Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1293–1307
https://doi.org/10.4134/BKMS.b160551
pISSN: 1015-8634 / eISSN: 2234-3016

ON A CLASS OF LOCALLY PROJECTIVELY FLAT GENERAL (α, β)-METRICS

Xiaohuan Mo and Hongmei Zhu

Abstract. General (α, β)-metrics form a rich class of Finsler metrics. They include many important Finsler metrics, such as Randers metrics, square metrics and spherically symmetric metrics. In this paper, we find equations which are necessary and sufficient conditions for such Finsler metric to be locally projectively flat. By solving these equations, we obtain all of locally projectively flat general (α, β)-metrics under certain condition. Finally, we manufacture explicitly new locally projectively flat Finsler metrics.

1. Introduction

Distance functions induced by a Finsler metric are regarded as smooth ones. The Hilbert Fourth Problem in the smooth case is characterize projectively flat Finsler metrics on an open subset in $\mathbb{R}^n$. A Finsler metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesics are straight in $\mathcal{U}$. A Finsler metric on a manifold M is said to be *locally projectively flat* if at any point, there is a local coordinate system (x^i) in which F is projectively flat. By the Beltrami's theorem, a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. However the situation is much more complicated for Finsler metrics. J. Douglas' a famous theorem said that a Finsler metric on a manifold M (dim $M \geq 3$) is locally projectively flat if and only if F is a Douglas metric of scalar curvature.

Recently, a great progress has been made in studying locally projectively flat Finsler metrics. Bácsó-Matsumoto showed that a Randers metric $F = \alpha + \beta$ is locally projectively flat if and only if α is of constant sectional curvature and β is closed [1, 2]. In [10], the authors told us that given a spherically symmetric

Received July 1, 2016; Revised September 26, 2016; Accepted November 3, 2016.
2010 *Mathematics Subject Classification.* 53B40, 53C60.
Key words and phrases. Finsler metric, general (α, β) type, locally projectively flat, flag curvature.
The first author was supported by the National Natural Science Foundation of China 11371032 and the second author was supported by the doctoral scientific research foundation of Henan Normal University 5101019170130 and Youth Science Fund of Henan Normal University (No.2015QK01).

Finsler metric $F(x, y) = |y|\phi\left(|x|, \frac{\langle x, y\rangle}{|y|}\right)$ on $\mathbb{B}^n(\nu)\,(n > 2)$, then F is locally projectively flat if and only if $\phi = \phi(r, s)$ satisfies

$$\left[(r^2 - s^2)Q - 1\right] r\phi_{ss} - s\phi_{rs} + \phi_r + rQ(\phi - s\phi_s) = 0, \tag{1.1}$$

where $Q = Q(r, s)$ is given by

$$Q(r, s) = f(r) + \frac{2rf(r)^2 + f'(r)}{r + 2f(r)r^3}s^2, \tag{1.2}$$

where $f = f(r)$ is a differentiable function.

A Randers metric can be expressed in the following navigation form [6]

$$F = \frac{\sqrt{(1-b^2)\alpha^2 + \beta^2}}{1-b^2} + \frac{\beta}{1-b^2},$$

where $(\alpha, \beta^\sharp)$ is the navigation data of F ($\beta^\sharp$ denotes the dual of β with respect to α) and $b := \|\beta\|_\alpha$ is the length of β. Both Randers metrics and spherically symmetric metrics belong to a larger class of the so-called general (α, β)-metrics, which are defined in the following form

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big), \tag{1.3}$$

where α is a Riemannian metric, β is a 1-form, $b := \|\beta\|_\alpha$ and $\phi\left(b^2, s\right)$ is a smooth function [7, 16]. Yu-Zhu found sufficient conditions for general (α, β)-metrics to be locally projectively flat [16]. They showed that for $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ where α and β satisfy

$${}^{\alpha}R^i{}_j = \mu(\alpha^2\delta^i{}_j - y^iy_j), \qquad b_{i|j} = c(x)a_{ij}, \tag{1.4}$$

the metric F is locally projectively flat if $\phi = \phi(b^2, s)$ satisfies

$$\phi_{22} = 2(\phi_1 - s\phi_{12}). \tag{1.5}$$

Here ϕ_1 means the derivation of ϕ with respect to the first variable b^2. Recently, Shen-Yu showed that if the second equation of (1.4) holds, then F is projectively equivalent to α if and only if $\phi = \phi(b^2, s)$ satisfies (1.5) [14]. Note that if α is locally projectively flat and F is projectively equivalent to α, then F is also locally projectively flat and therefore Shen-Yu's proposition implies Yu-Zhu's theorem.

The above results inspire us to study the locally projective flatness of general (α, β)-metrics under a weaker condition than (1.4). We shall make the following assumption:

A: the Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is an Einstein metric with Ricci constant μ, and $\beta = b_j(x)y^j$ is a 1-form satisfying

$${}^{\alpha}Ric = (n-1)\mu\alpha^2, \qquad b_{i|j} = ca_{ij}, \tag{1.6}$$

where $c = c(x)$ is a scalar function with $c^2 > 0$.

The condition **A** is natural [14]. Note that if α and β satisfy (1.6) with $c = 0$, then β is parallel with respect to α, in particular, $b =$ constant. In this case, F is actually an (α, β)-metrics. We shall only consider the case when $c^2 > 0$. We show the following:

Theorem 1.1. *Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α, β)-metric on an n-dimensional manifold M with $n \geq 3$, where α and β satisfy (1.6) with $c^2 > 0$. Then F is locally projectively flat if and only if*

(i) *$\phi = \phi(b^2, s)$ satisfies*

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = [f(b^2) + g(b^2)s^2][\phi - s\phi_2 + (b^2 - s^2)\phi_{22}], \tag{1.7}$$

where f and g are two arbitrary differentiable functions of b^2;

(ii) *α and β satisfy*

$$\begin{aligned} &{}^{\alpha}R_k{}^i{}_{jl}(x) + {}^{\alpha}R_l{}^i{}_{jk}(x) - 2\mu a_{kl}(x)\delta^i{}_j + \mu a_{jk}\delta^i{}_l + \mu a_{jl}\delta^i{}_k \\ &+ [(\kappa - \mu b^2)(2f' - g + 2f^2 + 2b^2 fg) - \mu f]S^i{}_{jkl}(x) = 0, \end{aligned} \tag{1.8}$$

where

$$\begin{aligned} S^i{}_{jkl}(x) := 2b_j b^i a_{kl} - \frac{2}{n-1}(b^2 a_{kl} - b_k b_l)\delta^i{}_j - \frac{1}{n-1} b_j(b_k\delta^i{}_l + b_l\delta^i{}_k) \\ - b^i(b_k a_{jl} + b_l a_{jk}) + \frac{b^2}{n-1}(a_{lj}\delta^i{}_k + a_{kj}\delta^i{}_l). \end{aligned} \tag{1.9}$$

For a proof of Theorem 1.1, see Section 3 below. In particular, we have the following result when α has constant sectional curvature.

Theorem 1.2. *Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α, β)-metric on a manifold M $(\dim M \geq 3)$ where α and β satisfy (1.4) with $c^2 > 0$. Then it is locally projectively flat if and only if $\phi = \phi(b^2, s)$ satisfies (1.7) where f and g satisfy*

$$\mu f = (\kappa - \mu b^2)(2f' - g + 2f^2 + 2b^2 fg). \tag{1.10}$$

Let us take a look at the special case: when $\alpha = |y|$, $\beta = \langle x, y\rangle$

$$\mu = 0, \qquad \kappa = 1.$$

Then we recover Theorem 1.1 in [10] (see (1.11) and (1.12) above).

Theorem 1.3. *Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α, β)-metric on a manifold M $(\dim M \geq 3)$ where α and β satisfy (1.4) with $c^2 > 0$. Then the general solution of (1.7) where f and g satisfy (1.10) is given by*

$$\phi(b^2, s) = s\left[h(b^2) - \int \frac{\Phi(\eta(b^2, s))}{s^2\sqrt{b^2 - s^2}} ds\right], \tag{1.11}$$

where h and Φ are arbitrary differentiable functions of b^2 and η respectively and

$$\eta(b^2, s) = \frac{b^2 - s^2}{e^{\int \xi db^2} - (b^2 - s^2)\int g e^{\int \xi db^2} db^2},$$

where

$$\xi := \frac{(\kappa - 2\mu b^2)f + 2b^2(\kappa - \mu b^2)f'}{(\kappa - \mu b^2)(1 - 2b^2 f)}, \tag{1.12}$$

$$g := \frac{2(\kappa - \mu b^2)(f' + f^2) - \mu f}{(\kappa - \mu b^2)(1 - 2b^2 f)}, \tag{1.13}$$

where f is arbitrary differentiable function of b^2. Moreover, the corresponding general (α, β)-metric is locally projectively flat.

After investigating (1.11), (1.12) and (1.13), we produce infinitely many new non-trivial examples satisfying the conditions and conclusions in Theorem 1.2. We have the following:

Theorem 1.4. *Let α be a Riemannian metric on a n-dimensional manifold M and β a 1-form on M satisfying $b := \|\beta\|_\alpha < b_0$ and (1.4) with $\mu = 0$. Let $n \geq 3$ and let $\phi(b^2, s)$ be functions defined by*

1. $m = 0$

$$\phi = h_1(b^2)s + (\varepsilon + \kappa)\frac{\omega}{1 - 2\lambda b}, \tag{1.14}$$

2. $m = 1$

$$\phi = h_2(b^2)s + \left(\varepsilon + \frac{\kappa b^2}{1 - 2\lambda b}\right)\frac{\omega}{1 - 2\lambda b}, \tag{1.15}$$

3. $m \geq 2$

If $\lambda \neq 0$, then

$$\phi = h_3(b^2)s + \left[\frac{\varepsilon}{1 - 2\lambda b} + \kappa\frac{(b^2 - s^2)^m}{\omega^{2m+2}}\right]\omega + \frac{(2m)!!}{(2m-1)!!}\frac{\kappa s^2}{(1 - 2\lambda b)\omega} \times (I), \tag{1.16}$$

where

$$\begin{aligned} (I) := {} & \frac{b^{2m-2}}{(1 - 2\lambda b)^m} + \frac{b(b^2 - s^2)}{2\lambda\omega^4}\left[\frac{\pi_1}{1 - 2\lambda b} + \frac{(2m-1)!!}{(2m)!!}\frac{2\lambda\pi_2}{b}\right] \\ & - \frac{b^2\omega^2 - s^2}{(1 - 2\lambda b)\omega^2}\left[\pi_3 - \frac{b^{2m-4}}{2(1 - 2\lambda b)^{m-1}}\right], \end{aligned} \tag{1.17}$$

where

$$\begin{aligned} \pi_1 &:= \sum_{j=0}^{m-3}\frac{(2m-2j-3)!!}{(2m-2j-2)!!}\frac{b^{2j}}{(1 - 2\lambda b)^j}\sum_{i=0}^{m-3-j}\left(-\frac{b}{2\lambda}\right)^i\left(\frac{b^2 - s^2}{\omega^2}\right)^{m-i-j-3}, \\ \pi_2 &:= \sum_{i=0}^{m-1}\left(-\frac{b}{2\lambda}\right)^{i-1}\left(\frac{b^2 - s^2}{\omega^2}\right)^{m-i-1}, \\ \pi_3 &:= \frac{b^{2i-2}}{(1 - 2\lambda b)^{i-1}} + \frac{(2m-1)!!}{(2m)!!}\left(-\frac{b}{2\lambda}\right)^{m-2} \end{aligned}$$

$$+\frac{b}{2\lambda(1-2\lambda b)}\sum_{i=1}^{m-2}\frac{(2m-2i-1)!!}{(2m-2i)!!}\left(-\frac{b}{2\lambda}\right)^{m-2}.$$

If $\lambda = 0$, then

$$\phi = \varepsilon + h_4(b^2)s + \kappa(b^2-s^2)^m$$
$$+\kappa s^2\frac{(2m)!!}{(2m-1)!!}\left[b^{2m-2}+\sum_{i=1}^{m-1}\frac{(2m-2i-1)!!}{(2m-2i)!!}b^{2i-2}(b^2-s^2)^{m-i}\right], \tag{1.18}$$

where κ and ε are positive constants, $\lambda < \frac{1}{2b_0}$, $h_j(b^2)$ are arbitrary differentiable functions, $j = 1, \ldots, 4$, $\omega := \sqrt{1-2\lambda(b-\frac{s^2}{b})}$. Then on M the following general (α, β)-metrics

$$F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$$

are locally projectively flat.

The condition (1.4) on α and β is equivalent to: (i) α has constant sectional curvature; (ii) β is closed and conformal with respect to α. Locally, it is easy to determine all these α and β following Y. Shen or Z. Shen and H. Xing [11, 13]. Let us take a look at a special case: when $\alpha = |y|$, $\beta = \langle x, y\rangle$

$$b = |x|.$$

Then α and β satisfy (1.4) with $\mu = 0$ and $c(x) = 1$. Such Finsler metrics are called *spherically symmetric metrics*.

Recently, Liu-Mo and Zhu have discovered some class of locally projectively flat spherically symmetric metrics [5, 18]. The metrics constructed by Liu-Mo (resp. Zhu) satisfy that $f + gs^2 = \frac{\lambda c}{a+cb^2}$ (resp. $\frac{1}{2} + \frac{\lambda s^2}{2(1-\lambda b^2)}$). From proof of Theorem 1.4, we see that Finsler metrics in Theorem 1.4 satisfy

$$f + gs^2 = \lambda\frac{b^2-s^2}{b^3}.$$

It follows that our metrics are not projectively related to Finsler metric constructed by Liu-Mo and Zhu because two spherically symmetric metrics are projectively related if and only if they have same $f + gs^2$ (see Theorem 1.1 in [8]). In particular, the Finsler metrics in Theorem 1.4 differ from the metrics in [5, 18].

Finally, we should point out that there are already several results on the locally projectively flat general (α, β)-metrics. For example, Z. Shen characterizes projectively flat (α, β)-metrics on an open subset $\mathcal{U}$ in the n-dimensional space $\mathbb{R}^n$ ([12], Theorem 1.1). Q. Xia gives an equivalent characterization for locally projectively flat general (α, β)-metrics on a manifold under the additional assumptions α is locally projectively flat and $\phi_1 \neq 0$ and determines the corresponding general solution [15]. Note that if $\phi_1 = 0$, then general (α, β)-metric is just a (α, β)-metric. On the other hand, locally projectively flat Finsler metrics on a manifold, in particular, on an open subset $\mathcal{U}$ in the n-dimensional space $\mathbb{R}^n$, form a broader class than projectively flat Finsler metrics on an open subset $\mathcal{U}$ [9]. Here we weaken Shen's condition on projective flatness in the case of $\phi_1 = 0$ and impose the second equation of (1.6) with $c^2 > 0$ instead. Comparing our Theorem 1.3 with Xia's Theorem 1.2, it seems that the explicit construction of locally projectively flat general (α, β)-metrics using our theorem is easier than Xia's theorem.

2. Preliminaries

In local coordinates, the geodesics of a Finsler metric $F = F(x, y)$ are characterized by

$$\frac{d^2x^i}{dt^2} + 2G^i\left(x,\, \frac{dx}{dt}\right) = 0,$$

where $G^i := \frac{1}{4}g^{il}\left[(F^2)_{x^k y^l}y^k - (F^2)_{x^l}\right]$ are the geodesic coefficients of F [2]. The *Riemann curvature* of F is a family of endomorphism $R_y = R^i_j dx^j \otimes \frac{\partial}{\partial x^i} : T_xM \to T_xM$, defined by

$$R^i{}_j := 2\frac{\partial G^i}{\partial x^j} - y^k\frac{\partial^2 G^i}{\partial x^k \partial y^j} + 2G^k\frac{\partial^2 G^i}{\partial y^k \partial y^j} - \frac{\partial G^i}{\partial y^k}\frac{\partial G^k}{\partial y^j}. \tag{2.1}$$

The *Ricci curvature* is the trace of the Riemann curvature, which is defined by

$$Ric = R^j{}_j.$$

A Finsler metric on a manifold M in the form (1.3) is said to be *general* $(\alpha,\, \beta)$ *type* where α is a Riemannian metric, β is a 1-form on M, $b = \|\beta\|_\alpha$ and $\phi(b^2,\, s)$ is a C^∞ function satisfying (see [7, 16])

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \quad |s| \leq b < b_o,$$

where $n \geq 3$ and ϕ_2 means the derivation of ϕ with respect to the second variable s or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \quad |s| \leq b < b_o,$$

where $n = 2$ [16]. In particular, we will say that F is a *Randers metric* if $\phi(b^2,\, s) = 1 + s$.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ and $\nabla\beta = b_{i|j}dx^i \otimes dx^j$. Let

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}), \quad r_{00} = r_{ij}y^iy^j, \quad s^i_0 = a^{ij}s_{jk}y^k,$$

$$r_i = b^jr_{ji}, \quad s_i = b^js_{ji}, \quad r_0 = r_iy^i, \quad s_0 = s_iy^i,$$
$$r^i = a^{ij}r_j, \quad s^i = a^{ij}s_j, \quad r = b^jr_j.$$

Lemma 2.1 (See [16]). *The geodesic coefficients G^i of a general (α, β)-metric $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ are given by*

$$G^i = {}^\alpha G^i + \alpha Q s^i{}_0 + \left\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Omega(r_0 + s_0)\right\}\frac{y^i}{\alpha}$$

$$+\left\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Pi(r_0 + s_0)\right\} b^i - \alpha^2 R(r^i + s^i), \tag{2.2}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2},$$

$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Psi = \frac{\phi_{22}}{2\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)},$$

$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

Suppose that β is conformal with respect to α and satisfies $d\beta = 0$, there is a scalar function $c = c(x)$ such that the second equation of (1.6) holds. By (2.2) and the second equation of (1.6), the geodesic coefficients G^i of F is given by

$$G^i = {}^{\alpha}G^i + Q^i, \tag{2.3}$$

where ${}^{\alpha}G^i$ are the geodesic coefficients of α; and

$$Q^i := c\psi\alpha y^i + c\zeta\alpha^2 b^i, \tag{2.4}$$

$$\zeta := \frac{\phi_{22} - 2(\phi_1 - s\phi_{12})}{2\big[\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big]}, \tag{2.5}$$

$$\psi := \frac{\phi_2 + 2s\phi_1}{2\phi} - \frac{\zeta}{\phi}[s\phi + (b^2 - s^2)\phi_2]. \tag{2.6}$$

3. Riemannian metric α with Ricci constant

In this section, we are going to determine the Weyl curvature of general (α, β)-metrics satisfy (1.6) with $c^2 > 0$ (see (3.33) below) and prove Theorem 1.1.

From (2.1) and (2.3), the Riemann curvature of F are related to that of α and given by

$$R^i{}_j = {}^{\alpha}R^i{}_j + 2Q^i{}_{|j} - y^k Q^i{}_{|k\cdot j} + 2Q^k Q^i{}_{\cdot k\cdot j} - Q^i{}_{\cdot k} Q^k{}_{\cdot j}, \tag{3.1}$$

where "$|$" and "$\cdot$" denote the horizontal covariant derivative and vertical covariant derivative with respect to α, respectively. Suppose that α and β satisfy (1.6). By Lemma 2.1 in [14], we have

$$c^2 = \kappa - \mu b^2 \tag{3.2}$$

for some constant κ. We assume that $c^2 > 0$. Then

$$c_{|j} = -\mu b_j. \tag{3.3}$$

Combining with (2.4), we get

$$\begin{aligned} Q^i{}_{|j} &= c^2(\zeta\alpha^2\delta^i{}_j + \psi_2 y_j y^i + \zeta_2\alpha y_j b^i + 2\psi_1\alpha b_j y^i + 2\zeta_1\alpha^2 b_j b^i) \\ &\quad - \mu\alpha b_j(\psi y^i + \zeta\alpha b^i), \end{aligned} \tag{3.4}$$

$$y^k Q^i_{|k\cdot j} = c^2\big\{(2s\psi_1 + \psi_2)\alpha^2\delta^i{}_j + [2s(\psi_1 - s\psi_{12}) + \psi_2 - s\psi_{22} + 2\zeta - s\zeta_2]y_j y^i$$

$$
\begin{aligned}
&+ [2s(2\zeta_1 - s\zeta_{12}) + 2\zeta_2 - s\zeta_{22}]\alpha y_j b^i + (2s\psi_{12} + \psi_{22} + \zeta_2)\alpha b_j y^i \\
&+ (2s\zeta_{12} + \zeta_{22})\alpha^2 b_j b^i\big\} - \mu s\big[\psi\alpha^2\delta^i{}_j + (\psi - s\psi_2)y_j y^i \\
&+ (2\zeta - s\zeta_2)\alpha y_j b^i + \psi_2\alpha b_j y^i + \zeta_2\alpha^2 b_j b^i\big],
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\frac{1}{c^2}Q^k Q^i{}_{\cdot k\cdot j} = {}& \big\{\psi^2 + \zeta[s\psi + (b^2 - s^2)\psi_2]\big\}\alpha^2\delta^i{}_j \\
&+ \big\{\psi(\psi - s\psi_2) - s\zeta[\psi - s\psi_2 + (b^2 - s^2)\psi_{22}]\big\}y_j y^i \\
&+ \big[\zeta(\psi - s\psi_2) + \psi(2\zeta - s\zeta_2) + (b^2 - s^2)\zeta(\zeta_2 - s\zeta_{22})\big]\alpha y_j b^i \\
&+ \big\{\psi\psi_2 + \zeta[\psi - s\psi_2 + (b^2 - s^2)\psi_{22}]\big\}\alpha b_j y^i \\
&+ \big\{\psi\zeta_2 + \zeta\psi_2 + \zeta[2\zeta + (b^2 - s^2)\zeta_{22}]\big\}\alpha^2 b_j b^i,
\end{aligned} \tag{3.6}
$$

$$
\begin{aligned}
\frac{1}{c^2}Q^i{}_{\cdot k}Q^k{}_{\cdot j} = {}& \psi^2\alpha^2\delta^i{}_j + \big\{3\psi(\psi - s\psi_2) + [s\psi + (b^2 - s^2)\psi_2](2\zeta - s\zeta_2)\big\}y_j y^i \\
&+ \big\{s\zeta_2(\psi - s\psi_2) + (2\zeta - s\zeta_2)[3\psi - s\psi_2 + 2s\zeta + (b^2 - s^2)\zeta_2]\big\}\alpha y_j b^i \\
&+ \big[3\psi\psi_2 + s\psi\zeta_2 + (b^2 - s^2)\zeta_2\psi_2\big]\alpha b_j y^i \\
&+ \big[2\zeta_2\psi + 2\zeta\psi_2 + 2s\zeta\zeta_2 + (b^2 - s^2)\zeta_2^2\big]\alpha^2 b_j b^i.
\end{aligned} \tag{3.7}
$$

Substituting (3.4), (3.5), (3.6) and (3.7) into (3.1) yields

$$
\begin{aligned}
R^i{}_j = {}& {}^{\alpha}R^i{}_j + \alpha^2(A - \mu)\delta^i{}_j + \alpha B b_j y^i - \alpha s C y_j b^i + \alpha^2 C b_j b^i \\
&+ (\mu - A - sB)y_j y^i,
\end{aligned} \tag{3.8}
$$

where

$$
A := \mu(1 + s\psi) + c^2\big\{\psi^2 - 2s\psi_1 - \psi_2 + 2\zeta[1 + s\psi + (b^2 - s^2)\psi_2]\big\}, \tag{3.9}
$$

$$
\begin{aligned}
B := {}& c^2\big\{2(2\psi_1 - s\psi_{12}) - \psi\psi_2 - \psi_{22} - \zeta_2[1 + s\psi + \psi_2(b^2 - s^2)] \\
&+ 2\zeta[\psi - s\psi_2 + \psi_{22}(b^2 - s^2)]\big\} - \mu(2\psi - s\psi_2),
\end{aligned} \tag{3.10}
$$

$$
\begin{aligned}
C := {}& c^2\big[2(2\zeta_1 - s\zeta_{12}) - \zeta_{22} + 2\zeta(2\zeta - s\zeta_2) + (2\zeta\zeta_{22} \\
&- \zeta_2^2)(b^2 - s^2)\big] - \mu(2\zeta - s\zeta_2).
\end{aligned} \tag{3.11}
$$

Thus we obtain the following:

Lemma 3.1. *Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α, β)-metric on a manifold M where α and β satisfy* (1.6) *with $c^2 > 0$. Then the Riemannian curvature of F are given by* (3.8).

Remark 3.2. When $\alpha = |y|$ and $\beta = \langle x, y\rangle$, (3.8) has been obtained in [5].

By (3.8) and the first equation of (1.6), we obtain

$$
\begin{aligned}
Ric &= {}^{\alpha}Ric + n\alpha^2 A - n\mu\alpha^2 + \alpha\beta B - \alpha\beta s C + \alpha^2 C b^2 + (\mu - A - sB)\alpha^2 \\
&= [(n-1)A + (b^2 - s^2)C]\alpha^2.
\end{aligned} \tag{3.12}
$$

Now we are going to determine the Weyl curvature of general $(\alpha,\ \beta)$-metrics satisfy (1.6) with $c^2 > 0$ (see (3.33) below). Let

$$\dot{R}^i{}_j := \alpha^2 A\delta^i{}_j + \alpha B b_j y^i - \alpha s C y_j b^i + \alpha^2 C b_j b^i + D y_j y^i, \tag{3.13}$$

where

$$D := -A - sB. \tag{3.14}$$

Combining this with (3.8) we have

$$R^i{}_j = {}^{\alpha}R^i{}_j + \dot{R}^i{}_j - \mu(\delta^i{}_j \alpha^2 + y_j y^i). \tag{3.15}$$

By simple calculations, we have

$$\frac{\partial \alpha^2}{\partial y^k} = 2y_k, \qquad \frac{\partial \alpha}{\partial y^k} = \frac{y_k}{\alpha}, \qquad \frac{\partial y_j}{\partial y^k} = a_{jk}. \tag{3.16}$$

Differentiating (3.13) with respect to y^k and using (3.13), we obtain

$$\begin{aligned}\frac{\partial \dot{R}^i{}_j}{\partial y^k} &= 2A\delta^i{}_j y_k + \alpha^2 A_2 s_{y^k}\delta^i{}_j + \frac{B}{\alpha} y_k b_j y^i + \alpha B_2 s_{y^k} b_j y^i + \alpha B b_j \delta^i{}_k \\ &- \frac{sC}{\alpha} y_k y_j b^i - \alpha(C + sC_2) s_{y^k} y_j y^i - \alpha s C a_{jk} b^i + 2C y_k b_j b^i \\ &+ \alpha^2 C_2 s_{y^k} b_j b^i + D_2 s_{y^k} y_j y^i + D a_{jk} y^i + D y_j \delta^i{}_k.\end{aligned} \tag{3.17}$$

By a straightforward computation, one obtains

$$s_{y^j} b^j = \frac{\alpha b_j - s y_j}{\alpha^2} b^j = \frac{b^2 - s^2}{\alpha}. \tag{3.18}$$

Note that s is positively homogeneous of degree 0. Hence,

$$s_{y^i} y^i = 0.$$

Taking this together with (3.17) and (3.18), we obtain

$$\sum_i \frac{\partial \dot{R}^i{}_j}{\partial y^i} = \alpha \mathcal{M} b_j + \mathcal{N} y_j, \tag{3.19}$$

where

$$\mathcal{M} := (n+1)B + A_2 + sC + C_2(b^2 - s^2)$$

and

$$\mathcal{N} := (1-n)A - (n+1)sB - sA_2 - Cs^2 - s(b^2 - s^2)C_2,$$

where we have used (3.14). By (3.13) and (3.14), we have

$$\dot{Ric} := \sum_{j=1}^{n} \dot{R}^j{}_j = \alpha^2 R = Ric, \tag{3.20}$$

where

$$R := (n-1)A + (b^2 - s^2)C,$$

and we have used (3.12). It follows that

$$\frac{\partial \dot{Ric}}{\partial y^j}=(2R-sR_2)y_j-\alpha R_2 b_j,$$

where we have made use of (3.16) and (3.18) and $R_2:=\frac{\partial R}{\partial s}$. Together with (3.13), (3.19) and (3.20), we obtain

$$\begin{aligned}
\dot{W}^i{}_j &:= \dot{R}^i{}_j-\frac{\dot{Ric}}{n-1}\delta^i{}_j-\frac{y^i}{n+1}\Big(\sum_k\frac{\partial \dot{R}^k{}_j}{\partial y^k}-\frac{1}{n-1}\frac{\partial \dot{Ric}}{\partial y^j}\Big)\\
&=\alpha^2 A\delta^i{}_j+\alpha B b_j y^i-\alpha s C y_j b^i+\alpha^2 C b_j b^i+D y_j y^i-\frac{\alpha^2}{n-1}R\delta^i{}_j\\
&\quad -\frac{y^i}{n+1}\big\{\alpha(\mathcal{M}-\frac{R_2}{n-1})b_j+[\mathcal{N}-\frac{1}{n-1}(2R-sR_2)]y_j\big\}\\
&=\alpha^2 W_1\delta^i{}_j+\alpha W_2 b_j y^i-\alpha s W_3 y_j b^i+\alpha^2 W_3 b_j b^i+W_4 y_j y^i, \qquad (3.21)
\end{aligned}$$

where

$$W_1:=A-\frac{1}{n-1}R=-\frac{b^2-s^2}{n-1}C, \qquad (3.22)$$

$$W_2:=B-\frac{1}{n+1}(\mathcal{M}-\frac{R_2}{n-1})=-\frac{1}{n-1}\big[sC+\frac{n-2}{n+1}C_2(b^2-s^2)\big], \qquad (3.23)$$

$$W_3:=C, \qquad (3.24)$$

$$\begin{aligned}
W_4 &:= D-\frac{1}{n+1}\big[\mathcal{N}-\frac{1}{n-1}(2R-sR_2)\big]\\
&=\frac{1}{n-1}\big[b^2C+\frac{n-2}{n+1}s(b^2-s^2)C_2\big], \qquad (3.25)
\end{aligned}$$

where we have used the following

$$R_2=(n-1)A_2-2sC+(b^2-s^2)C_2.$$

By (3.15) and (3.21), we see that the (*projective*) *Weyl curvature* $W_y=W^i{}_j\frac{\partial}{\partial x^i}\otimes dx^j$ is given by

$$\begin{aligned}
W^i{}_j &= R^i{}_j-\frac{Ric}{n-1}\delta^i{}_j-\frac{y^i}{n+1}\Big(\sum_k\frac{\partial R^k{}_j}{\partial y^k}-\frac{1}{n-1}\frac{\partial Ric}{\partial y^j}\Big)\\
&={}^{\alpha}W^i{}_j+\dot{W}^i{}_j\\
&={}^{\alpha}W^i{}_j+\alpha^2 W_1\delta^i{}_j+\alpha W_2 b_j y^i-\alpha s W_3 y_j b^i+\alpha^2 W_3 b_j b^i+W_4 y_j y^i, \qquad (3.26)
\end{aligned}$$

where ${}^{\alpha}W^i{}_j$ is the Weyl curvature of α.

Now we compute the Weyl curvature ${}^{\alpha}W^i{}_j$ of α where α satisfies the first equation of (1.6). Since α is a Riemannian metric, we conclude that $R^i{}_j=R^i{}_j(x,y)$ are quadratic in $y\in T_xM$ [4]. We obtain

$$ {}^{\alpha}R^i{}_j={}^{\alpha}R_k{}^i{}_{jl}(x)y^k y^l, \qquad {}^{\alpha}Ric=\sum_j {}^{\alpha}R_k{}^j{}_{jl}(x)y^k y^l, \qquad (3.27)$$

where ${}^{\alpha}R_k{}^i{}_{jl}(x)$ is the Riemannian curvature of α. We conclude that

$$ {}^{\alpha}A^i{}_j := {}^{\alpha}R^i_j - \frac{Ric}{n-1}\delta^i{}_j = \big(R_k{}^i{}_{jl} - \frac{\delta^i{}_j}{n-1}R_k{}^m{}_{ml}\big)y^k y^l $$

and

$$ \sum_k \frac{\partial^{\alpha}A^k{}_j}{\partial y^k} = (R_p{}^k{}_{jq} - \frac{\delta^k{}_j}{n-1}R_p{}^m{}_{mq})\frac{\partial}{\partial y^k}(y^p y^q) = -\frac{n+1}{n-1}R_{jp}y^p, $$

where we have used the following facts

$$ R_k{}^k{}_{jp} + R_j{}^k{}_{pk} + R_p{}^k{}_{kj} = 0, \qquad R_p{}^k{}_{kj} = R_{pj} = R_{jp}. $$

Thus the Weyl curvature ${}^{\alpha}W^i{}_j$ of α is given by

$$ \begin{aligned} {}^{\alpha}W^i{}_j &= {}^{\alpha}A^i{}_j - \frac{1}{n+1}\sum_k \frac{\partial A^k{}_j}{\partial y^k}y^i \\ &= \big[{}^{\alpha}R_k{}^i{}_{jl}(x) - \mu a_{kl}(x)\delta^i{}_j + \mu a_{jk}\delta^i{}_l\big]y^k y^l. \end{aligned} \tag{3.28} $$

Plugging (3.28) into (3.26) yields

$$ \begin{aligned} W^i{}_j = {} & \big[{}^{\alpha}R_k{}^i{}_{jl}(x) - \mu a_{kl}(x)\delta^i{}_j + \mu a_{jk}\delta^i{}_l\big]y^k y^l \\ & + \alpha^2 W_1\delta^i{}_j + \alpha W_2 b_j y^i - \alpha s W_3 y_j b^i + \alpha^2 W_3 b_j b^i + W_4 y_j y^i. \end{aligned} \tag{3.29} $$

Now we consider Douglas metric $F = \alpha\phi(b^2, \frac{\beta}{\alpha})$, where α and β satisfy (1.6) with $c^2 > 0$. By Theorem 1.1 in [17],

$$ \zeta = \frac{1}{2}\left[f(b^2) + g(b^2)s^2\right], \tag{3.30} $$

where f and g are two arbitrary differentiable functions of b^2. It follows that

$$ \zeta_1 = \frac{1}{2}(f' + g's^2),\ \zeta_2 = gs,\ \zeta_{12} = g's,\ \zeta_{22} = g. $$

Plugging these into (3.11) yields

$$ C = c^2(2f' - g + 2f^2 + 2b^2 fg) - \mu f. \tag{3.31} $$

Recall that a Finsler metric on a manifold M is called a *Douglas metric* if it has vanishing Douglas curvature. Plugging (3.2) into (3.31) we have

$$ C = (\kappa - \mu b^2)(2f' - g + 2f^2 + 2b^2 fg) - \mu f = C(b^2). \tag{3.32} $$

It follows that

$$ C_2 = 0. $$

Plugging this into (3.23) and (3.25), and using (3.22) and (3.24) we obtain

$$ \big(W_1, W_2, W_3, W_4\big) = C\left(-\frac{b^2 - s^2}{n-1},\ -\frac{s}{n-1},\ 1,\ \frac{b^2}{n-1}\right). $$

Substituting this and (3.32) into (3.29) yields

(3.33)

$$ W^i{}_j := \big\{{}^{\alpha}R_k{}^i{}_{jl}(x) - \mu a_{kl}(x)\delta^i{}_j + \mu a_{jk}\delta^i{}_l + [(\kappa - \mu b^2)(2f' - g + 2f^2 + 2b^2 fg) $$

$$- \mu f] T^i{}_{jkl} \} y^k y^l,$$

where

$$T^i{}_{jkl}(x) := b_j b^i a_{kl} - \frac{1}{n-1}(b^2 a_{kl} - b_k b_l)\delta^i{}_j - \frac{1}{n-1} b_j b_k \delta^i{}_l \\ - b^i b_k a_{jl} + \frac{b^2}{n-1} a_{lj} \delta^i{}_k. \tag{3.34}$$

Proof of Theorem 1.1. According to Douglas' result, Finsler metric $F(x, y)$ on an n-dimensional manifold M with $n \geq 3$ is locally projectively flat if and only if F has vanishing Weyl curvature and Douglas curvature [3]. On the other hand, note that $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ is a general (α, β)-metric on M where α and β satisfy (1.6) with $c^2 > 0$, we have the following:

(i) F has vanishing Douglas curvature if and only if $\phi = \phi(b^2, s)$ satisfies (1.7).

(ii) If $\phi = \phi(b^2, s)$ satisfies (1.7) where f and g are two arbitrary differentiable functions of b^2, then F has vanishing Weyl curvature if and only if (1.8) holds where $S^i{}_{jkl}(x)$ satisfy (1.9) and we have used (3.33) and (3.34). Now our theorem is an immediate conclusion of (i), (ii) and Douglas' result. □

4. Riemannian metric α with constant sectional curvature

In this section, we assume that α is of constant sectional curvature μ. Then we have

$${}^{\alpha}R^i{}_j = \mu(\alpha^2 \delta^i{}_j - y^i y_j) = \mu\left[a_{kl}(x)\delta^i{}_j - a_{jk}(x)\delta^i{}_l\right] y^k y^l. \tag{4.1}$$

Plugging this into (3.8) yields

$$R^i{}_j = \alpha^2 A \delta^i{}_j + \alpha B b_j y^i - \alpha s C y_j b^i + \alpha^2 C b_j b^i + D y_j y^i, \tag{4.2}$$

where D satisfies (3.14). Combing (4.1) with the first equation of (3.27) yields

$${}^{\alpha}R_k{}^i{}_{jl}(x) + {}^{\alpha}R_l{}^i{}_{jk}(x) = \mu\left[2a_{kl}(x)\delta^i{}_j - a_{jl}(x)\delta^i{}_k - a_{jk}(x)\delta^i{}_l\right]$$

and

$${}^{\alpha}R_k{}^i{}_{jl}(x) y^k y^l = \mu\left[a_{kl}(x)\delta^i{}_j - a_{jk}(x)\delta^i{}_l\right] y^k y^l. \tag{4.3}$$

Plugging (4.3) into (3.29) yields

$$W^i{}_j = \alpha^2 W_1 \delta^i{}_j + \alpha W_2 b_j y^i - \alpha s W_3 y_j b^i + \alpha^2 W_3 b_j b^i + W_4 y_j y^i, \tag{4.4}$$

where W_1, W_2, W_3, W_4 are given by (3.22), (3.23), (3.24) and (3.25), respectively. Thus we have the following.

Proposition 4.1. *Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a general (α, β)-metric on a manifold M where α and β satisfy* (1.4) *with $c^2 > 0$. Then F is a Weyl metric if and only if*

$$C := c^2\left[2(2\zeta_1 - s\zeta_{12}) - \zeta_{22} + 2\zeta(2\zeta - s\zeta_2) + (2\zeta\zeta_{22} - \zeta_2^2)(b^2 - s^2)\right] \\ - \mu(2\zeta - s\zeta_2) = 0, \tag{4.5}$$

where ζ *is given in* (2.5).

Recall that a Finsler metric is called a *Weyl metric* if it has vanishing Weyl curvature. We have the following interesting result [5]: A Finsler metric has vanishing Weyl curvature if and only if it is of scalar curvature. It follows that the Weyl curvature gives a measure of the failure of a Finsler metric to be of scalar (flag) curvature.

Proof of Theorem 1.2. Suppose that (1.7) and (1.10) hold. Then F is a Douglas metric by Proposition 2.3(b) in [14]. Substituting (1.10) into (3.31) and using (3.2) we obtain $C = 0$. Hence F is of scalar curvature. Douglas' result implies that F is locally projectively flat.

Conversely, suppose that F is locally projectively flat. Then it is a Douglas metric with scalar curvature. Thus F satisfies (1.7). Moreover $C = 0$ by Proposition 4.1. Together with (3.2) and (3.31) we have (1.10). $\square$

Note that when $\alpha = |y|$ and $\beta = \langle x, y\rangle$ recover Theorem 1.1 in [10].

Proof of Theorem 1.3. (1.13) can be obtained from (1.10). It follows that

$$\begin{aligned}\xi := f + gb^2 &= f + \frac{2(\kappa - \mu b^2)(f' + f^2) - \mu f}{(\kappa - \mu b^2)(1 - 2b^2 f)} b^2\\ &= \frac{(\kappa - 2\mu b^2)f + 2b^2(\kappa - \mu b^2)f'}{(\kappa - \mu b^2)(1 - 2b^2 f)}.\end{aligned}$$

Now our result can be obtained from (1.12), (1.13) and Theorem 1.2 in [17]. $\square$

Proof of Theorem 1.4. Let us consider the special case of (1.12) and (1.4) in $\mu = 0$ and $f = \frac{\lambda}{b}$. In this case,

$$(\xi,\, g) = \left(0,\, -\frac{\lambda}{b^3}\right),$$

$$\eta(b^2,\, s) = \frac{b^2 - s^2}{1 - (b^2 - s^2)\int\left(-\frac{\lambda}{b^3}\right) db^2} = \frac{b^2 - s^2}{\omega^2},$$

where $\omega := \sqrt{1 - 2\lambda(b - \frac{s^2}{b})}$. Plugging this into (1.11) yields

$$\phi = s\left[h(b^2) - \int \frac{\Phi(\frac{b^2 - s^2}{\omega^2})}{s^2\sqrt{b^2 - s^2}} ds\right] = s\left[h(b^2) - \int \frac{\varphi\left(\frac{b^2 - s^2}{\omega^2}\right)}{\omega s^2} ds\right], \tag{4.6}$$

where $\varphi(x) := \frac{\Phi(x)}{\sqrt{x}}$. A direct calculation yields

$$\phi - s\phi_s = \frac{\varphi\left(\frac{b^2 - s^2}{\omega^2}\right)}{\omega}. \tag{4.7}$$

Differentiating (4.7) with respect to s, we obtain

$$\phi_{22} = \frac{2\lambda}{b\omega^3}\varphi(x) + \frac{2}{\omega^5}\varphi'(x),$$

where $x := \frac{b^2-s^2}{\omega^2}$. Taking this together with (4.7) yields

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} = \frac{1}{\omega^3}\left[\varphi(x) + 2x\varphi'(x)\right]. \tag{4.8}$$

Note that $\lambda \le \frac{1}{2b_0}$ because of $\omega > 0$. Considering $F(x,\, y) = \alpha\phi\left(b^2,\, \frac{\beta}{\alpha}\right)$ where ϕ satisfies (4.6), from (4.7) and (4.8), we obtain that F is a Finsler metric if and only if the positive function ϕ satisfies

$$\varphi(x) > 0, \quad \varphi(x) + 2x\varphi'(x) > 0, \tag{4.9}$$

where $x \ge 0$.

Taking $\varphi(x) = \kappa x^m + \varepsilon$ where $m \in \{0,\, 1,\, 2, \ldots\}$, κ and ε are positive constants, we see that φ satisfies (4.9). Moreover

$$\begin{aligned}\phi(b^2,\, s) &= s\left\{h(b^2) - s\int\left[\kappa\left(\frac{b^2-s^2}{\omega^2}\right)^m + \varepsilon\right]\frac{ds}{\omega s^2}\right\}\\ &= \tilde{h}_(b^2)s + \frac{\omega}{1-2\lambda b} - \kappa s\int\frac{(b^2-s^2)^m}{s^2\omega^{2m+1}}ds.\end{aligned} \tag{4.10}$$

We require the following result in [18].

Lemma 4.2. *Let*

$$T_m := \int\frac{(b^2-s^2)^m}{s^2\omega^{2m+1}}ds,$$

where $\omega := \sqrt{1-2\lambda(b-\frac{s^2}{b})}$, λ *is a constant and* b *is independent of* s. *Then for* $m \ge 2$, *we have*

1) $\lambda \neq 0$

$$T_m = -\frac{(2m)!!}{(2m-1)!!}\frac{s}{(1-2\lambda b)\omega}\times(I) - \frac{(b^2-s^2)^m}{s\omega^{2m+1}} + t_1(b^2),$$

where (I) *is given in* (1.17).

2) $\lambda = 0$

$$T_m = -\frac{(b^2-s^2)^m}{s} - \frac{(2m)!!s}{(2m-1)!!}\left[b^{2m-2} + \sum_{i=1}^{m-1}\frac{(2m-2i-1)!!}{(2m-2i)!!}b^{2i-2}(b^2-s^2)^{m-i}\right] + t_2(b^2).$$

Furthermore,

$$T_0 := -\frac{\omega}{s(1-2\lambda b)} + t_3(b^2), \quad T_1 := -\frac{b^2\omega^2+s^2}{s\omega(1-2\lambda b)^2} + t_4(b^2),$$

where $1-2\lambda b > 0$ *and* $t_j(b^2)$ *are arbitrary differentiable functions,* $j = 1, \ldots, 4$.

Now Theorem 1.4 is an immediate consequence of (4.10) and Lemma 4.2. □

References

[1] S. Bácsó and M. Matsumoto, *On Finsler spaces of Douglas typea generalization of the notion of Berwald space*, Publ. Math. Debrecen **51** (1997), no. 3-4, 385–406.
[2] S. S. Chern and Z. Shen, *Riemann-Finsler geometry*, Nankai Tracts in Mathematics, 6. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005. x+192pp.
[3] J. Douglas, *The general geometry of paths*, Ann. Math. **29** (1927/1928), no. 1-4, 143–168.
[4] B. Li and Z. Shen, *On Randers metrics of quadratic Riemann curvature*, Internat. J. Math. **20** (2009), no. 3, 369–376.
[5] H. Liu and X. Mo, *Examples of Finsler metrics with special curvature properties*, Math. Nachr. **288** (2015), no. 13, 1527–1537.
[6] X. Mo, *A global classification result for Randers metrics of scalar curvature on closed manifolds*, Nonlinear Anal. **69** (2008), 2996–3004.
[7] ———, *Finsler metrics with constant* (*or scalar*) *flag curvature*, Proc. Indian Acad. Sci. Math. Sci. **122** (2012), no. 3, 411–427.
[8] X. Mo and X. Wang, *On projectively related spherically symmetric Finsler metrics*, Publ. Math. Debrecen. **88** (2016), no. 1-2, 249–259.
[9] X. Mo, L. Zhou, and H. Zhu, *On a class of Finsler metrics of constant curvature*, accepted for publication in: Houston Journal of Mathematics.
[10] X. Mo and H. Zhu, *On a class of locally projectively flat Finsler metrics*, to appear in: Bulletin of the Iran Mathematical Society.
[11] Y. Shen, *General solutions of some differential equations on Riemannian manifolds*, (Chinese) J. Math. Res. Exposition **2** (1982), no. 2, 55–60.
[12] Z. Shen, *On projectively flat* (α, β)*-metrics*, Canad. Math. Bull. **52** (2009), no. 1, 132–144.
[13] Z. Shen and H. Xing, *On Randers metrics with isotropic* S*-curvature*, Acta Math. Sin. (Engl. Ser.) **24** (2008), no. 5, 789–796.
[14] Z. Shen and C. Yu, *On a class of Einstein Finsler metrics*, Internat. J. Math. **25** (2014), no. 4, 1450030, 18 pages.
[15] Q. Xia, *On a class of projectively flat Finsler metrics*, Differential Geom. Appl. **44** (2016), 1–16.
[16] C. Yu and H. Zhu, *On a new class of Finsler metrics*, Differential Geom. Appl. **29** (2011), no. 2, 244–254.
[17] H. Zhu, *On general* (α, β)*-metrics with vanishing Douglas curvature*, Internat. J. Math. **26** (2015), no. 9, 1550076, 16 pages.
[18] ———, *A class of Finsler metrics of scalar flag curvature*, Differential Geom. Appl. **40** (2015), 321–331.

Xiaohuan Mo
School of Mathematical Sciences
Peking University
Beijing 100871, P. R. China
E-mail address: moxh@pku.edu.cn

Hongmei Zhu
College of Mathematics and Information Science
Henan Normal University
Xinxiang, 453007, P. R. China
E-mail address: zhm403@163.com