Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1281–1291
https://doi.org/10.4134/BKMS.b160541
pISSN: 1015-8634 / eISSN: 2234-3016

A NOTE ON MINIMAL PRIME IDEALS

Rasul Mohammadi, Ahmad Moussavi, and Masoome Zahiri

Abstract. Let R be a strongly 2-primal ring and I a proper ideal of R. Then there are only finitely many prime ideals minimal over I if and only if for every prime ideal P minimal over I, the ideal $P/\sqrt{I}$ of $R/\sqrt{I}$ is finitely generated if and only if the ring $R/\sqrt{I}$ satisfies the ACC on right annihilators. This result extends "D. D. Anderson, A note on minimal prime ideals, *Proc. Amer. Math. Soc.* 122 (1994), no. 1, 13–14." to large classes of noncommutative rings. It is also shown that, a 2-primal ring R only has finitely many minimal prime ideals if each minimal prime ideal of R is finitely generated. Examples are provided to illustrate our results.

1. Introduction

Throughout this paper, R denotes an associative ring with identity. In the case when R is a commutative ring with identity and I an ideal of R, it was first appeared in [9, Theorem 88], that if R satisfies the ACC (ascending chain condition) on radical ideals, then R has finitely many prime ideals minimal over I. Of course, for a Noetherian ring R every prime ideal minimal over I is finitely generated.

Anderson [2] proved that whenever R is a commutative ring, I a proper ideal and every prime ideal minimal over I is finitely generated, then R has only finitely many prime ideals minimal over I. By [5, Lemma 1.16], if R is a reduced ring with ACC on annihilators, then R has finitely many minimal prime ideals. C. Huh, N. K. Kim and Y. Lee [8] generalized the Anderson's Theorem to a class of noncommutative rings.

A ring R is called 2-*primal* if the lower nil radical, $Nil_*(R)$, coincides with the set of nilpotent elements $Nil(R)$ of R. Shin in [16, Proposition 1.11] showed that a ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime (i.e., R/P is a domain). Birkenmeier, Heatherly and Lee in [3, Proposition 2.6] proved that R is 2-primal if and only if the polynomial ring $R[x]$ is 2-primal.

Received June 28, 2016; Accepted September 21, 2016.
2010 *Mathematics Subject Classification.* 16D25, 16N60.
Key words and phrases. minimal prime ideal, strongly 2-primal ring, duo ring.

In this note, we extend the Anderson's theorem and prove that, when R is a strongly 2-primal ring and I a proper ideal of R, then there are only finitely many prime ideals minimal over I if and only if for every prime ideal P minimal over I, the ideal $P/\sqrt{I}$ of $R/\sqrt{I}$ is finitely generated.

We also show that, a 2-primal ring R has only finitely many minimal prime ideals if and only if $R/Nil_*(R)$ has the *ACC* on right annihilators if and only if every minimal prime ideal of R is finitely generated.

Let R be a 2-primal ring with an endomorphism σ. If every minimal prime ideal of R is finitely generated, then there are only finitely many minimal prime ideals in the ring $S(R,n,\sigma)$. This result enables us to produce a big class of noncommutative rings which satisfy our results.

1.1. The results

Definition ([10])**.** An ideal I of R is called 2-primal if $Nil_*(R/I) = Nil(R/I)$. A ring R is *strongly* 2-*primal* whenever every proper ideal of R is 2-primal.

The strong 2-primal property has the following equivalent conditions:

Proposition 1.1 ([10])**.** *For a ring R the following statements are equivalent*:
(1) *R is a strongly 2-primal ring.*
(2) *Every prime ideal of R is completely prime.*
(3) *Every semiprime ideal of R is completely semiprime.*
(4) *$R/Nil_*(R)$ is a strongly 2-primal ring.*

Note that every strongly 2-primal ring is 2-primal but not conversely, see [10]. Several authors investigated the condition 2-primal rings that to be strongly 2-primal. N. K. Kim and Y. Lee [10], proved that whenever $R/Nil_*(R)$ is a weakly π-regular ring, then R is 2-primal if and only if R is strongly 2-primal. Also by [7], whenever R is a 2-primal ring, then $R/Nil_*(R)$ is weakly π-regular if and only if every prime ideal is maximal. According to [7], in every one sided perfect ring the notions of 2-primal and strongly 2-primal rings coincide.

We wire $\sqrt{I} = \{r \in R \,|\, \text{there exists an integer } k \text{ for which } r^k \in I\}$, for every proper ideal I of R. A ring R satisfies *IFP* (*the insertion-of-factors-property*), if the right annihilator of each element of R is an ideal (equivalently, if for all $a, b \in R$ we have $ab = 0$ implies $aRb = 0$).

Lemma 1.2. *The following statements are equivalent*:
(i) *R is a strongly 2-primal ring*;
(ii) *$\sqrt{P} = P$ has IFP for every prime ideal P of R.*

Proof. (i)$\Rightarrow$(ii) Suppose that P is a prime ideal of R. As R is strongly 2-primal, $0 = Nil_*(R/P) = Nil(R/P) = \sqrt{P}/P$ yields that $\sqrt{P} = P$. If $r_1r_2 \in P$, then $(r_2r_1)^2 \in P = \sqrt{P}$ implies that $r_2r_1 \in P$. Hence $r_2r_1R \subseteq P$ and from $(r_1Rr_2)^2 \subseteq P = \sqrt{P}$ we get $r_1Rr_2 \subseteq P$.

(ii)$\Rightarrow$(i) As every prime ideal P of R is completely prime, applying Proposition 1.1, the result follows. $\square$

Lemma 1.3. *Let R be a strongly 2-primal ring and I an ideal of R. Then for every $a, b \in R$, $ab \in \sqrt{I}$ if and only if $ba \in \sqrt{I}$.*

Proof. Assume that $a, b \in R$ such that $ab \in \sqrt{I}$ but $ba \notin \sqrt{I}$. As R/I is 2-primal, $\sqrt{I}/I = Nil(R/I) = Nil_*(R/I)$. Since $(ba + I) \notin \sqrt{I}/I$, there exists a prime ideal P minimal over I with $ba \notin P$. As P/I is completely prime in R/I, from $ab \in \sqrt{I}$ we get $(ab + I) \in P/I$ and so $a + I \in P/I$ or $b + I \in P/I$, which is a contradiction. □

Lemma 1.4. *Let I, J be ideals of R. Then*

$$\sqrt{I}\sqrt{J} \subseteq \sqrt{IJ}.$$

Proof. Assume that $\sum_{i=1}^{n} a_i b_i \in \sqrt{I}\sqrt{J}$, where $a_i \in \sqrt{I}$ and $b_i \in \sqrt{J}$ for $1 \leq i \leq n$. Then $\sum_{i=1}^{n} a_i b_i \in \sqrt{I} \cap \sqrt{J}$ and so there exists an integer k for which $(\sum_{i=1}^{n} a_i b_i)^k \in I \cap J$. It follows that $(\sum_{i=1}^{n} a_i b_i)^{2k} \in IJ$ which shows that $\sum_{i=1}^{n} a_i b_i \in \sqrt{IJ}$. □

Theorem 1.5. *Let R be a strongly 2-primal ring and I a proper ideal of R. Then there are only finitely many prime ideals minimal over I if and only if for every prime ideal P minimal over I, the ideal $P/\sqrt{I}$ of $R/\sqrt{I}$ is finitely generated.*

Proof. Our arguments are inspired by the techniques developed in [2, Theorem]. First assume to the contrary that there are infinitely many prime ideals minimal over I, and set

$$S = \{P_1 \cdots P_n \mid \text{each } P_i \text{ is a prime ideal minimal over } I,\ n = 1, 2, \ldots\}.$$

Assume that for each $C \in S$, $C \nsubseteq \sqrt{I}$. It is easy to see that the set

$$T = \{\sqrt{J} \mid J \text{ is an ideal of } R \text{ with } I \subseteq J \text{ and } C \nsubseteq \sqrt{J} \text{ for each } C \in S\},$$

is partially ordered by inclusion. Now we show that every chain in T has a maximal element. For this, take a chain $U = \{\sqrt{K_\alpha} \mid \alpha \in A\}$ in T and put $K = \bigcup_{\alpha \in A} \sqrt{K_\alpha}$. We claim that $K \in T$. If not, then there exists an element $C = P_1 \cdots P_n \in S$ for which $C \in K$. As for any $1 \leq i \leq n$, $P_i/\sqrt{I}$ is a finitely generated ideal of $R/\sqrt{I}$, we can write $P_i = \sum_{t=1}^{m_i} Rp(i)_t R + \sqrt{I}$. It follows that every $p(1)_{b_1} \cdots p(n)_{b_n} \in C$ is contained in K where $p(i)_{b_i} \in P_i$, $1 \leq b_i \leq m_i$. Since the cardinality of $V = \{p(1)_{b_1} \cdots p(n)_{b_n} \mid p(i)_{b_i} \in P_i\}$ is less than or equal to $m_1 \cdots m_n$, we get V is finite. Thus there exists $K_\alpha \in U$ with $V \in K_\alpha$. Now since R is strongly 2-primal, by Lemma 1.2, $\sqrt{K_\alpha}$ has the IFP. From this and that $p(1)_{b_1} \cdots p(n)_{b_n} \in K_\alpha$, we get $p(1)_{b_1} R \cdots Rp(n)_{b_n} \in \sqrt{K_\alpha}$, entailing $C \subseteq \sqrt{K_\alpha} \in T$, a contradiction. Thus $K \in T$, so by Zorn's lemma, there exists a maximal element Q in T. By definition of T, $Q = \sqrt{Q}$. Now we show that Q is a prime ideal of R. If not, there exist elements $a, b \notin Q$ such that $aRb \subseteq Q$. Then by the maximality of Q, there exist $C_1, C_2 \in S$ such that $C_1 \subseteq \sqrt{Q + RaR}$ and $C_2 \subseteq \sqrt{Q + RbR}$. But $C_1 C_2 = \sqrt{(Q + RaR)}\sqrt{(Q + RbR)}$, by Lemma 1.4,

$C_1C_2 \subseteq \sqrt{(Q+RaR)(Q+RbR)} = \sqrt{Q} = Q$ and since $C_1C_2 \in S$ we get a contradiction. Thus Q is a prime ideal of R and $I \subseteq Q$, and so by [6, Proposition 2.3] there exists a prime ideal P minimal over I with $P \subseteq Q$. Then $P \in S$ with $P \subseteq Q \in T$, a contradiction. Thus the result follows.

Conversely, assume on the contrary that there are finitely many prime ideals minimal over proper ideal I, namely $\{P_1, \ldots, P_n\}$, whereas there exists a minimal prime ideal P_k, $1 \leq k \leq n$, such that $\overline{P_k} = \sum_{i=1}^{\infty} \overline{R}\overline{q_i}\overline{R}$, where $\overline{P_k} = P_k/\sqrt{I}$, $\overline{R} = R/\sqrt{I}$ and $q_i \in P_k/\sqrt{I}$. Take $T = \bigcap_{i=1, i\neq k}^{n} P_i$. Then $P_kT \subseteq \sqrt{I}$, so for every i, $\overline{q_i}$ is a right annihilator in $\overline{R}$. Thus we get a chain

$$\overline{R}\overline{q_1}\overline{R} \subseteq \overline{R}\overline{q_1}\overline{R} + \overline{R}\overline{q_2}\overline{R} \subseteq \cdots$$

of right annihilators in $\overline{R}$. As there are finitely many prime ideals minimal over the proper ideal I, by Theorem 1.17, $\overline{R}$ satisfies ACC on right annihilator. Hence there exists an integer s for which $P_k = \sum_{i=1}^{s} \overline{R}\overline{q_i}\overline{R}$, as needed. □

An ideal I of R is said to have *IFP*, if for every $a, b \in R$, $ab \in I$ implies that $aRb \subseteq I$. By C. Huh, N. K. Kim, and Y. Lee [8], a ring R is said to have *homomorphically IFP*, if all ideals of R have *IFP*.

Corollary 1.6 ([8])**.** *Let R be a homomorphically IFP ring and I a proper ideal of R. If all the prime ideals minimal over I are finitely generated, then there are only finitely many prime ideals minimal over I.*

Corollary 1.7 ([2])**.** *Let R be a commutative ring and I an ideal of R. If all the prime ideals minimal over I are finitely generated, then there are only finitely many prime ideals minimal over I.*

Corollary 1.8. *Let R be a 2-primal ring whose prime ideals are maximal and I be a proper ideal of R. Then there are finitely many prime ideals minimal over I if and only if for every prime ideal P minimal over I, $P/\sqrt{I}$ is finitely generated.*

Proof. It follows from the facts that $R/Nil(R)$ is weakly π-regular if and only if prime ideals are maximal and every 2-primal ring with $R/Nil(R)$ weakly π-regular is strongly 2-primal. □

Theorem 1.9. *Let R be a 2-primal ring. Then there are only finitely many minimal prime ideals in R if and only if for every minimal prime ideal P of R, $P/Nil(R)$ is a finitely generated ideal of $R/Nil(R)$.*

Proof. The proof is an exact copy of that of Theorem 1.5 only taking $I = 0$ and using the fact that in any 2-primal ring minimal prime ideals are completely prime. □

The following example shows that Theorem 1.5 properly generalizes the main results of [8] and [2].

Example 1.10. Consider the ring $R = k[x_1, x_2, x_3, \ldots]/\langle x_1^2, x_2^3, x_3^4, \ldots\rangle$, where k is a field. Then R is a commutative local ring with the unique minimal prime ideal $J(R) = Nil_*(R)$, where $J(R)$ is the Jacobson radical of R. Hence $J(R)$ is the unique prime ideal minimal over proper ideal I of R, but $J(R)/I$ is not finitely generated, for every finitely generated proper ideal I of R. However $Nil_*(R) = \sqrt{I}$ and $J(R)/Nil_*(R)$ is zero and so finitely generated in the ring $R/Nil_*(R)$.

Motivated by the above example, we consider duo rings R, whose prime ideals are maximal, and show that if R has the ACC on direct summands, then for every proper ideal I of R, the ring R has finitely many prime ideals minimal over I.

Theorem 1.11. *Let R be a duo ring whose prime ideals are maximal and I a proper ideal of R. Then R has finitely many prime ideals minimal over I if and only if R has the ACC on direct summands.*

Proof. Assume that R has finitely many prime ideals minimal over I, and applying Theorem 1.5, it suffices to show that $R/Nil(R)$ is left Noetherian. This is because, we have $Nil(R) \subseteq \sqrt{I}$ for every proper ideal I of R, so we get $R/\sqrt{I}$ is a left Noetherian ring, so for every prime ideal P which is minimal over I, $P/\sqrt{I}$ is finitely generated.

As R is a 2-primal ring and prime ideals are maximal by [7, Corollary 3.6], $R/Nil(R)$ is a right weakly regular ring and every prime factor ring of R is a simple domain. Hence every prime factor of $R/Nil(R)$ is a simple domain. Now for every prime ideal P of R, R/P is a simple domain. Let $a + P$ be a nonzero element of R/P. Then $RaR + P = R$, because R/P is a simple ring. As R is left duo, $Ra + P = R$. This shows that R/P is a division ring and so a von Neumann regular ring. Applying [18, 4.6], it yields that $R/Nil(R)$ is a von Neumann regular ring. If $R/Nil(R)$ is not left Noetherian, then there exists a left ideal $K/Nil(R)$ of $R/Nil(R)$ for which $K/Nil(R)$ is not finitely generated. Assume that $\{k_i\}_{i=1}^{\infty}$ generates $K/Nil(R)$ and $K_j = \sum_{i=1}^{j} Rk_i$ for $j = 1, 2, \ldots$. Then for every j, $(K_j + Nil(R))/Nil(R)$ is a direct summand of $R/Nil(R)$. Now we get an infinite chain of direct summands

$$(K_1+Nil(R))/Nil(R) \subseteq (K_2+Nil(R))/Nil(R) \subseteq (K_3+Nil(R))/Nil(R) \subseteq \cdots$$

of $R/Nil(R)$. As idempotents lift module $Nil(R)$, then there exists an infinite set of direct summands in R, which is a contradiction. Therefore $R/Nil(R)$ is left Noetherian as desired, and the proof follows.

Conversely, assume on the contrary that there exists an infinite chain of direct summand

$$e_1R \subset e_2R \subset e_nR \subset \cdots$$

in R, where $e_i = e_i^2$ are nontrivial idempotents in R for $i = 1, 2, \ldots$. As R is a duo ring, $e_1 \neq 1$, we can consider $I = e_1R$ as a proper ideal of R. Since R has finitely many prime ideals minimal over I, the ring $R/\sqrt{I}$ has ACC on right annihilators, by Theorem 1.17. Thus the ring $R/\sqrt{I}$ and so the $\frac{R/I}{\sqrt{I}/I}$ satisfies ACC on direct summand. As direct summands of the ring R/I lift module to $Nil(R/I) = \sqrt{I}/I$, we get the ring R/I satisfies ACC on direct summand. It not hard to see that, for any $i = 2, 3, \ldots$, e_iR/I is a direct summand of the ring R/I. As

$$e_2R/I \subset e_3R/I \subset \cdots$$

is a chain of direct summand of R/I, there exists an integer k for which $e_kR/I = e_{k+i}R/I$, for $i \geq 1$. From this we get $e_kR = e_{k+1}R = \cdots$, a contradiction. Hence R has the ACC on direct summands. □

Lemma 1.12. *Let R be a reduced ring. Then the following statements are equivalent*:

(1) *R has finitely many minimal prime ideals*;
(2) *R satisfies the ACC on right* (*left*) *annihilators*;
(3) *any minimal prime ideal is finitely generated*;
(4) *R has finite left* (*right*) *uniform dimension.*

Proof. (1) ⇔ (2) Note that when R is a reduced ring, then ACC on left annihilators and ACC on right annihilators coincide. If R satisfies ACC on right annihilators, then by [5, Lemma 1.16], R has only finitely many minimal prime ideals.

For the converse assume that the number of minimal prime ideals of R is n, namely $\{P_1, \ldots, P_n\}$. Assume on the contrary that R does not satisfy ACC on right annihilators. Then by [12, Exercise 6.22] there are $t_i, s_i, i = 1, 2, \ldots$ with $t_is_i \neq 0$ for all positive integers i while $t_is_j = 0$ for every $j < i$. Hence there exists a minimal prime ideal P_i for which $s_i, t_i \notin P_i$, for every $i = 1, 2, \ldots$, so there exist distinct positive integers p, q and $1 \leq r \leq n$ such that $s_p, s_q, t_p, t_q \notin P_r$. We may assume that $p < q$. Then we get $0 = t_qs_p \in P_r$. Since R is reduced, minimal prime ideals of R are completely prime, so we get $t_q \in P_r$ or $s_p \in P_r$, which is a contradiction, and the result follows.

(1) ⇔ (3) This follows from Theorem 1.9.

(2) ⇔ (4) This is well known. □

Theorem 1.13. *Let R be a 2-primal ring. Then R has finitely many minimal prime ideals if and only if $R/Nil(R)$ satisfies the ACC on right annihilators.*

Proof. Note that there exists a one to one correspondence between the minimal prime ideals in the rings R and $R/Nil(R)$ and 2-primal condition forces that $R/Nil(R)$ to be a reduced ring. Now in view of Lemma 1.12, we get $R/Nil(R)$ satisfies the ACC on right annihilators if and only if R has finitely many minimal prime ideals. □

Corollary 1.14. *Let R be a 2-primal ring. If every minimal prime ideal is finitely generated, then $R/Nil(R)$ satisfies the ACC on right annihilators.*

Corollary 1.15. *Let R be a 2-primal ring. If every minimal prime ideal of R is finitely generated, then there are only finitely many minimal prime ideals in $R[x]$.*

Proof. This follows from the fact that each minimal prime ideal of $R[x]$ is of the form $P[x]$ for some minimal prime ideal P of R. □

Using Theorems 1.11 and 1.13, we deduce.

Corollary 1.16. *Let R be a commutative reduced and weakly π-regular ring. Then R satisfies the ACC on direct summands if and only if R has the ACC on annihilators.*

Theorem 1.17. *Let R be a strongly 2-primal ring and I a proper ideal of R. Then there exist finitely many prime ideals minimal over I if and only if the ring $R/\sqrt{I}$ has the ACC on right annihilators.*

Proof. Note that $R/\sqrt{I}$ is a reduced ring for every proper ideal I of R, because if there exists an element $a \in R$ such that $(a+\sqrt{I})^2 = \sqrt{I}$. Then we must have $a^2 \in \sqrt{I}$. Thus $a \in \sqrt{I}$, which implies that $a+\sqrt{I}$ is zero in $R/\sqrt{I}$. Now from the fact that every prime ideal minimal over I must contain $\sqrt{I}$, we get the number of prime ideals minimal over I and the number of minimal prime ideals of $R/\sqrt{I}$ are equal. Now repeated using of 1.12 completes the proof. □

Corollary 1.18. *Let R be a strongly 2-primal ring and I a proper ideal of R. Then the following statements are equivalent*:

(1) *for every prime ideal P minimal over I, $P/\sqrt{I}$ is a finitely generated ideal of $R/\sqrt{I}$*;

(2) *the ring $R/\sqrt{I}$ satisfies the ACC on right annihilators*;

(3) *there exist finitely many prime ideals minimal over I.*

Proof. It follows from Theorems 1.5 and 1.17. □

According to [1], a ring R is said to be a *right q.f.d.-ring* if every cyclic right R-module has finite Goldie dimension. Left *q.f.d.-rings* are defined symmetry. Right *q.f.d-rings* have been studied in [1], [11], [15] and [17]. This is equivalent to the requirement for every cyclic right R-module to have finitely generated socle [11]. Examples of right *q.f.d.-rings* include rings with left Krull dimension. In particular, left noetherian rings are also right *q.f.d.-rings.* R is a *right q.f.d.-ring* if and only if every finitely generated R-module has finite Goldie dimension [4].

Theorem 1.19. *Let R be a strongly 2-primal left q.f.d.-ring and I a proper ideal of R. Then there exist finitely many prime ideals minimal over I.*

Proof. As prime ideals minimal over I and over $\sqrt{I}$ are the same, we work on the ideal $\sqrt{I}$. Since R is a left q.f.d. ring, $U.\dim(R/\sqrt{I}) < \infty$ as a left R-module. Hence the ring $R/\sqrt{I}$ has the finite left uniform dimension. Now from [5, Lemma 1.14], the ring $R/\sqrt{I}$ has ACC on right annihilators. Applying Theorem 1.17 guarantees the finiteness of number of prime ideals minimal over $\sqrt{I}$. □

Corollary 1.20. *Let R be a strongly 2-primal ring with Krull dimension. Then for every ideal I of R, $R/\sqrt{I}$ is a semiprime left Goldie ring.*

Proof. From Theorem 1.19, there are finitely many prime ideals minimal over I. Applying Theorem 1.17, $R/\sqrt{I}$ satisfies ACC on right annihilators. As $R/\sqrt{I}$ is a reduced ring, $R/\sqrt{I}$ satisfies ACC on left annihilators if and only if $R/\sqrt{I}$ satisfies ACC on right annihilators. On the other hand, Lemma 1.12 implies that $R/\sqrt{I}$ has finite left uniform dimension. Hence $R/\sqrt{I}$ is a semiprime left Goldie ring. □

As mentioned above strongly 2-primal rings play an important role in the present paper, so we provide examples of such rings. Also every 2-primal ring with $R/Nil(R)$ being weakly π-regular is strongly 2-primal, so we use this fact to construct more examples of noncommutative strongly 2-primal rings.

Note that 2-primal rings, perfect rings and weakly π-regular rings transfer to the triangular matrix rings $T_n(R), R_n(R), T(R, n)$, so we can construct several non-commutative strongly 2-primal rings.

Let R be a ring and σ an endomorphism of R with $\sigma(1) = 1$. We denote the identity matrix, unit matrices and the (i, j)-entry, in the full matrix ring $M_n(R)$, by I_n and e_{ij}, respectively. In [14], T. K. Lee and Y. Zhou introduced a subring of the *skew triangular matrix* ring as a set of all triangular matrices $T_n(R)$, with addition pointwise and a new multiplication subject to the condition $e_{ij}r = \sigma^{j-i}(r)e_{ij}$. So $(a_{ij})(b_{ij}) = (c_{ij})$, where $c_{ij} = a_{ii}b_{ij} + a_{i,i+1}\sigma(b_{i+1,j}) + \cdots + a_{ij}\sigma^{j-i}(b_{jj})$, for each $i \leq j$ and denoted it, by $T_n(R, \sigma)$.

The subring of the skew triangular matrices with constant diagonals is denoted by $S(R, n, \sigma)$. Then

$$S(R,n,\sigma) = \left\{ \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{11} & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1,n} \\ 0 & 0 & \cdots & a_{11} \end{pmatrix} \mid a_{ij} \in R,\ 1 \leq i, j \leq n \right\}$$

is a ring with addition pointwise and multiplication given by:

$$re_{ij}se_{jk} = r\sigma^{j-i}(s)e_{ik} \quad \text{for every } r, s \in R.$$

Proposition 1.21. *If R is a strongly 2-primal ring with an endomorphism σ, then so is the ring $S(R, n, \sigma)$.*

Proof. As $R/Nil(R) \simeq S(R, n, \sigma)/Nil(S(R, n, \sigma))$, applying Proposition 1.1, it implies that $S(R, n, \sigma)$ is strongly 2-primal. □

Lemma 1.22. *Let R be a 2-primal ring and assume that $R/Nil(R)$ is a weakly π-regular ring. Then $S(R,n,\sigma)$ is a 2-primal ring and*

$$S(R,n,\sigma)/Nil(S(R,n,\sigma))$$

is a weakly π-regular ring. Especially $S(R,n,\sigma)$ is a strongly 2-primal ring.

Proof. Note that

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{11} & \ddots & \vdots \\ \vdots & \vdots & \ddots & a_{n-1,n} \\ 0 & 0 & \cdots & a_{11} \end{pmatrix} \in Nil_*(S(R,n,\sigma))$$

if and only if a_{11} is a strongly nilpotent element of R, so $Nil_*(S(R,n,\sigma))$ consists exactly those elements that their main diagonal is in $Nil_*(R)$. It is easy to check that $Nil(S(R,n,\sigma))$ consists exactly those elements which their main diagonal are nilpotent. Since R is a 2-primal ring we get $Nil(S(R,n,\sigma)) = Nil_*(S(R,n,\sigma))$, which shows that $S(R,n,\sigma)$ is 2-primal. On the other hand from the ring isomorphism

$$R/Nil(R) \simeq S(R,n,\sigma)/Nil(S(R,n,\sigma)),$$

we get $S(R,n,\sigma)/Nil(S(R,n,\sigma))$ is weakly π-regular. □

Proposition 1.23. *Let R be a 2-primal ring. If every minimal prime ideal of R is finitely generated, then there are only finitely many minimal prime ideals in $S(R,n,\sigma)$.*

Proof. We will show that Q is a prime ideal in $S(R,n,\sigma)$ if and only if $Q = PI_n + \sum_{i<j} Re_{ij}$ for some prime ideal P of R. For, assume that Q is a prime ideal of $S(R,n,\sigma)$. In view of Lemma 1.22, $S(R,n,\sigma)$ is 2-primal and so $\sum_{i<j} Re_{ij} \subseteq Nil(S(R,n,\sigma)) \subseteq Q$. If I, J are ideals of R with $IJ \subseteq Q_1$, where Q_1 is the set of elements of R which occur in the main diagonal of elements in Q. Then $(II_n + \sum_{i<j} Re_{ij})(JI_n + \sum_{i<j} Re_{ij}) \subseteq Q$, where I_n is the identity matrix in $S(R,n,\sigma)$. As Q is a prime ideal of $S(R,n,\sigma)$ we get $(II_n + \sum_{i<j} Re_{ij}) \subseteq Q$ or $(JI_n + \sum_{i<j} Re_{ij}) \subseteq Q$. Hence either $I \subseteq Q_1$ or $J \subseteq Q_1$ and so the set of all main diagonal of Q is a prime ideal of R.

Conversely, one can show that if P is a prime ideal of R, then $Q = PI_n + \sum_{i<j} Re_{ij}$ is a prime ideal of $S(R,n,\sigma)$. □

Corollary 1.24. *Let R be a commutative ring. If every minimal prime ideal of R is finitely generated, then there are only finitely many minimal prime ideals in $S(R,n,\sigma)$.*

Lemma 1.25. *Let I be an ideal of the ring $S(R,n,\sigma)$. Then for an ideal L of R*

$$\sqrt{I} = \sqrt{L} I_n + \sum_{i<j,\ 1\le i\le n,\ 2\le j\le n} Re_{ij}.$$

Proof. Assume that I is an ideal of $S(R,n,\sigma)$. Then as $\sum_{i<j,\ 1\leq i\leq n,\ 2\leq j\leq n} Re_{ij}$ is a nilpotent ideal of $S(R,n,\sigma)$, $\sum_{i<j,\ 1\leq i\leq n,\ 2\leq j\leq n} Re_{ij} \subseteq \sqrt{I}$.

Let L be the set of all elements of R which occur in the main diagonal elements of I. Then L is an ideal of R such that $\sqrt{L}I_n \subseteq \sqrt{I}$. Hence

$$\sqrt{L}I_n + \sum_{i<j,\ 1\leq i\leq n,\ 2\leq j\leq n} Re_{ij} \subseteq \sqrt{I}.$$

If $A \in \sqrt{I}$, then there exists an integer n such that $A^n \in I$. So we have

$$A \in \sqrt{L}I_n + \sum_{i<j,\ 1\leq i\leq n,\ 2\leq j\leq n} Re_{ij}, \text{ i.e., } \sqrt{I} = \sqrt{L}I_n + \sum_{i<j,\ 1\leq i\leq n,\ 2\leq j\leq n} Re_{ij}.$$

□

Corollary 1.26. *Let L be a proper ideal of R. Then the right uniform dimensions of the rings $R/\sqrt{L}$ and $S(R,n,\sigma)/\sqrt{I}$ are equal, where I is an ideal of $S(R,n,\sigma)$ with $\sqrt{I\cap I_n} = \sqrt{L}I_n$, where I_n is the identity matrix.*

Corollary 1.27. *Let L be a proper ideal of R. If there exist finitely many minimal primes over L, then there are finitely many minimal primes over the ideal I of $S(R,n,\sigma)$, where $\sqrt{I\cap I_n} = \sqrt{L}I_n$.*

References

[1] A. H. Al-Huzali, S. K. Jain, and S. R. López-Permouth-Permouth, *Rings whose cyclics have finite Goldie dimension*, J. Algebra. **153** (1992), no. 1, 37–40.

[2] D. D. Anderson, *A note on minimal prime ideals*, Proc. Amer. Math. Soc. **122** (1994), no. 1, 13–14.

[3] G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee, *Completely prime ideals and associated radicals, in Ring Theory*, eds. S. K. Jain and S. T. Rizvi, 102–129, World Scientific, Singapore, 1993.

[4] V. P. Camillo, *Modules whose quotients have finite Goldie dimension*, Pacific J. Math. **69** (1977), no. 2, 337–338.

[5] A. W. Chatters and C. R. Hajarnavis, *Rings with Chain Conditions*, Pitman Advanced Publishing Program, Boston, 1980.

[6] K. R. Goodearl and Jr. R. B. Warfield, *An introduction to Noncommutative Noetherian Rings*, London Mathematical Society Student Texts, 16. Cambridge University Press, Cambridge, 1989.

[7] C. Y. Hong, Y. C. Jeon, K. H. Kim, N. K. Kim, and Y. Lee, *Weakly regular rings with ACC on annihilators and maximality of strongly prime ideals of weakly regular rings*, J. Pure Appl. Algebra **207** (2006), no. 3, 565–574.

[8] C. Huh, N. K. Kim, and Y. Lee, *An Anderson's Theorem on noncommutative rings*, Bull. Korean Math. Soc. **45** (2008), no. 4, 797–800.

[9] I. Kaplansky, *Commutative Rings*, Allyn and Bacon, Boston, 1970.

[10] N. K. Kim and Y. Lee, *On rings whose prime ideals are completely prime*, J. Pure Appl. Algebra **170** (2002), no. 2-3, 255–265.

[11] R. P. Kurshan, *Rings whose cyclic modules have finitely generated socle*, J. Algebra **15** (1970), 376–386.

[12] T. Y. Lam, *Lectures on Modules and Rings*, Springer-Verlag New York, Inc, 1998.

[13] ______, *A First Course in Noncommutative Rings*, Springer-Verlag New York, Inc, 1990.

[14] T. K. Lee and Y. Zhou, *A unified approach to the Armendariz property of polynomial rings and power series rings*, Colloq. Math. **113** (2008), no. 1, 151–169.
[15] S. R. López-Permouth, *Rings characterized by their weakly injective modules*, Glasgow Math. J. **34** (1992), no. 3, 349–353.
[16] G. Shin, *Prime ideals and sheaf representations of a pseudo symmetric ring*, Trans. Amer. Math. Soc. **184** (1973), 43–60.
[17] R. C. Shock, *Dual generalization of the Artinian and Noetherian conditions*, Pacific J. Math. **54** (1974), no. 2, 227–235.
[18] A. Tuganbaev and Semiregular, *Weakly regular, and π-regular rings*, J. Math. Sci. (New York) **109** (2002), no. 3, 1509–1588.

Rasul Mohammadi
Department of pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
E-mail address: mohamadi.rasul@yahoo.com

Ahmad Moussavi
Department of pure Mathematics
Faculty of Mathematical Sciences
Tarbiat Modares University
P.O.Box:14115-134, Tehran, Iran
E-mail address: moussavi.a@modares.ac.ir, moussavi.a@gmail.com

Masoome Zahiri
Department of Mathematics
Faculty of Sciences
Higher Education Center of Eghlid
Eghlid, Iran
E-mail address: m.zahiri86@gmail.com