for all $t \ge 0$, $\varepsilon > 0$ and $x \in \mathbb{R}^n$

$$\|g(t, x, \varepsilon)\| \le \delta_1(\varepsilon)\nu(t) + \delta_2(\varepsilon)\|x\|, \tag{3.13}$$

such that $\delta_1(\varepsilon)$, $\delta_2(\varepsilon) > 0$ and $\delta_1(\varepsilon)$, $\delta_2(\varepsilon) \ \to 0$ as $\varepsilon \to 0^+$ and ν is a nonnegative continuous function. Then, we have the following theorem:

Theorem 3.2. *Suppose* (H) *holds, the system* (3.12) *is* ε_1*-uniformly practically Mittag Leffler stable for some* $\varepsilon_1 > 0$ *if there exists a symmetric and positive definite matrix* P, $\eta > 0$ *and* $\lambda \in]0, \eta[$ *such that*

$$A^T P + PA + \eta I < 0, \tag{3.14}$$

and

$$t \mapsto \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}\Big(-\frac{(\eta-\lambda)}{\lambda_{\max}(P)}(t-s)^\alpha\Big)\nu^2(s)ds \quad \textit{is a bounded function.}$$

Proof. It follows from (3.13) that:

$$\begin{aligned} 2x^T P g(t, x, \varepsilon) &\le 2\|x\|\|P\|\|g(t, x, \varepsilon)\| \\ &\le 2\|x\|\|P\|\big(\delta_1(\varepsilon)\nu(t) + \delta_2(\varepsilon)\|x\|\big). \end{aligned} \tag{3.15}$$

Let $0 < \lambda_1 < \lambda$, we have

$$2\|x\|\|P\|\delta_1(\varepsilon)\nu(t) \le \lambda_1\|x\|^2 + \frac{\|P\|^2\delta_1(\varepsilon)^2\nu^2(t)}{\lambda_1}. \tag{3.16}$$

Substituting (3.16) into (3.15) yields

$$2x^T P g(t, x, \varepsilon) \le \big(\lambda_1 + 2\delta_2(\varepsilon)\|P\|\big)\|x\|^2 + \frac{\|P\|^2\delta_1(\varepsilon)^2\nu^2(t)}{\lambda_1}.$$

Since $\delta_2(\varepsilon) \ \to 0$ as $\varepsilon \to 0^+$ then there exists $\varepsilon_1 > 0$ such that for all $\varepsilon \in]0, \varepsilon_1]$, $2\delta_2(\varepsilon)\|P\| + \lambda_1 < \lambda$.

Then, $\forall \varepsilon \in]0, \varepsilon_1]$, we have

$$2x^T P g(t, x, \varepsilon) \le \lambda\|x\|^2 + \frac{\|P\|^2\delta_1(\varepsilon)^2\nu^2(t)}{\lambda_1}. \tag{3.17}$$

Let $\varepsilon \in]0, \varepsilon_1]$. Choose a Lyapunov function $V(t,x) = x^T P x$.

It follows from Lemma 6 that:

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t)) &\leq 2x_\varepsilon(t)^T P\, {}^C D^{\alpha}_{t_0,t} x_\varepsilon(t) \\ &\leq \big[Ax_\varepsilon + g(t, x_\varepsilon, \varepsilon)\big]^T P x_\varepsilon + x_\varepsilon^T P \big[Ax_\varepsilon + g(t, x_\varepsilon, \varepsilon)\big] \\ &\leq x_\varepsilon^T \big(A^T P + PA\big) x_\varepsilon(t) + 2x_\varepsilon^T P g(t, x_\varepsilon, \varepsilon). \end{aligned} \tag{3.18}$$

By (3.17) and (3.18) we have

$${}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t)) \leq x_\varepsilon(t)^T \big(A^T P + PA\big) x_\varepsilon(t) + \lambda \|x_\varepsilon(t)\|^2 + \frac{\|P\|^2 \delta_1(\varepsilon)^2 \nu^2(t)}{\lambda_1}.$$

Hence by (3.14),

$${}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t)) \leq -\eta_2 \|x_\varepsilon(t)\|^2 + \frac{\|P\|^2 \delta_1(\varepsilon)^2 \nu^2(t)}{\lambda_1},$$

with $\eta_2 = \eta - \lambda$.

Then, all hypothesis of Proposition 1 are satisfied. Therefore, the system (3.12) is ε_1-uniformly practically Mittag Leffler stable. □

Example 3.3. Consider the following fractional system:

$$(E) \qquad \begin{cases} {}^C D^{\alpha}_{0,t} x_1 = -2x_1 + x_2 + \varepsilon e^{-t} x_1 + \varepsilon^2 \frac{1}{1+t^2} \\ {}^C D^{\alpha}_{0,t} x_2 = x_1 - x_2 + \varepsilon e^{-t} x_2 + \varepsilon^2 \frac{2t}{1+t^2} \end{cases}$$

where, $0 < \alpha < 1$ and $x(t) = \big(x_1(t), x_2(t)\big) \in \mathbb{R}^2$.

This system has the same from of (3.12) with

$$A = \begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$$

and

$$g(t, x, \varepsilon) = \varepsilon e^{-t}(x_1, x_2) + (\frac{\varepsilon^2}{1+t^2}, \frac{2\varepsilon^2 t}{1+t^2}).$$

The perturbed term g satisfies (H) with $\delta_1(\varepsilon) = \varepsilon^2$, $\delta_2(\varepsilon) = \varepsilon$ and $\nu(t) = \sqrt{2}$. Select $P = 2I$, since

$$A^T P + PA + I = \begin{pmatrix} -7 & 4 \\ 4 & -3 \end{pmatrix} < 0,$$

then, the assumptions of Theorem 3.2 are satisfied.

We have $\eta = 1$, we choose $\lambda = \frac{3}{4}$ and $\lambda_1 = \frac{1}{2}$. So, $2\varepsilon\|P\| + \lambda_1 < \lambda$, $\forall \varepsilon \in (0, \varepsilon_1]$ with $0 < \varepsilon_1 < \frac{1}{16}$. Hence, the system (E) is ε_1-uniformly practically Mittag Leffler stable. Note that in this case, the state approaches the origin, for some sufficiently small ε, when t tends to infinity.

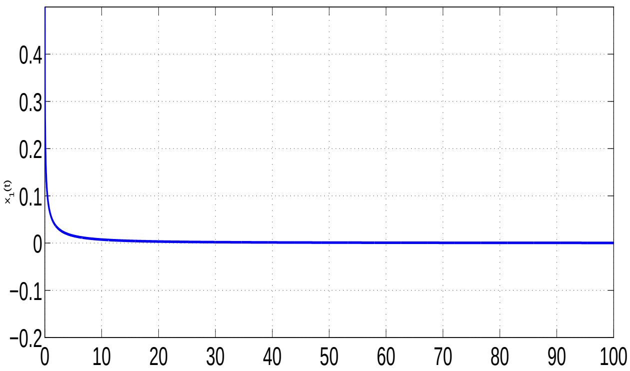

FIGURE 1. Time evolution of the state $x_1(t)$ of system (E) with $\varepsilon = 0.01$

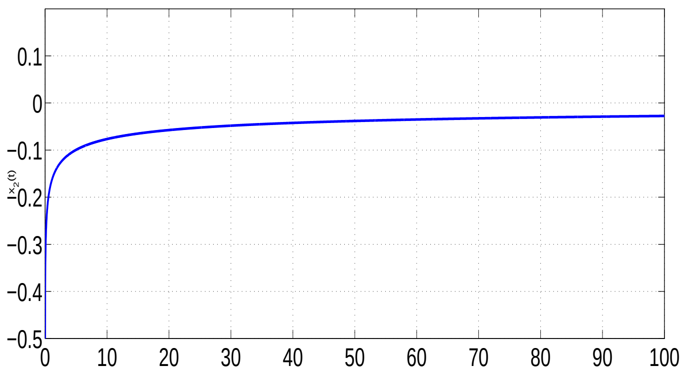

FIGURE 2. Time evolution of the state $x_2(t)$ of system (E) with $\varepsilon = 0.01$

3.2. Practical Mittag Leffler stability of a class of nonlinear fractional differential equations with uncertainties

In this section we discuss the problem of stabilization for a class of nonlinear fractional-order systems with uncertainties.

Consider the system

$$ {}^{C}D^{\alpha}_{t_0,t}x(t) = Ax + B\big(\phi(x,u) + u\big), \tag{3.19} $$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^q$, A and B are respectively $(n \times n)$, $(n \times q)$ constant matrices, $\phi : \mathbb{R}^n \times \mathbb{R}^q \longrightarrow \mathbb{R}^q$.

Assume that the following assumptions are satisfied:
(H_1) There exist a symmetric and positive definite matrix P and $\eta > 0$ such that the following inequality holds:

$$A^T P + PA + \eta I < 0. \tag{3.20}$$

(H_2) There exists a nonnegative continuous function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ such that:

$$\|\phi(x,u)\| \leq \psi(x), \ \forall x \in \mathbb{R}^n, \ \forall u \in \mathbb{R}^q. \tag{3.21}$$

Theorem 3.4. *Suppose that assumptions* (H_1), (H_2) *hold, then the feedback law*

$$u(\varepsilon, x) = -\frac{B^T P x \psi(x)^2}{\|B^T P x\| \psi(x) + \rho(\varepsilon)}, \tag{3.22}$$

where $\rho(\varepsilon) \to 0$ *as* $\varepsilon \to 0^+$, $\rho(\varepsilon) > 0$, $\forall \varepsilon > 0$, *uniformly practically Mittag Leffler stabilizes the system* (3.19).

Proof. Choose a Lyapunov function $V(t,x) = x^T P x$. It follows from Lemma 2.6 that:

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t)) &\leq 2x_\varepsilon(t)^T P {}^C D^{\alpha}_{t_0,t} x_\varepsilon(t) \\ &\leq 2x_\varepsilon^T P \big[A x_\varepsilon + B\big(\phi(x_\varepsilon, u) + u\big)\big] \\ &\leq x_\varepsilon^T \big(A^T P + PA\big) x_\varepsilon - \frac{2x_\varepsilon^T PBB^T P x_\varepsilon \psi(x_\varepsilon)^2}{\|B^T P x_\varepsilon\| \psi(x_\varepsilon) + \rho(\varepsilon)} \\ &\quad + 2x_\varepsilon^T PB\phi(x_\varepsilon, u). \end{aligned} \tag{3.23}$$

Thus, by (H_1) and (H_2) we have

$$\begin{aligned} {}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t)) &\leq -\eta \|x_\varepsilon\|^2 - \frac{2x_\varepsilon^T PBB^T P x_\varepsilon \psi(x_\varepsilon)^2}{\|B^T P x_\varepsilon\| \psi(x_\varepsilon) + \rho(\varepsilon)} + 2\|B^T P x_\varepsilon\| \psi(x_\varepsilon) \\ &\leq -\eta \|x_\varepsilon\|^2 + \frac{2\|B^T P x_\varepsilon\| \psi(x_\varepsilon) \rho(\varepsilon)}{\|B^T P x_\varepsilon\| \psi(x_\varepsilon) + \rho(\varepsilon)}. \end{aligned} \tag{3.24}$$

Using the following inequality

$$\frac{\|B^T P x_\varepsilon\| \psi(x_\varepsilon) \rho(\varepsilon)}{\|B^T P x_\varepsilon\| \psi(x_\varepsilon) + \rho(\varepsilon)} \leq \rho(\varepsilon),$$

then,

$${}^C D^{\alpha}_{t_0,t} V(t, x_\varepsilon(t)) \leq -\eta \|x_\varepsilon(t)\|^2 + 2\rho(\varepsilon). \tag{3.25}$$

Hence, all hypothesis of Proposition 3.1 are satisfied. Therefore the uncertain closed-loop dynamical system (3.19) is ε^*-uniformly practically Mittag Leffler stable for some $\varepsilon^* > 0$. □

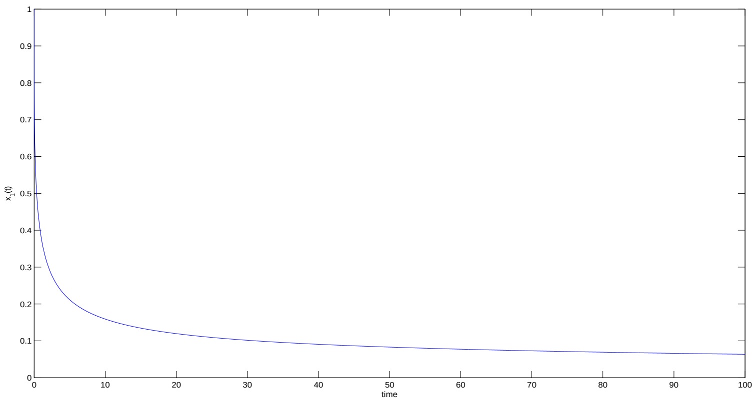

FIGURE 3. Time evolution of the state $x_1(t)$ of system (E') with $\varepsilon = 0.1$

Example 3.5. Consider the following fractional system:

$$(E') \qquad \begin{cases} {}^C D^{\alpha}_{0,t} x_1 = -4x_1 + 2x_2 + u + \cos(u)x_2 \\ {}^C D^{\alpha}_{0,t} x_2 = 2x_1 - 3x_2 \end{cases}$$

where, $0 < \alpha < 1$, $x(t) = \big(x_1(t), x_2(t)\big) \in \mathbb{R}^2$.

This system has the same from of (3.19) with $A = \begin{pmatrix} -4 & 2 \\ 2 & -3 \end{pmatrix}$, $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\phi(x, u) = \cos(u)x_2$.

Select $P = 2I$, since

$$A^T P + PA + I = \begin{pmatrix} -15 & 8 \\ 8 & -11 \end{pmatrix} < 0,$$

then, the assumptions of Theorem 3.4 are satisfied. Hence, we obtain the ε^*-practical Mittag Leffler stability of the closed loop fractional-order system (E') for some $\varepsilon^* > 0$ with

$$u = \frac{2x_1 x_2^2}{2|x_1 x_2| + \varepsilon^2}.$$

4. Conclusion

In this paper, we have introduced a notion of practical Mittag Leffler stability for fractional differential equations depending on a parameter. Sufficiently conditions are given by using Lyapunov theory. These results are applied to the analysis of some fractional systems.

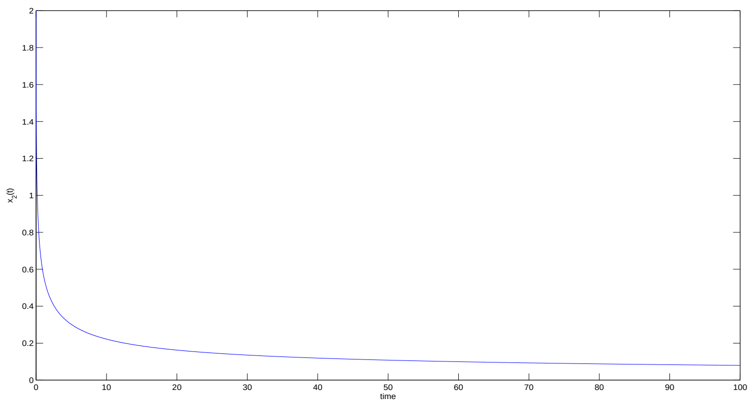

FIGURE 4. Time evolution of the state $x_2(t)$ of system (E') with $\varepsilon = 0.1$

References

[1] B. Ben Hamed, Z. Haj Salem, and M. A. Hammami, *Stability of nonlinear time-varying perturbed differential equations*, Nonlinear Dynam. **73** (2013), no. 3, 1353–1365.

[2] A. Ben Makhlouf and M. A. Hammami, *The convergence relation between ordinary and delay-integro-differential equations*, Int. J. Dyn. Syst. Differ. Equ. **5** (2015), no. 3, 236–247.

[3] S. Burov and E. Barkai, *Fractional Langevin equation: overdamped, underdamped, and critical behaviors*, Phys. Rev. (3) **78** (2008), no. 3, 031112, 18 pp.

[4] L. Chen, Y. He, Y. Chai, and R. Wu, *New results on stability and stabilization of a class of nonlinear fractional-order systems*, Nonlinear Dynam. **75** (2014), no. 4, 633–641.

[5] L. Chen, Y. He, R. Wu, and L. Yin, *Robust finite time stability of fractional-order linear delayed systems with nonlinear perturbations*, Int. J. Control Autom. **12** (2014), no. 3, 697–702.

[6] ______, *New result on finite-time stability of fractional-order nonlinear delayed systems*, J. Comput. Nonlin. Dyn. **10** (2015), 064504.

[7] ______, *Finite-time stability criteria for a class of fractional-order neural networks with delay*, Neural Comput. Appl. **27** (2016), 549–556.

[8] S. K. Choi, B. Kang, and N. Koo, *Stability for Caputo fractional differential systems*, Abstr. Appl. Anal. (2014), Article ID 610547, 6 pages.

[9] M. Corless and G. Leitmann, *Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamic systems*, IEEE Trans. Automat. Control **26** (1981), no. 5, 1139–1144.

[10] S. Das and P. Gupta, *A mathematical model on fractional Lotka-Volterra equations*, J. Theoret. Biol. **277** (2011), 1–6.

[11] D. Ding, D. Qi, J. Peng, and Q. Wang, *Asymptotic pseudo-state stabilization of commensurate fractional-order nonlinear systems with additive disturbance*, Nonlinear Dynam. **81** (2015), no. 1-2, 667–677.

[12] D. Ding, D. Qi, and Q. Wang, *Non-linear Mittag-Leffler stabilisation of commensurate fractional-order non-linear systems*, IET Control Theory Appl. **9** (2015), no. 5, 681–690.

[13] M. A. Duarte-Mermoud, N. Aguila-Camacho, J. A. Gallegos, and R. Castro-Linares, *Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems*, Commun. Nonlinear Sci. Numer. Simul. **22** (2015), no. 1-3, 650–659.

[14] B. Ghanmi, *Stability of impulsive systems depending on a parameter*, Mathematical Methods in the Applied Sciences, DOI: 10.1002/mma.3717.(2015)

[15] M. A. Hammami, *On the Stability of Nonlinear Control Systems with Uncertainty*, J. Dynam. Control Systems **7** (2001), no. 2, 171–179.

[16] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[17] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Application of Fractional Differential Equations*, Elsevier, New York, 2015.

[18] A. Leung, X. Li, Y. Chu, and X. Rao, *Synchronization of fractional-order chaotic systems using unidirectional adaptive full-state linear error feedback coupling*, Nonlinear Dynam. **82** (2015), no. 1-2, 185–199.

[19] Y. Li, Y. Q. Chen, and I. Podlubny, *Mittag-Leffler stability of fractional order nonlinear dynamic systems*, Automatica J. IFAC **45** (2009), no. 8, 1965–1969.

[20] Y. Lin, D. Sontag, and Y. Wang, *A smooth converse Lyapunov theorem for Robust Stability*, SIAM J. Control Optim. **34** (1996), no. 1, 124–160.

[21] S. Liu, W. Jiang, X. Li, and X. Zhou, *Lyapunov stability analysis of fractional nonlinear systems*, Appl. Math. Lett. **51** (2016), 13–19.

[22] M. S. Mahmoud, *Improved robust stability and feedback stabilization criteria for time-delay systems*, IMA J. Math. Control Inform. **26** (2009), no. 4, 451–466.

[23] D. Matignon, *Stability result on fractional differential equations with applications to control processing*, Proceedings of IMACS-SMC, 963–968, 1996.

[24] K. Miller and S. Samko, *A note on the complete monotonicity of the generalized Mittag-Leffler function*, Real Anal. Exchange **23** (1999), no. 2, 753–755.

[25] O. Naifar, A. Ben Makhlouf, and M. A. Hammami, *Comments on Lyapunov stability theorem about fractional system without and with delay*, Commun. Nonlinear Sci. Numer. Simul. **30** (2016), no. 1-3, 360–361.

[26] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego California, 1999.

[27] D. Qian, C. Li, R. P. Agarwal, and P. Wong, *Stability analysis of fractional differential system with Riemann-Liouville derivative*, Math. Comput. Modelling **52** (2010), no. 5-6, 862–874.

[28] Z. Qu, *Exponential stability of a class of nonlinear dynamical systems with uncertainties*, Systems Control Lett. **18** (1992), 301–307.

[29] Y. Ryabov and A. Puzenko, *Damped oscillation in view of the fractional oscillator equation*, Phys. Rev. **66** (2002), 184–201.

[30] Y. Wei, Y. Chen, S. Liang, and Y. Wang, *A novel algorithm on adaptive backstepping control of fractional order systems*, Math. Comput. Modelling **165** (2015), 395–402.

[31] H. Wu and K. Mizukami, *Exponential stability of a class of nonlinear dynamical systems with uncertainties*, Systems and Control Letters **21** (2016), 307–313.

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