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SOME RESULTS ON PARAMETRIC EULER SUMS

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Abstract. In this paper we present a new family of identities for parametric Euler sums which generalize a result of David Borwein et al. [2]. We then apply it to obtain a family of identities relating quadratic and cubic sums to linear sums and zeta values. Furthermore, we also evaluate several other series involving harmonic numbers and alternating harmonic numbers, and give explicit formulas.

1. Introduction

The study of special values of harmonic number functions concerns itself with relations between values at positive integer vectors $(p_1,\ldots,p_r,q)$ of sums of the form [11]

$$S_{p_1,\ldots,p_r;q}:=\sum_{n=1}^{\infty}\frac{H_n^{(p_1)}\cdots H_n^{(p_r)}}{n^q}\quad (q>1) \tag{1.1}$$

commonly referred to as classical Euler sums [11,22,23]. Here $H_n^{(p)}$ denotes the harmonic number of order p defined by ([9,11])

$$H_n^{(p)}:=\sum_{k=1}^{n}\frac{1}{k^p}\ \left(p,n\in\mathbb{N}:=\{1,2,3,\ldots\}\right), \tag{1.2}$$

with $H_n\equiv H_n^{(1)}$ and the empty sum $H_0^{(m)}$ is conventionally understood to be zero. Similarly, the generalized alternating harmonic numbers of order p are given by [11]

$$\bar{H}_n^{(p)}:=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^p}\ \left(p,n\in\mathbb{N}:=\{1,2,3,\ldots\}\right), \tag{1.3}$$

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with $\bar{H}_n \equiv \bar{H}_n^{(1)}$ and $\bar{H}_0^{(p)} = 0$. In [17], Sofo discovered the following formulae for harmonic numbers and alternating harmonic numbers

$$\bar{H}_n^{(k)} = H_{2[(n+1)/2]-1}^{(k)} - \frac{1}{2^k}\left(H_{[n/2]}^{(k)} + H_{[(n-1)/2]}^{(k)}\right),\ k \in \mathbb{N} \setminus \{1\} := \{2,3,4,\ldots\},$$

$$\bar{H}_n = H_n - H_{[n/2]},$$

where $[x]$ is the integer part of x. Next, we introduce the generalized Euler sums which involve harmonic numbers and alternating harmonic numbers. For integers $q, p_1, \ldots, p_m$ with $q \geq 2$, we define the generalized Euler sums as

$$S(p_1, p_2, \ldots, p_m; q) := \sum_{n=1}^{\infty} \frac{X_n(p_1) X_n(p_2) \cdots X_n(p_m)}{n^q}, \tag{1.4}$$

where $X_n(p)$ is defined by

$$X_n(p) := \begin{cases} H_n^{(p)}, & p > 0, \\ 1, & p = 0, \\ \bar{H}_n^{(-p)}, & p < 0. \end{cases} \tag{1.5}$$

In below, if $p < 0$, we will denote it by $\overline{-p}$. For example,

$$S(2, \bar{3}, 5, \bar{7}; q) = \sum_{n=1}^{\infty} \frac{H_n^{(2)} \bar{H}_n^{(3)} H_n^{(5)} \bar{H}_n^{(7)}}{n^q}.$$

Similarly, for $q \in \mathbb{N}$, we define

$$S(p_1, p_2, \ldots, p_m; \bar{q}) := \sum_{n=1}^{\infty} \frac{X_n(p_1) X_n(p_2) \cdots X_n(p_m)}{n^q} (-1)^{n-1}. \tag{1.6}$$

The integers $w := |p_1| + |p_2| + \cdots + |p_m| + |q|$ and m are called the weight and the depth (or length) of a Euler sums $S(p_1, p_2, \ldots, p_m; q)$. Hence, by the definition of $S(p_1, p_2, \ldots, p_m; q)$, we know that the linear sums are altogether four types:

$$S(p;q) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q},\ S(\bar{p};q) = \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q},$$
$$S(p;\bar{q}) = \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} (-1)^{n-1},\ S(\bar{p};\bar{q}) = \sum_{n=1}^{\infty} \frac{\bar{H}_n^{(p)}}{n^q} (-1)^{n-1}.$$

A good deal of work on Euler sums has focused on the problem of determining when complicated sums can be expressed in terms of simpler sums. Thus, researchers are interested in determining which sums can be expressed in terms of other sums of lesser depth. It has been discovered in the course of the years that many Euler sums admit expressions involving finitely the zeta values, that is to say, values of the following Riemann zeta function:

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}\ (\Re(s) > 1) \tag{1.7}$$

with positive integer arguments. The alternating Riemann zeta function is defined by

$$\bar{\zeta}(s) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} = (1 - 2^{1-s})\zeta(s)\ (\Re(s) > 1), \tag{1.8}$$

with $\bar{\zeta}(1) = \ln 2$. While, there are also many Euler sums which need not only zeta values but also linear sums and polylogarithms. Namely, many non-linear Euler sums (depth≥ 2) are reducible to polynomials in zeta values, to polylogarithms and to linear sums. For example, in [11], Philippe Flajolet and Bruno Salvy gave explicit reductions to zeta values for all linear sums $S(p;q), S(\bar{p};q), S(p;\bar{q}), S(\bar{p};\bar{q})$ with $w := p+q$ odd. Moreover, they proved the following conclusion: If $p_1 + p_2 + q$ is even, and $p_1 > 1, p_2 > 1, q > 1$, the quadratic sums

$$S(p_1, p_2; q) = \sum_{n=1}^{\infty} \frac{H_n^{(p_1)} H_n^{(p_2)}}{n^q}$$

are reducible to linear sums (see Theorem 4.2 in the reference [11]). In [22] and [24], we showed that all quadratic sums $S(p_1, p_2; q)$ with $|p_1|+|p_2|+|q| \leq 4$ were reducible to zeta values and polylogarithms, and in [23], we proved that all Euler sums of the form $S(1, p; q)$ for weights $p+q+1 \in \{4, 5, 6, 7, 9\}$ with $p \geq 1$ and $q \geq 2$ are expressible polynomially in terms of zeta values. For weight 8, all such sums are the sum of a polynomial in zeta values and a rational multiple of $S(2; 6)$. Here the polylogarithm function is defined as follows

$$\mathrm{Li}_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^p}\ (\Re(p) > 1,\ |x| \leq 1) \tag{1.9}$$

with $\mathrm{Li}_1(x) = -\ln(1-x),\ x \in [-1, 1)$. The relationship between the values of the Riemann zeta function and generalized Euler sums has been studied by many authors. The interested reader is referred to [1,3,5–8,10–14,16,18,22–25], and references therein.

So far, surprisingly little work has been done on parametric Euler sums. Similarly as in (1.4) and (1.6), the generalized parametric Euler sums are defined by the series

$$S\left(\begin{array}{c} p_1, p_2, \cdots, p_m \\ a_1, a_2, \cdots, a_r \end{array}\right) := \sum_{n=1}^{\infty} \frac{X_n(p_1) X_n(p_2) \cdots X_n(p_m)}{(n+a_1)(n+a_2)\cdots(n+a_r)} \quad (r \in \mathbb{N} \setminus \{1\}, m \in \mathbb{N}), \tag{1.10}$$

$$\bar{S}\left(\begin{array}{c} p_1, p_2, \cdots, p_m \\ a_1, a_2, \cdots, a_r \end{array}\right) := \sum_{n=1}^{\infty} \frac{X_n(p_1) X_n(p_2) \cdots X_n(p_m)}{(n+a_1)(n+a_2)\cdots(n+a_r)} (-1)^{n-1} \quad (r \in \mathbb{N} \setminus \{1\}, m \in \mathbb{N}), \tag{1.11}$$

where $a_i \notin \mathbb{N}^- := \{-1, -2, -3, \ldots\}$ $(i = 1, 2, \ldots, r)$ and $X_n(p)$ is defined by (1.5). We are primarily interested in positive integer values of the arguments

$(p_1, \ldots, p_m)$, in which case it is easily seen that $r > 1$ is both necessary and sufficient for the sum (1.10) to converge. By $\{a_1, a_2, \ldots, a_j\}_m$ we denote the sequence of length mj with m repetitions of $(a_1, a_2, \ldots, a_j)$. To avoid confusion with the notion of analytic continuation, we shall henceforth adopt the notation of [5], in which each p_j in (1.10) and (1.11) is replaced by $\overline{p_j}$ when $p_j < 0$. Thus, for example,

$$S\left(\begin{array}{c}\{1\}_3\\ \{a\}_m\end{array}\right) := \sum_{n=1}^{\infty} \frac{H_n^3}{(n+a)^m},\quad \bar{S}\left(\begin{array}{c}1,\bar{1}\\ \{a\}_m\end{array}\right) := \sum_{n=1}^{\infty} \frac{H_n \bar{H}_n}{(n+a)^m}(-1)^{n-1},$$

$$S\left(\begin{array}{c}2m\\ 0,a,-a\end{array}\right) := \sum_{n=1}^{\infty} \frac{H_n^{(2m)}}{n\left(n^2-a^2\right)}, \bar{S}\left(\begin{array}{c}\bar{1}\\ a,-a\end{array}\right) := \sum_{n=1}^{\infty} \frac{\bar{H}_n}{n^2-a^2}(-1)^{n-1}.$$

There are fewer results for sums of the type (1.10) and (1.11). Some earlier attempts at evaluating parametric Euler sums are due to Jonathan M. Borwein et al. [4] and David Borwein et al. [2]. Jonathan M. Borwein gave the following closed form of alternating parametric Euler sums

$$\begin{aligned}\bar{S}\left(\begin{array}{c}1\\ a,-a\end{array}\right) &= \sum_{n=1}^{\infty} \frac{H_n}{n^2-a^2}(-1)^{n-1}\\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1} n}{\left(n^2-a^2\right)^2} - \sum_{n=1}^{\infty} \frac{1}{(2n-1)\left((2n-1)^2-a^2\right)}\\ &\quad + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n\left(n^2-a^2\right)} - a^2\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-a^2}\right)\left(\sum_{n=1}^{\infty} \frac{1}{n\left(n^2-a^2\right)}\right).\end{aligned} \tag{1.12}$$

(Note that the closed form of $\bar{S}\left(\begin{smallmatrix}1\\ a,-a\end{smallmatrix}\right)$ is in error in [4].) David Borwein et al. proved that the parametric sums

$$S\left(\begin{array}{c}2m-1\\ a,-a\end{array}\right) := \sum_{n=1}^{\infty} \frac{H_n^{(2m-1)}}{n^2-a^2} \quad (m \in \mathbb{N})$$

can be expressed as a rational linear combination of several given rational series. In this paper we are interested in parametric Euler sums with harmonic numbers and alternating harmonic numbers $S\left(\begin{smallmatrix}p_1,p_2,\ldots,p_m\\ a_1,a_2,\ldots,a_r\end{smallmatrix}\right)$ and $\bar{S}\left(\begin{smallmatrix}p_1,p_2,\ldots,p_m\\ a_1,a_2,\ldots,a_r\end{smallmatrix}\right)$. Such series could be of interest in analytic number theory. We show that many sums are related to the values of the Hurwitz zeta function and alternating Hurwitz zeta function, which are defined by

$$\zeta(s,a+1) := \sum_{n=1}^{\infty} \frac{1}{(n+a)^s}, \bar{\zeta}(s,a+1) := \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n+a)^s}, \tag{1.13}$$

with $\Re(s) > 1,\ a \neq -1, -2, \ldots$. Finally, we also evaluate several other series involving the partial sums $\zeta_n(p, a+1)$ (also called the parametric harmonic numbers) for $p \geq 1$ of Hurwitz zeta function defined by

$$\zeta_n(p,a+1) := \sum_{k=1}^{n} \frac{1}{(k+a)^p} \ (p \in \mathbb{N}, a \notin \mathbb{N}^-).$$

The plan of the paper is as follows. In the second section, we give some lemmas, which will be useful in the development of the main theorems. In the third section we establish some relations involving two or more parametric Euler sums by the method of constructing the Tornheim-type series function $T_{s,t}^{(p_1,p_2)}(x,y;a)$ defined by

$$T_{s,t}^{(p_1,p_2)}(x,y;a) := \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} \frac{X_n(p_1)\,X_m(p_2)}{(n+a)^s(m+a)^t(m-n)} x^n y^n, \tag{1.14}$$

where $s,t \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}, s+t \geq 1, x,y \in (-1,1), a \notin \mathbb{N}^-$ and $X_n(p)$ is defined by (1.5). For example, we prove that the quadratic combination

$$\frac{1}{2}\bar{S}\left(\begin{array}{c} 1,1 \\ \{a\}_s \end{array}\right) - \bar{S}\left(\begin{array}{c} 1,\bar{1} \\ \{a\}_s \end{array}\right)$$

is expressible in terms of parametric linear sums and rational series. Moreover, we also consider the types of parametric linear sums involving harmonic numbers

$$S\left(\begin{array}{c} 2m-1 \\ \{0\}_{2s},a,-a \end{array}\right) := \sum_{n=1}^{\infty} \frac{H_n^{(2m-1)}}{n^{2s}(n^2-a^2)},\ S\left(\begin{array}{c} 2m \\ 0,a,-a \end{array}\right) := \sum_{n=1}^{\infty} \frac{H_n^{(2m)}}{n(n^2-a^2)}$$

by using the method of Contour Integral Representations and the Theorem of Residues. In the fourth section, we give some specific series associated with harmonic numbers (For result on classical Tornheim series, see [19–21]). In the last fifth section we use certain integral representations of series to evaluate several other sums with parametric harmonic numbers.

2. Lemmas and proofs

The following lemmas will be useful in the development of the main results of the paper.

Lemma 2.1. *Let n be a positive integers. Then the following identities hold*:

$$\sum_{k=1}^{n-1} \frac{\bar{H}_k}{n-k}(-1)^{k-1} = 2(-1)^{n-1}\sum_{k=1}^{n} \frac{H_k}{k}(-1)^{k-1} - 2(-1)^{n-1}\bar{H}_n^{(2)}, \tag{2.1}$$

$$\sum_{m=n+1}^{\infty} \frac{\bar{H}_m}{m-n}(-1)^{m-n} = \sum_{k=1}^{n} \frac{H_k}{k}(-1)^{k-1} - \sum_{j=1}^{\infty} \frac{\bar{H}_j}{j}(-1)^{j-1}, \tag{2.2}$$

$$(-1)^{n-1}\sum_{k=1}^{n-1} \frac{\bar{H}_k}{n-k} = \sum_{k=1}^{n-1} \frac{H_k}{n-k}(-1)^k, \tag{2.3}$$

$$(2.4)\quad \sum_{m=n+1}^{\infty} \frac{H_m}{m-n}(-1)^{m-n} = \frac{\bar{H}_n^2 + H_n^{(2)}}{2} - \ln 2\left(H_n + \bar{H}_n\right) - \sum_{n=1}^{\infty} \frac{H_j}{j}(-1)^{j-1},$$

where $\sum_{n=1}^{\infty} \frac{H_n(-1)^{n-1}}{n} = \frac{\zeta(2)-\ln^2 2}{2}$, $\sum_{n=1}^{\infty} \frac{\bar{H}_n(-1)^{n-1}}{n} = \frac{\zeta(2)+\ln^2 2}{2}$.

Proof. To prove identity (2.1), using the Cauchy product formula of power series, we have

$$(2.5)\qquad \frac{\ln(1+x)}{1-x} = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n\right)\left(\sum_{n=1}^{\infty} x^{n-1}\right) = \sum_{n=1}^{\infty} \bar{H}_n x^n.$$

Furthermore, by (2.5) and Cauchy product formula again, we deduce that

$$(2.6)\qquad \frac{\ln^2(1+x)}{1-x} = \sum_{n=1}^{\infty} (-1)^{n+1} \left(\sum_{k=1}^{n} \frac{\bar{H}_k}{n-k+1}(-1)^{k-1}\right) x^{n+1}.$$

On the other hand, from [9], we obtain

$$(2.7)\qquad \ln^2(1+x) = 2\left\{\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} x^n - \sum_{n=1}^{\infty} \frac{H_n}{n}(-1)^{n-1} x^n\right\}.$$

Hence, applying (2.7), then we have

$$(2.8)\qquad \frac{\ln^2(1+x)}{1-x} = 2\sum_{n=1}^{\infty}\sum_{k=1}^{n} \left\{\bar{H}_k^{(2)} - \frac{H_k}{k}(-1)^{k-1}\right\} x^n.$$

Comparing (2.6) and (2.8), we can obtain (2.1). Now, we prove identity (2.2). We see that we may rewrite the left hand side of (2.2) as

$$\begin{aligned}
\sum_{m=n+1}^{\infty} \frac{\bar{H}_m}{m-n}(-1)^{m-n} &= \sum_{j=1}^{\infty} \frac{\bar{H}_{n+j}}{j}(-1)^j \\
&= \sum_{j=1}^{\infty} \frac{(-1)^j}{j}\left\{\sum_{k=1}^{j} \frac{(-1)^{k-1}}{k} + \sum_{k=j+1}^{n+j} \frac{(-1)^{k-1}}{k}\right\} \\
&= \sum_{j=1}^{\infty} \frac{\bar{H}_j}{j}(-1)^j + \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} \sum_{j=1}^{\infty}\left(\frac{1}{j} - \frac{1}{j+k}\right) \\
(2.9)\qquad &= \sum_{k=1}^{n} \frac{H_k}{k}(-1)^{k-1} - \sum_{j=1}^{\infty} \frac{\bar{H}_j}{j}(-1)^{j-1}.
\end{aligned}$$

Similarly as in the proofs of (2.1) and (2.2), we can prove (2.3) and (2.4). □

Lemma 2.2. *For integers* $m, p \in \mathbb{N} \setminus \{1\}$ *and reals* $a, b \notin \mathbb{N}^-$, *we have*

$$(2.10)\qquad \sum_{n=1}^{\infty} \left\{\frac{\zeta(m,b+1)\zeta_n(p,a+1)}{n+a} - \frac{\zeta(p,a+1)\zeta_n(m,b+1)}{n+b}\right\}$$
$$= \zeta(m,b+1)\zeta(p,a+1)\left(\psi(b+1) - \psi(a+1)\right)$$

$$+\zeta(p,a+1)\sum_{n=1}^{\infty}\frac{\zeta_n(1,b+1)}{(n+b)^m}+\zeta(m,b+1)\zeta(p+1,a+1)$$
$$-\zeta(p,a+1)\zeta(m+1,b+1)-\zeta(m,b+1)\sum_{n=1}^{\infty}\frac{\zeta_n(1,a+1)}{(n+b)^p},$$

where $\psi(z)$ denotes the digamma function (or called Psi function) which is defined as the logarithmic derivative of the well known gamma function ([15]):

$$\psi(z):=\frac{d}{dz}(\ln\Gamma(z))=\frac{\Gamma'(z)}{\Gamma(z)}=-\gamma-\frac{1}{z}+\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+z}\right). \tag{2.11}$$

In general, the polygamma function of order m is defined as the $(m+1)$-th derivative of the logarithm of the gamma function:

$$\psi^{(m)}(z):=\frac{d^m}{dz^m}\psi(z)=\frac{d^{m+1}}{dz^{m+1}}\ln(\Gamma(z))\quad(m\in\mathbb{N}_0,z\notin\mathbb{N}_0^-:=\mathbb{N}^-\cup\{0\}). \tag{2.12}$$

Proof. To prove the identity we consider the N-th partial sum

$$\sum_{n=1}^{N}\left\{\frac{\zeta(m,b+1)\zeta_n(p,a+1)}{n+a}-\frac{\zeta(p,a+1)\zeta_n(m,b+1)}{n+b}\right\}$$
$$=\zeta(m,b+1)\sum_{n=1}^{N}\frac{\zeta_n(p,a+1)}{n+a}-\zeta(p,a+1)\sum_{n=1}^{N}\frac{\zeta_n(m,b+1)}{n+b}$$
$$=\zeta(m,b+1)\sum_{n=1}^{N}\sum_{k=1}^{n}\frac{1}{(n+a)(k+a)^p}-\zeta(p,a+1)\sum_{n=1}^{N}\sum_{k=1}^{n}\frac{1}{(n+b)(k+b)^m}$$
$$=\zeta(m,b+1)\sum_{k=1}^{N}\sum_{n=k}^{N}\frac{1}{(k+a)^p(n+a)}-\zeta(p,a+1)\sum_{k=1}^{N}\sum_{n=k}^{N}\frac{1}{(k+b)^m(n+b)}$$
$$=\zeta(m,b+1)\sum_{k=1}^{N}\frac{\zeta_N(1,a+1)-\zeta_{k-1}(1,a+1)}{(k+a)^p}$$
$$-\zeta(p,a+1)\sum_{k=1}^{N}\frac{\zeta_N(1,b+1)-\zeta_{k-1}(1,b+1)}{(k+b)^p}$$
$$=\zeta(m,b+1)\zeta_N(1,a+1)\zeta_N(p,a+1)-\zeta(p,a+1)\zeta_N(1,b+1)\zeta_N(m,b+1)$$
$$+\zeta(p,a+1)\sum_{k=1}^{N}\frac{\zeta_k(1,b+1)}{(k+b)^p}-\zeta(p,a+1)\zeta_N(m+1,b+1)$$
$$-\zeta(m,b+1)\sum_{k=1}^{N}\frac{\zeta_k(1,a+1)}{(k+a)^p}+\zeta(m,b+1)\zeta_N(p+1,a+1).$$

Letting $N \to \infty$ in above equation, and using the following formula

$$\begin{aligned}&\lim_{N\to\infty}\{\zeta(m,b+1)\zeta_N(1,a+1)\zeta_N(p,a+1)\\&\quad-\zeta(p,a+1)\zeta_N(1,b+1)\zeta_N(m,b+1)\}\\&=\zeta(m,b+1)\zeta(p,a+1)\sum_{n=1}^{\infty}\left\{\frac{1}{n+a}-\frac{1}{n+b}\right\}\\&=\zeta(m,b+1)\zeta(p,a+1)\left(\psi(b+1)-\psi(a+1)\right),\end{aligned}$$

we may easily deduce the result. □

3. Identities for the parametric Euler sums

We begin by establishing some relations between parametric Euler sums and certain rational series by using the methods of constructing function $T_{s,t}^{(p_1,p_2)}(x, y; a)$, and contour integral representations and residues computations. First, we give a simple theorem.

Theorem 3.1. *For $s \in \mathbb{N}$, $a \notin \mathbb{N}^-$ and $x, y \in [-1, 1)$. Then the following identity holds*:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{y^n\sum\limits_{j=1}^{n-1}\frac{x^{n-j}}{j}+x^n\sum\limits_{j=1}^{n-1}\frac{y^{n-j}}{j}}{(n+a)^s}&=s\mathrm{Li}_{s+1}(a,xy)-\sum_{j=1}^{s}\mathrm{Li}_j(a,x)\mathrm{Li}_{s+1-j}(a,y)\\&\quad+\mathrm{Li}_s(a,xy)\left(\mathrm{Li}_1(x)+\mathrm{Li}_1(y)\right),\end{aligned}\tag{3.1}$$

where $\mathrm{Li}_s(a,x)$ denotes the parametric polylogarithm function defined by

$$\mathrm{Li}_s(a,x) := \sum_{n=1}^{\infty}\frac{x^n}{(n+a)^s},\ \Re(s)\ge 1, -1\le x<1.$$

Obviously, setting $x = 1$ in above equation, then

$$\mathrm{Li}_s(a,1) = \zeta(s,a+1).$$

If $a = 0$, then the function $\mathrm{Li}_s(a,x)$ are reducible to the classical polylogarithm function $\mathrm{Li}_s(x)$ which is defined by

$$\mathrm{Li}_s(x) := \sum_{n=1}^{\infty}\frac{x^n}{n^s},\ \Re(s)>1,\ x\in[-1,1],$$

with $\mathrm{Li}_1(x) = -\ln(1-x)$, $x \in [-1, 1)$.

Proof. For real $-1 \le x, y < 1$ and non-negative integers s and t, we consider the function

$$T_{s,t}^{(0,0)}(x,y;a) = \sum_{\substack{m,n=1\\m\ne n}}^{\infty}\frac{x^n y^m}{(n+a)^s(m+a)^t(m-n)}$$

$$= \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^n y^m (m-n+n+a)}{(n+a)^s (m+a)^{t+1} (m-n)}$$
$$= \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^n y^m}{(n+a)^s (m+a)^{t+1}}$$
$$+ \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^n y^m}{(n+a)^{s-1} (m+a)^{t+1} (m-n)}$$
$$= \sum_{n=1}^{\infty} \frac{x^n}{(n+a)^s} \left\{ \sum_{m=1}^{\infty} \frac{y^m}{(m+a)^{t+1}} - \frac{y^n}{(n+a)^{t+1}} \right\}$$
$$+ T_{s-1,t+1}(x,y)$$
$$= \mathrm{Li}_s(a,x)\,\mathrm{Li}_{t+1}(a,y) - \mathrm{Li}_{s+t+1}(a,xy) + T^{(0,0)}_{s-1,t+1}(x,y;a).$$

Telescoping this gives

$$T^{(0,0)}_{s,t}(x,y;a) = T^{(0,0)}_{0,s+t}(x,y;a) - s\mathrm{Li}_{s+t+1}(a,xy) + \sum_{j=1}^{s} \mathrm{Li}_j(a,x)\,\mathrm{Li}_{s+t+1-j}(a,y), \quad s \in \mathbb{N}_0.$$

With $t = 0$ in the above equation, this becomes

$$T^{(0,0)}_{s,0}(x,y;a) = T^{(0,0)}_{0,s}(x,y;a) - s\mathrm{Li}_{s+1}(a,xy) + \sum_{j=1}^{s} \mathrm{Li}_j(a,x)\,\mathrm{Li}_{s+1-j}(a,y), \quad s \in \mathbb{N}. \tag{3.2}$$

On the other hand, for any integers s and t, we have

$$T^{(0,0)}_{t,s}(x,y;a) = \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^n y^m}{(n+a)^t (m+a)^s (m-n)}$$
$$= -\sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{y^m x^n}{(m+a)^s (n+a)^t (n-m)} \tag{3.3}$$
$$= -T^{(0,0)}_{s,t}(y,x;a).$$

Moreover, from the definition of $T^{(0,0)}_{s,t}(x,y;a)$, we can deduce that

$$T^{(0,0)}_{s,0}(x,y;a) = \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{x^n y^m}{(n+a)^s (m-n)}$$

$$= \sum_{n=1}^{\infty} \frac{x^n y^n}{(n+a)^s} \sum_{m=n+1}^{\infty} \frac{y^{m-n}}{m-n} - \sum_{n=1}^{\infty} \frac{x^n}{(n+a)^s} \sum_{m=1}^{n-1} \frac{y^m}{n-m} \tag{3.4}$$

$$= \mathrm{Li}_s(a,xy)\,\mathrm{Li}_1(y) - \sum_{n=1}^{\infty} \frac{x^n}{(n+a)^s} \sum_{j=1}^{n-1} \frac{y^{n-j}}{j}.$$

Hence, substituting formulae (3.3) and (3.4) into (3.2) yields the desired result. Thus, we complete the proof of Theorem 3.1. □

In the same manner, we can get the following general theorem.

Theorem 3.2. *For $s \in \mathbb{N}$, $a, b \notin \mathbb{N}^-$ and $x, y \in (-1,1)$ with $a-b \notin \mathbb{N} \cup \mathbb{N}^-$. Then the following identity holds*:

$$\sum_{n=1}^{\infty} \frac{x^n}{(n+a)^s} \sum_{j=1}^{n-1} \frac{y^{n-j}}{j+a-b} + \sum_{n=1}^{\infty} \frac{y^n}{(n+b)^s} \sum_{j=1}^{n-1} \frac{x^{n-j}}{j+b-a}$$
$$= \mathrm{Li}_s(a,xy)\,\mathrm{Li}_1(b-a,y) + \mathrm{Li}_s(b,xy)\,\mathrm{Li}_1(a-b,x)$$
$$+ \sum_{j=1}^{s} \{\mathrm{Li}_{j,s+1-j}(a,b,xy) - \mathrm{Li}_j(a,x)\,\mathrm{Li}_{s+1-j}(b,y)\}, \tag{3.5}$$

where $\mathrm{Li}_{s,t}(a,b,x)$ denotes the two parametric polylogarithm function defined by

$$\mathrm{Li}_{s,t}(a,b,x) := \sum_{n=1}^{\infty} \frac{x^n}{(n+a)^s (n+b)^t}, \quad |x| < 1,$$

with $s \in \mathbb{N}_0, t \in \mathbb{N}_0$ and $s+t \geq 1$.

Proof. To prove (3.5), we consider the following Tornheim-type series

$$T_{s,t}^{(0,0)}(x,y;a,b) := \sum_{\substack{m,n=1 \\ m \neq n}}^{\infty} \frac{x^n y^m}{(n+a)^s (m+b)^t (m+b-n-a)}.$$

Then applying the same arguments as in the proof of Theorem 3.1, we may easily deduce the result. □

It is clear that Theorem 3.1 is immediate corollary of Theorem 3.2. Next, we establish some relations between parametric quadratic sums and parametric linear sums.

Theorem 3.3. *Let $s \geq 1$ be integer with $a \notin \mathbb{N}^-$. Then the following identities hold*:

$$\frac{1}{2}\bar{S}\begin{pmatrix} 1,1 \\ \{a\}_s \end{pmatrix} - \bar{S}\begin{pmatrix} 1,\bar{1} \\ \{a\}_s \end{pmatrix}$$
$$= \frac{3}{2}\bar{S}\begin{pmatrix} 2 \\ \{a\}_s \end{pmatrix} + s\bar{S}\begin{pmatrix} 1 \\ \{a\}_{s+1} \end{pmatrix} + a\bar{\zeta}(s,a+1)\,S\begin{pmatrix} 1 \\ 0,a \end{pmatrix} \tag{3.6}$$

$$-\ln 2\bar{S}\begin{pmatrix} 1 \\ \{a\}_s \end{pmatrix} - \bar{S}\begin{pmatrix} 1 \\ 0, \{a\}_s \end{pmatrix}$$
$$-\sum_{j=2}^{s}\bar{\zeta}(s+1-j,a+1)\,S\begin{pmatrix} 1 \\ \{a\}_j \end{pmatrix},$$

$$\frac{3}{2}\left(S\begin{pmatrix} 1,1 \\ \{a\}_s \end{pmatrix} - S\begin{pmatrix} 2 \\ \{a\}_s \end{pmatrix}\right)$$
$$= sS\begin{pmatrix} 1 \\ \{a\}_{s+1} \end{pmatrix} + S\begin{pmatrix} 1 \\ 0, \{a\}_s \end{pmatrix} + S\begin{pmatrix} 1 \\ \{a\}_s \end{pmatrix}(\psi(a+1)+\gamma) \tag{3.7}$$
$$+ a\zeta(s,a+1)\,S\begin{pmatrix} 1 \\ 0, a \end{pmatrix} - \sum_{j=2}^{s-1}\zeta(s+1-j,a+1)\,S\begin{pmatrix} 1 \\ \{a\}_j \end{pmatrix},$$

where $s>1$ *in* (3.7).

Proof. Similarly as the proof of Theorem 3.1, we consider the function

$$T_{s,t}^{(1,0)}(x,y;a) = \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} \frac{H_n x^n y^m}{(n+a)^s (m+a)^t (m-n)}$$
$$= H_s^{(1)}(a,x)\,\mathrm{Li}_{t+1}(a,y) - H_{s+t+1}^{(1)}(a,xy) + T_{s-1,t+1}^{(1,0)}(x,y;a) \tag{3.8}$$
$$= T_{0,s+t}^{(1,0)}(x,y;a) - sH_{s+t+1}^{(1)}(a,xy)$$
$$+ \sum_{j=1}^{s} H_j^{(1)}(a,x)\,\mathrm{Li}_{s+t+1-j}(a,y),$$

where the function $H_s^{(m)}(a,x)$ is defined by

$$H_s^{(m)}(a,x) := \sum_{n=1}^{\infty} \frac{H_n^m}{(n+a)^s} x^n,\ \ s\in\mathbb{N}, m\in\mathbb{N}_0, x\in[-1,1).$$

Taking $t=0$ in (3.8), we have

$$T_{s,0}^{(1,0)}(x,y;a) = T_{0,s}^{(1,0)}(x,y;a) - sH_{s+1}^{(1)}(a,xy)$$
$$+ \sum_{j=1}^{s} H_j^{(1)}(a,x)\,\mathrm{Li}_{s+1-j}(a,y). \tag{3.9}$$

Furthermore, by using the definition of $T_{s,t}^{(1,0)}(x,y;a)$, we can find that

$$T_{s,0}^{(1,0)}(x,y;a) = \sum_{\substack{m,n=1 \\ m\neq n}}^{\infty} \frac{H_n x^n y^m}{(n+a)^s (m-n)}$$
$$= H_s^{(1)}(a,xy)\,\mathrm{Li}_1(y) - \sum_{n=1}^{\infty} \frac{H_n x^n}{(n+a)^s}\left(\sum_{j=1}^{n-1}\frac{y^{n-j}}{j}\right), \tag{3.10}$$

$$T_{0,s}^{(1,0)}(x,y;a) = \sum_{\substack{m,n=1\\ m\neq n}}^{\infty} \frac{H_n x^n y^m}{(m+a)^s (m-n)}$$

$$= -\mathrm{Li}_s(a,xy) \sum_{n=m+1}^{\infty} \frac{H_n}{n-m} x^{n-m} + \sum_{m=1}^{\infty} \frac{y^m}{(m+a)^s} \sum_{n=1}^{m-1} \frac{H_n}{m-n} x^n. \tag{3.11}$$

Hence, substituting (3.11) and (3.10) into (3.9) respectively, then letting $x \to 1$ and $y \to -1$ with the help of formula (2.4), we can deduce the formula (3.6). To prove the second identity of our theorem we note that

$$\sum_{m=n+1}^{\infty} \frac{H_m x^{m-n}}{m-n} = \sum_{j=1}^{\infty} \frac{H_j}{j} x^j + \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{\infty} x^j \left(\frac{1}{j} - \frac{1}{k+j}\right). \tag{3.12}$$

Then putting $x = y$ in (3.9)-(3.11) and combining (3.12), we obtain

$$\sum_{n=1}^{\infty} \frac{H_n x^n}{(n+a)^s} \left(\sum_{j=1}^{n-1} \frac{x^{n-j}}{j}\right) + \sum_{n=1}^{\infty} \frac{x^n}{(n+a)^s} \left(\sum_{j=1}^{n-1} \frac{H_j x^j}{n-j}\right)$$

$$= sH_{s+1}^{(1)}\left(a,x^2\right) + \sum_{n=1}^{\infty} \frac{x^{2n}}{(n+a)^s} \left(\sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{\infty} x^j \left(\frac{1}{j} - \frac{1}{k+j}\right)\right)$$

$$- \sum_{j=2}^{s-1} H_j^{(1)}(a,x)\,\mathrm{Li}_{s+1-j}(a,x)$$

$$+ \left(H_s^{(1)}\left(a,x^2\right)\mathrm{Li}_1(x) - H_s^{(1)}(a,x)\,\mathrm{Li}_1(a,x)\right)$$

$$+ \left(\mathrm{Li}_s\left(a,x^2\right) H_1^{(1)}(0,x) - \mathrm{Li}_s(a,x) H_1^{(1)}(a,x)\right). \tag{3.13}$$

Moreover, by a direct calculation, we arrive at the conclusion that

$$\lim_{x\to 1} \left(H_s^{(1)}\left(a,x^2\right)\mathrm{Li}_1(x) - H_s^{(1)}(a,x)\,\mathrm{Li}_1(a,x)\right)$$

$$= \left(\sum_{n=1}^{\infty} \frac{H_n}{(n+a)^s}\right)(\psi(a+1)+\gamma), \tag{3.14}$$

$$\lim_{x\to 1} \left(\mathrm{Li}_s\left(a,x^2\right) H_1^{(1)}(0,x) - \mathrm{Li}_s(a,x)\,H_1^{(1)}(a,x)\right)$$

$$= a\zeta(s,a+1)\left(\sum_{n=1}^{\infty} \frac{H_n}{n(n+a)}\right), \tag{3.15}$$

$$\lim_{x\to 1} \sum_{k=1}^{n} \frac{1}{k} \sum_{j=1}^{\infty} x^j \left(\frac{1}{j} - \frac{1}{k+j}\right) = \sum_{k=1}^{n} \frac{H_k}{k}, \tag{3.16}$$

$$\sum_{j=1}^{n-1}\frac{H_j}{n-j}=H_n^2-H_n^{(2)},\ \sum_{k=1}^{n}\frac{H_k}{k}=\frac{H_n^2+H_n^{(2)}}{2}. \tag{3.17}$$

So, taking the limit in (3.13) and combining (3.14)-(3.17) yields the desired result. This completes the proof of Theorem 3.2. □

Letting $x\to -1$ and $y\to 1$ in (3.9)-(3.11), we also obtain that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sum\limits_{k=1}^{n-1}\frac{H_k}{n-k}(-1)^k}{(n+a)^s}&=\bar{S}\begin{pmatrix}1,1\\ \{a\}_s\end{pmatrix}+\frac{1}{2}\left(\zeta(2)-\ln^2 2\right)\bar{\zeta}(s,a+1)\\&+\ln 2\left(\bar{S}\begin{pmatrix}1\\ \{a\}_s\end{pmatrix}+\bar{S}\begin{pmatrix}\bar{1}\\ \{a\}_s\end{pmatrix}\right)\\&+\sum_{j=1}^{s-1}\zeta(s+1-j,a+1)\,\bar{S}\begin{pmatrix}1\\ \{a\}_j\end{pmatrix}-\bar{S}\begin{pmatrix}1\\ 0,\{a\}_s\end{pmatrix}\\&-\frac{1}{2}\left(\bar{S}\begin{pmatrix}\bar{1},\bar{1}\\ \{a\}_s\end{pmatrix}+\bar{S}\begin{pmatrix}2\\ \{a\}_s\end{pmatrix}\right)-s\bar{S}\begin{pmatrix}1\\ \{a\}_{s+1}\end{pmatrix}\\&-\bar{S}\begin{pmatrix}1\\ \{a\}_s\end{pmatrix}(\psi(a+1)+\gamma),\end{aligned} \tag{3.18}$$

where γ denotes the Euler-Mascheroni constant, defined by

$$\gamma:=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\ln n\right)=-\psi(1)\approx 0.577215664901532860606512\cdots.$$

Proceeding in a similar fashion to the evaluation of Theorem 3.3, it is possible to establish other parametric relations involving harmonic numbers. In fact, by constructing the following Tornheim-type series

$$T_{s,t}^{(1,\bar{1})}(x,y;a)=\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{H_n\bar{H}_m x^n y^m}{(n+a)^s(m+a)^t(m-n)},$$

$$T_{s,t}^{(\bar{1},0)}(x,y;a)=\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{\bar{H}_n x^n y^m}{(n+a)^s(m+a)^t(m-n)},$$

$$T_{s,t}^{(1,1)}(x,x;a)=\sum_{\substack{m,n=1\\ m\neq n}}^{\infty}\frac{H_n H_m x^{n+m}}{(n+a)^s(m+a)^t(m-n)},$$

and by a similar argument as in the proof of Theorem 3.2, we can give some further results. See the following Theorems 3.4-3.6.

Theorem 3.4. *For integer $s \in \mathbb{N}$ and real $a \notin \mathbb{N}^-$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{H_n}{(n+a)^s}(-1)^{n-1} \left\{ \sum_{k=1}^{n} \frac{H_k}{k}(-1)^{k-1} - 2\bar{H}_n^{(2)} \right\} \\
&+ \frac{1}{2}\left(\bar{S}\left(\begin{array}{c} 1,1,\bar{1} \\ \{a\}_s \end{array} \right) - 3\bar{S}\left(\begin{array}{c} \bar{1},2 \\ \{a\}_s \end{array} \right) \right) \\
&= a\bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_s \end{array} \right) S\left(\begin{array}{c} 1 \\ 0,a \end{array} \right) + s\bar{S}\left(\begin{array}{c} 1,\bar{1} \\ \{a\}_{s+1} \end{array} \right) \\
&- \frac{\ln^2 2 + \zeta(2)}{2}\bar{S}\left(\begin{array}{c} 1 \\ \{a\}_s \end{array} \right) \\
&- \sum_{j=2}^{s} S\left(\begin{array}{c} 1 \\ \{a\}_j \end{array} \right)\bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_{s+1-j} \end{array} \right).
\end{aligned} \tag{3.19}$$

Theorem 3.5. *For integer $s \in \mathbb{N} \setminus \{1\}$ and real $a \notin \mathbb{N}^-$, we have*

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sum\limits_{k=1}^{n} \frac{H_k}{k}(-1)^{k-1}}{(n+a)^s} &= sS\left(\begin{array}{c} \bar{1} \\ \{a\}_{s+1} \end{array} \right) + S\left(\begin{array}{c} \bar{1},\bar{1} \\ \{a\}_s \end{array} \right) + 2S\left(\begin{array}{c} \bar{2} \\ \{a\}_s \end{array} \right) \\
&- \ln 2 S\left(\begin{array}{c} \bar{1} \\ \{a\}_s \end{array} \right) - \bar{S}\left(\begin{array}{c} \bar{1} \\ 0,\{a\}_s \end{array} \right) \\
&- \frac{\ln^2 2 + \zeta(2)}{2}\zeta(s,a+1) \\
&- \sum_{j=1}^{s} \bar{\zeta}(s+1-j,a+1)\,\bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_j \end{array} \right),
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\sum\limits_{k=1}^{n-1} \frac{\bar{H}_k}{n-k}(-1)^{n-1}}{(n+a)^s} &= \sum_{n=1}^{\infty} \frac{\sum\limits_{k=1}^{n} \frac{\bar{H}_k}{k}}{(n+a)^s}(-1)^{n-1} + s\bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_{s+1} \end{array} \right) \\
&+ a\bar{\zeta}(s,a+1)\,S\left(\begin{array}{c} \bar{1} \\ 0,a \end{array} \right) - \ln 2\bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_s \end{array} \right) \\
&- S\left(\begin{array}{c} \bar{1} \\ 0,\{a\}_s \end{array} \right) - \ln 2\left(\bar{S}\left(\begin{array}{c} 1 \\ \{a\}_s \end{array} \right) + \bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_s \end{array} \right) \right) \\
&+ \bar{S}\left(\begin{array}{c} \bar{1},\bar{1} \\ \{a\}_s \end{array} \right) - \sum_{j=2}^{s} \bar{\zeta}(s+1-j,a+1)\,S\left(\begin{array}{c} \bar{1} \\ \{a\}_j \end{array} \right),
\end{aligned} \tag{3.21}$$

$$\sum_{n=1}^{\infty} \frac{\sum\limits_{k=1}^{n} \frac{H_k}{k}(-1)^{k-1}}{(n+a)^s}(-1)^{n-1} = s\bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_{s+1} \end{array} \right) + \bar{S}\left(\begin{array}{c} \bar{1} \\ \{a\}_s \end{array} \right)(\psi(a+1)+\gamma)$$

$$+\bar{S}\begin{pmatrix}\bar{1}\\0,\{a\}_s\end{pmatrix}+2\bar{S}\begin{pmatrix}\bar{2}\\\{a\}_s\end{pmatrix}$$

$$-\sum_{j=1}^{s-1}\zeta(s+1-j,a+1)\bar{S}\begin{pmatrix}\bar{1}\\\{a\}_j\end{pmatrix} \tag{3.22}$$

$$-\bar{S}\begin{pmatrix}1,\bar{1}\\\{a\}_s\end{pmatrix}-\frac{1}{2}\left(\zeta(2)+\ln^2 2\right)\bar{\zeta}(s,a+1).$$

Theorem 3.6. *For integer $s\in\mathbb{N}\setminus\{1\}$ and real $a\notin\mathbb{N}^-$, we have*

$$S\begin{pmatrix}1,1,1\\\{a\}_s\end{pmatrix}-3S\begin{pmatrix}1,2\\\{a\}_s\end{pmatrix}$$

$$=sS\begin{pmatrix}1,1\\\{a\}_{s+1}\end{pmatrix}-\sum_{j=2}^{s-1}S\begin{pmatrix}1\\\{a\}_j\end{pmatrix}S\begin{pmatrix}1\\\{a\}_{s+1-j}\end{pmatrix} \tag{3.23}$$

$$+2aS\begin{pmatrix}1\\\{a\}_s\end{pmatrix}S\begin{pmatrix}1\\0,a\end{pmatrix}.$$

Combining (2.3), (3.18) and (3.21), we can get the following corollary.

Corollary 3.7. *Let $s\geq 1$ be integer with $a\notin\mathbb{N}^-$. Then the following identity holds*:

$$\sum_{n=1}^{\infty}\frac{\sum\limits_{k=1}^{n}\frac{\bar{H}_k}{k}}{(n+a)^s}(-1)^{n-1}=\bar{S}\begin{pmatrix}1,1\\\{a\}_s\end{pmatrix}+2\ln 2\left(\bar{S}\begin{pmatrix}1\\\{a\}_s\end{pmatrix}+\bar{S}\begin{pmatrix}\bar{1}\\\{a\}_s\end{pmatrix}\right)$$

$$-\frac{3}{2}\bar{S}\begin{pmatrix}\bar{1},\bar{1}\\\{a\}_s\end{pmatrix}-\frac{1}{2}\bar{S}\begin{pmatrix}2\\\{a\}_s\end{pmatrix}$$

$$+\sum_{j=1}^{s-1}\zeta(s+1-j,a+1)\bar{S}\begin{pmatrix}1\\\{a\}_j\end{pmatrix}$$

$$+\sum_{j=1}^{s}\bar{\zeta}(s+1-j,a+1)S\begin{pmatrix}\bar{1}\\\{a\}_j\end{pmatrix}$$

$$-\bar{S}\begin{pmatrix}1\\0,\{a\}_s\end{pmatrix}-s\bar{S}\begin{pmatrix}1\\\{a\}_{s+1}\end{pmatrix}-s\bar{S}\begin{pmatrix}\bar{1}\\\{a\}_{s+1}\end{pmatrix} \tag{3.24}$$

$$+\ln 2\bar{S}\begin{pmatrix}\bar{1}\\\{a\}_s\end{pmatrix}+S\begin{pmatrix}\bar{1}\\0,\{a\}_s\end{pmatrix}$$

$$+\frac{1}{2}\left(\zeta(2)-\ln^2 2\right)\bar{\zeta}(s,a+1)$$

$$-\bar{S}\begin{pmatrix}1\\\{a\}_s\end{pmatrix}(\psi(a+1)+\gamma)$$

$$-a\bar{\zeta}(s,a+1)S\begin{pmatrix}\bar{1}\\0,a\end{pmatrix}.$$

Next, we use certain contour integral representations to evaluate several parametric Euler series with harmonic numbers. We need the following lemma.

Lemma 3.8 ([11]). *Let $\xi(s)$ be a kernel function and let $r(s)$ be a rational function which is $O(s^{-2})$ at infinity. Then*

$$\sum_{\alpha\in E}\mathrm{Res}(r(s)\,\xi(s))_{s=\alpha}+\sum_{\beta\in S}\mathrm{Res}(r(s)\,\xi(s))_{s=\beta}=0, \tag{3.25}$$

where S is the set of poles of $r(s)$ and E is the set of poles of $\xi(s)$ that are not poles $r(s)$. Here $\mathrm{Res}(r(s))_{s=\alpha}$ denotes the residue of $r(s)$ at $s=\alpha$. The kernel function $\xi(s)$ is meromorphic in the whole complex plane and satisfies $\xi(s)=o(s)$ over an infinite collection of circles $|z|=\rho_k$ with $\rho_k\to\infty$.

Moreover, Flajolet and Salvy [11] gave the following formulas:

$$\pi\cot(\pi s)\overset{s\to n}{=}\frac{1}{s-n}-2\sum_{k=1}^{\infty}\zeta(2k)\,(s-n)^{2k-1},$$

$$\psi(-s)+\gamma\overset{s\to n}{=}\frac{1}{s-n}+H_n+\sum_{k=1}^{\infty}\left((-1)^kH_n^{(k+1)}-\zeta(k+1)\right)(s-n)^k,\ \ n\geq 0$$

$$\psi(-s)+\gamma\overset{s\to -n}{=}H_{n-1}+\sum_{k=1}^{\infty}\left(H_{n-1}^{(k+1)}-\zeta(k+1)\right)(s+n)^k,\ n>0$$

$$\frac{\psi^{(p-1)}(-s)}{(p-1)!}\overset{s\to n}{=}\frac{1}{(s-n)^p}\left(1+(-1)^p\sum_{i\geq p}\binom{i-1}{p-1}\left(\zeta(i)+(-1)^iH_n^{(i)}\right)(s-n)^i\right),\ n\geq 0,\ p>1$$

$$\frac{\psi^{(p-1)}(-s)}{(p-1)!}\overset{s\to -n}{=}(-1)^p\sum_{i\geq 0}\binom{p-1+i}{p-1}\left(\zeta(p+i)-H_{n-1}^{(p+i)}\right)(s+n)^i,\ n>0,\ p>1.$$

Furthermore, by the definition of polygamma function, we can deduce that Using differentiation n times, we obtain

$$\frac{\psi^{(n)}(-s)}{n!}=\frac{1}{s^{n+1}}-(-1)^n\sum_{k=1}^{\infty}\binom{n+k-1}{k-1}\zeta(k+n)\,s^{k-1}\quad(0<|s|<1). \tag{3.26}$$

Now we state our main results.

Theorem 3.9. *Let m be positive integers with $a\notin\mathbb{N}_0^-$. Then the following parametric linear sums are reducible to zeta values and rational function series,*

$$\begin{aligned}S\begin{pmatrix}2m\\0,a,-a\end{pmatrix}&\\=\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}(n^2-a^2)}&-\frac{\pi\cot(\pi a)}{4a^2}\left(\zeta(2m,1-a)-\zeta(2m,1+a)\right)\\&+\frac{1}{2a^2}\sum_{k=1}^{m}\zeta(2k)\left\{\zeta(2m-2k+1,1+a)+\zeta(2m-2k+1,1-a)\right\}\end{aligned} \tag{3.27}$$

$$-\frac{1}{a^2}\sum_{k=1}^{m}\zeta(2k)\,\zeta(2m-2k+1)+\frac{m}{a^2}\zeta(2m+1)$$
$$-\frac{1}{4a^2}\left\{\zeta(2m+1,1+a)+\zeta(2m+1,1-a)-2\zeta(2m+1)\right\}.$$

Proof. The theorem follows by applying the kernel function

$$\xi(z)=\frac{\pi\cot(\pi z)\,\psi^{(2m-1)}(-z)}{(2m-1)!}$$

to the base function $r(z)=z^{-1}\left(z^2-a^2\right)^{-1}$. The only singularities are poles at the integers and $\pm a$. At a negative integer $-n$ the pole is simple and the residue is

$$\frac{H_n^{(2m)}-\zeta(2m)}{n\left(n^2-a^2\right)}-\frac{1}{n^{2m+1}\left(n^2-a^2\right)}.$$

At a positive integer n, the pole has order $2m+1$ and the residue is

$$\frac{H_n^{(2m)}+\zeta(2m)}{n\left(n^2-a^2\right)}+\frac{1}{2a^2}\left\{\frac{1}{(n+a)^{2m+1}}+\frac{1}{(n-a)^{2m+1}}-\frac{2}{n^{2m+1}}\right\}$$
$$-\frac{1}{a^2}\sum_{k=1}^{m}\zeta(2k)\left\{\frac{1}{(n+a)^{2m-2k+1}}+\frac{1}{(n-a)^{2m-2k+1}}-\frac{2}{n^{2m-2k+1}}\right\}.$$

The residue of the pole at $\pm a$ is

$$\frac{\pi\cot(\pi a)}{2a^2}\sum_{n=1}^{\infty}\left\{\frac{1}{(n-a)^{2m}}-\frac{1}{(n+a)^{2m}}\right\}.$$

Finally the residue of the pole of order $2m+2$ at 0 is found to be

$$-\frac{2m}{a^2}\zeta(2m+1).$$

Summing these four contributions yields the statement of the theorem. □

In the same manner we also obtain the following theorems.

Theorem 3.10. *For all $a\notin\mathbb{N}_0^-$, the quadratic sums $S\left(\begin{smallmatrix}1,1\\0,a,-a\end{smallmatrix}\right)$ reduce to linear sums*:

$$S\left(\begin{array}{c}1,1\\0,a,-a\end{array}\right)-S\left(\begin{array}{c}2\\0,a,-a\end{array}\right) \tag{3.28}$$
$$=\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n\left(n^2-a^2\right)}+\frac{1}{2a^2}\left\{S\left(\begin{array}{c}1\\a,a\end{array}\right)+S\left(\begin{array}{c}1\\-a,-a\end{array}\right)-4\zeta(3)\right\}$$
$$-\frac{1}{6a^2}\left\{\zeta(3,1+a)+\zeta(3,1-a)-2\zeta(3)\right\}-\frac{\zeta(3)}{a^2}$$
$$-\frac{(\psi(-a)+\gamma)^3+(\psi(a)+\gamma)^3}{6a^2}.$$

Proof. The proof is based on the cubic kernel

$$\xi(z) = (\psi(-z) + \gamma)^3$$

and the usual residue computation. When applied to an arbitrary rational function $r(z)$ satisfying the conditions: (i) $r(z)$ is $O(z^{-2})$ at infinity, (ii) $r(z)$ has no pole in $\mathbb{Z} \setminus \{-a, 0, a\}$, it yields the following summatory formula

$$\begin{aligned} &3\sum_{n=1}^{\infty} r(n)\left(H_n^2 - H_n^{(2)}\right) + 3\sum_{n=1}^{\infty} H_n r'(n) + \sum_{n=1}^{\infty}\left(\frac{1}{2}r''(n) - 3r(n)\zeta(2)\right) \\ &= -\sum_{\alpha\in S\cup\{0\}} \mathrm{Res}(\xi(z)\,r(z))_{z=\alpha}. \end{aligned}$$

The specialization to $r(z) = \frac{1}{z(z^2-a^2)}$ gives the statement. □

Theorem 3.11. *For integer $m \geqslant 0$ and real $a \notin \mathbb{N}_0^-$, the parametric linear sums are reducible to zeta values and rational function series*:

$$\begin{aligned} &S\begin{pmatrix} 2m+1 \\ a, -a \end{pmatrix} \\ &= \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^{2m+1}(n^2-a^2)} \\ &\quad + \frac{\pi\cot(\pi a)}{4a}\{\zeta(2m+1, 1-a) + \zeta(2m+1, 1+a) - 2\zeta(3)\} \\ &\quad - \frac{1}{2a}\sum_{k=0}^{m} \zeta(2k)\{\zeta(2m-2k+2, 1-a) - \zeta(2m-2k+2, 1+a)\}. \end{aligned} \tag{3.29}$$

Proof. Similarly as in the proof of Theorem 2.3. The theorem results from applying the kernel function

$$\xi(z) = \frac{\pi\cot(\pi z)\,\psi^{(2m)}(-z)}{(2m)!}$$

to the base function $r(z) = (z^2 - a^2)^{-1}$. Note that $r(z)$ can be rewritten as

$$r(z) = \frac{1}{z^2 - a^2} = \frac{1}{2a}\left(\frac{1}{z-a} - \frac{1}{z+a}\right). \tag{3.30}$$

Hence, by a similar argument as in the proof of Theorem 2.3, we may easily deduce the desired result. □

Note that formula above was also proved in [2] by another method. Moreover, by applying the partial fraction decomposition

$$\frac{1}{z^{2s}(z^2-a^2)} = \frac{1}{a^{2s}}\frac{1}{z^2-a^2} - \sum_{j=1}^{s} \frac{1}{a^{2s+2-2j}} \cdot \frac{1}{z^{2j}}$$

to (3.27) and (3.29), we have the following results

(3.31)
$$S\begin{pmatrix} 2m+1 \\ \{0\}_{2s}, a, -a \end{pmatrix} = \frac{1}{a^{2s}} S\begin{pmatrix} 2m+1 \\ a, -a \end{pmatrix} - \sum_{j=1}^{s} \frac{1}{a^{2s+2-2j}} S(2m+1, 2j),$$

(3.32)
$$S\begin{pmatrix} 2m \\ \{0\}_{2s+1}, a, -a \end{pmatrix} = \frac{1}{a^{2s}} S\begin{pmatrix} 2m \\ 0, a, -a \end{pmatrix} - \sum_{j=1}^{s} \frac{1}{a^{2s+2-2j}} S(2m, 2j+1).$$

Therefore, from Theorem 3.9, Theorem 3.11, formulas (3.31) and (3.32), we obtain the following description.

Theorem 3.12. *For integers $m \in \mathbb{N}, s \in \mathbb{N}_0$ and real $a \notin \mathbb{N}_0^-$. Then the linear sums*
$$S\begin{pmatrix} 2m \\ \{0\}_{2s+1}, a, -a \end{pmatrix}, \quad S\begin{pmatrix} 2m-1 \\ \{0\}_{2s}, a, -a \end{pmatrix}$$
are reducible to zeta values and rational series.

4. Some examples

From Theorems 3.1-3.6 and Theorems 3.9-3.11, we can give the following examples.

$$S\begin{pmatrix} 1 \\ \{a\}_s \end{pmatrix} = \frac{s}{2}\zeta(s+1, a+1) - \frac{1}{2}\sum_{j=1}^{s-2} \zeta(s-j, a+1)\zeta(j+1, a+1)$$
$$+ \zeta(s, a+1)(\psi(a+1)+\gamma) + \frac{\psi(a+1)+\gamma}{a^s} - \sum_{j=2}^{s} \frac{\zeta(j, a+1)}{a^{s+1-j}}, \tag{4.1}$$

$$S\begin{pmatrix} \bar{1} \\ \{a\}_s \end{pmatrix} = \frac{1}{2}\sum_{j=1}^{s-2} \bar{\zeta}(s-j, a+1)\bar{\zeta}(j+1, a+1) - \frac{s}{2}\zeta(s+1, a+1)$$
$$+ \zeta(s, a+1)\ln 2 + \bar{\zeta}(s, a+1)\bar{\zeta}(1, a+1) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(n+a)^s}, \tag{4.2}$$

$$S\begin{pmatrix} 1 \\ a, -a \end{pmatrix}$$
$$= \sum_{n=1}^{\infty} \frac{n}{(n^2-a^2)^2} + \left(1 - a^2\left(\sum_{n=1}^{\infty} \frac{1}{n^2-a^2}\right)\right)\left(\sum_{n=1}^{\infty} \frac{1}{n(n^2-a^2)}\right), \tag{4.3}$$

$$S\begin{pmatrix} \bar{1} \\ a, -a \end{pmatrix} = \left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-a^2}\right)\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n+a}\right) + a\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2-a^2}\right)^2$$

$$+\ln 2\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}+\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n\left(n^2-a^2\right)}-\sum_{n=1}^{\infty}\frac{n}{\left(n^2-a^2\right)^2}, \tag{4.4}$$

$$\begin{aligned}\bar{S}\left(\begin{array}{c}\bar{1}\\ a,-a\end{array}\right)=&\sum_{n=1}^{\infty}\frac{1}{n\left(n^2-a^2\right)}+\ln 2\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2-a^2}\\&+\left(\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}\right)\left(\sum_{n=1}^{\infty}\frac{n}{n^2-a^2}(-1)^{n-1}\right)\\&-\sum_{n=1}^{\infty}\frac{1}{(2n-1)\left((2n-1)^2-a^2\right)}-\sum_{n=1}^{\infty}\frac{n}{\left(n^2-a^2\right)^2}(-1)^{n-1},\end{aligned} \tag{4.5}$$

$$\begin{aligned}S\left(\begin{array}{c}2\\ 0,a,-a\end{array}\right)=&\frac{\zeta(3)}{a^2}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3\left(n^2-a^2\right)}+\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n\left(n^2-a^2\right)}\\&-\frac{1}{4a^2}\left\{\zeta(3,1+a)+\zeta(3,1-a)-2\zeta(3)\right\}\\&-\frac{\pi\cot(\pi a)}{4a^2}\left\{\zeta(2,1-a)-\zeta(2,1+a)\right\},\end{aligned} \tag{4.6}$$

$$\begin{aligned}S\left(\begin{array}{c}1,1\\ 0,a,-a\end{array}\right)=&2\zeta(2)\sum_{n=1}^{\infty}\frac{1}{n\left(n^2-a^2\right)}+\frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^3\left(n^2-a^2\right)}\\&+\frac{\zeta(3,1+a)+\zeta(3,1-a)}{12a^2}\\&+\left(\frac{1}{a}-\frac{\pi\cot(\pi a)}{2}\right)\frac{\zeta(2,1-a)-\zeta(2,1+a)}{2a^2}-\frac{7}{6a^2}\zeta(3)\\&+\frac{\zeta(2,1+a)(\psi(a+1)+\gamma)+\zeta(2,1-a)(\psi(1-a)+\gamma)}{2a^2}\\&+\frac{\psi(a+1)+\psi(1-a)+2\gamma}{2a^4}-\frac{(\psi(-a)+\gamma)^3+(\psi(a)+\gamma)^3}{6a^2},\end{aligned} \tag{4.7}$$

$$\begin{aligned}S\left(\begin{array}{c}1\\ 0,a,-a\end{array}\right)=&\frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{\left(n^2-a^2\right)^2}+\sum_{n=1}^{\infty}\frac{1}{n^2\left(n^2-a^2\right)}-\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}\right)^2\\&-\frac{a^2}{2}\sum_{n=1}^{\infty}\frac{1}{n^2\left(n^2-a^2\right)^2}-\frac{a^2}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2\left(n^2-a^2\right)}\right)^2,\end{aligned} \tag{4.8}$$

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{H_n}{n^4-a^4}=&2\sum_{n=1}^{\infty}\frac{n^3}{\left(n^4-a^4\right)^2}\\&-\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n\left(n^2-a^2\right)}\right)\end{aligned}$$

$$+\sum_{n=1}^{\infty}\frac{1}{n\left(n^4-a^4\right)}$$

$$-\frac{1}{2}\left(\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}\right)\left(\sum_{n=1}^{\infty}\frac{1}{n\left(n^2+a^2\right)}\right), \tag{4.9}$$

$$\sum_{n=1}^{\infty}\frac{nH_n}{\left(n^2-a^2\right)^2}=\frac{1}{2}\sum_{n=1}^{\infty}\frac{3n^2+a^2}{\left(n^2-a^2\right)^3}-\frac{1}{4}\zeta\left(2,1-a\right)\frac{\psi\left(1-a\right)+\gamma}{a}$$
$$+\sum_{n=1}^{\infty}\frac{1}{\left(n^2-a^2\right)^2}-\frac{1}{4}\zeta\left(2,1+a\right)\frac{\psi\left(a+1\right)+\gamma}{a}, \tag{4.10}$$

$$\bar{S}\left(\begin{array}{c}\bar{1}\\ \{a\}_s\end{array}\right)-\bar{S}\left(\begin{array}{c}1\\ \{a\}_s\end{array}\right)=\frac{\psi\left(a+1\right)+\gamma}{a^s}-\sum_{j=2}^{s}\frac{\zeta\left(j,a+1\right)}{a^{s+1-j}}$$
$$+\ln 2\bar{\zeta}\left(s,a+1\right)$$
$$+\sum_{j=2}^{s}\zeta\left(j,a+1\right)\bar{\zeta}\left(s+1-j,a+1\right)$$
$$-\sum_{n=1}^{\infty}\frac{\left(-1\right)^{n-1}}{n\left(n+a\right)^s}$$
$$-s\bar{\zeta}\left(s+1,a+1\right)-\bar{\zeta}\left(s,a+1\right)\left(\psi\left(a+1\right)+\gamma\right). \tag{4.11}$$

5. Other parametric Euler sums involving parametric harmonic numbers

In this section we use certain integral representations of series to evaluate several other Euler-type sums with parametric harmonic numbers.

Theorem 5.1. *For integers $m,p\in\mathbb{N}$ and reals $a,b\notin\mathbb{N}_0^-$ with $a+b\notin\mathbb{N}^-$. Then the following identity holds*:

$$\sum_{n=1}^{\infty}\left\{\frac{\left(-1\right)^{p-1}}{\left(n+a\right)^{m+p}}-\frac{\left(-1\right)^{m-1}}{\left(n+b\right)^{m+p}}\right\}\left(\sum_{k=1}^{n}\frac{x^{k+a+b}}{k+a+b}\right)$$
$$=\sum_{k=1}^{m-1}\left(-1\right)^{k-1}{}_aH_{m+1-k}\left(x\right){}_bH_{p+k}\left(x\right) \tag{5.1}$$
$$-\sum_{k=1}^{p-1}\left(-1\right)^{k-1}{}_bH_{p+1-k}\left(x\right){}_aH_{m+k}\left(x\right)$$
$$+\left(-1\right)^{m-1}\left({}_bH_{p+m}\left(x\right){}_aH_1\left(x\right)-{}_bH_{p+m}\left(1\right){}_{a+b}H_1\left(x\right)\right)$$
$$-\left(-1\right)^{p-1}\left({}_aH_{p+m}\left(x\right){}_bH_1\left(x\right)-{}_aH_{p+m}\left(1\right){}_{a+b}H_1\left(x\right)\right),$$

where the function ${}_aH_m\left(x\right)$ *is defined by*

$$ {}_aH_m\left(x\right) := \sum_{n=1}^{\infty} \frac{x^{n+a}}{\left(n+a\right)^m},\ \Re(m) \ge 1,\ a \notin \mathbb{N}_0^-,\ x \in [-1,1)\,. \tag{5.2} $$

Proof. From the definition of ${}_aH_m\left(x\right)$, and using integration by parts, we obtain the following recurrence identity

$$ \begin{aligned} &\int_0^x {}_aH_m\left(t\right) t^{n+b-1} dt \\ &= \sum_{k=1}^{m-1} \frac{\left(-1\right)^{k-1} x^{n+b}}{\left(n+b\right)^k} {}_aH_{m+1-k}\left(x\right) \\ &\quad + \frac{\left(-1\right)^{m-1}}{\left(n+b\right)^m} \left\{ x^{n+b} {}_aH_1\left(x\right) + \sum_{k=1}^{n} \frac{x^{k+a+b}}{k+a+b} - \sum_{k=1}^{\infty} \frac{x^{k+a+b}}{k+a+b} \right\}. \end{aligned} \tag{5.3} $$

Hence, by considering the following integral

$$ \begin{aligned} \int_0^x \frac{{}_aH_m\left(t\right) {}_bH_p\left(t\right)}{t} dt &= \sum_{n=1}^{\infty} \frac{1}{\left(n+a\right)^m} \int_0^x {}_bH_p\left(t\right) t^{n+a-1} dt \\ &= \sum_{n=1}^{\infty} \frac{1}{\left(n+b\right)^p} \int_0^x {}_aH_m\left(t\right) t^{n+b-1} dt. \end{aligned} \tag{5.4} $$

Then with the help of formula (5.3) we may easily deduce the result. □

Letting $x, y \to 1$ and $s = 2m+1$ ($m \in \mathbb{N}_0$) in Theorem 3.2, and $x \to 1$, $p = m+1$ in Theorem 5.1, we get

$$ \begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{\left(n+a\right)^{2m+1}} \sum_{k=1}^{n} \frac{1}{k+a-b} + \sum_{n=1}^{\infty} \frac{1}{\left(n+b\right)^{2m+1}} \sum_{k=1}^{n} \frac{1}{k+b-a} \\ &= \sum_{n=1}^{\infty} \frac{1}{\left(n+a\right)^{2m+1} \left(n+a-b\right)} + \sum_{n=1}^{\infty} \frac{1}{\left(n+b\right)^{2m+1} \left(n+b-a\right)} \\ &\quad + \sum_{j=1}^{2m+1} \mathrm{Li}_{j,2m+2-j}\left(a,b,1\right) - \sum_{j=2}^{2m} \mathrm{Li}_j\left(a,1\right) \mathrm{Li}_{2m+2-j}\left(b,1\right) \\ &\quad + a\zeta\left(2m+1,a+1\right) \mathrm{Li}_{1,1}\left(b,b-a,1\right) \\ &\quad + b\zeta\left(2m+1,b+1\right) \mathrm{Li}_{1,1}\left(a,a-b,1\right), \end{aligned} \tag{5.5} $$

$$ \begin{aligned} &\sum_{n=1}^{\infty} \frac{1}{\left(n+b\right)^{2m+1}} \sum_{k=1}^{n} \frac{1}{k+a+b} + \sum_{n=1}^{\infty} \frac{1}{\left(n+a\right)^{2m+1}} \sum_{k=1}^{n} \frac{1}{k+b+a} \\ &= \sum_{k=1}^{m} \left(-1\right)^{m+k} \zeta\left(m+2-k,b+1\right) \zeta\left(k+m,a+1\right) \end{aligned} \tag{5.6} $$

$$\begin{aligned}&-\sum_{k=1}^{m-1}(-1)^{m+k}\zeta(m+1-k,a+1)\zeta(k+m+1,b+1)\\&-a\zeta(2m+1,a+1)\operatorname{Li}_{1,1}(b,a+b,1)\\&-b\zeta(2m+1,b+1)\operatorname{Li}_{1,1}(a,a+b,1).\end{aligned}$$

Combining (3.1) and (3.2), we can get the following identity

(5.7)

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{(n+a)^{2m+1}}\sum_{k=1}^{n}\frac{2b}{(k+a)^2-b^2}+\sum_{n=1}^{\infty}\frac{1}{(n+b)^{2m+1}}\sum_{k=1}^{n}\frac{2a}{(k+b)^2-a^2}\\&=\operatorname{Li}_{2m+1,1}(a,a-b,1)+\operatorname{Li}_{2m+1,1}(b,b-a,1)+\sum_{j=1}^{2m+1}\operatorname{Li}_{j,2m+2-j}(a,b,1)\\&\quad-\sum_{j=2}^{2m}\zeta(j,a+1)\zeta(2m+2-j,b+1)\\&\quad-\sum_{k=1}^{m}(-1)^{m+k}\zeta(m+2-k,b+1)\zeta(k+m,a+1)\\&\quad+\sum_{k=1}^{m-1}(-1)^{m+k}\zeta(m+1-k,a+1)\zeta(k+m+1,b+1)\\&\quad+2a\zeta(2m+1,a+1)\sum_{n=1}^{\infty}\frac{1}{(n+b)^2-a^2}\\&\quad+2b\zeta(2m+1,b+1)\sum_{n=1}^{\infty}\frac{1}{(n+a)^2-b^2}.\end{aligned}$$

Taking $a=0$ (or $b=0$) in (5.7), we obtain the following Theorem 5.2.

Theorem 5.2. *For integer $m\in\mathbb{N}_0$ and real $a\notin\mathbb{N}_0^-$, the parametric sums*

$$\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\sum_{k=1}^{n}\frac{1}{k^2-a^2}$$

are reducible to Hurwitz zeta values and rational series.

On the other hand, by a direct calculation, we deduce that

$$\begin{aligned}&\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}}\sum_{k=1}^{n}\frac{1}{k^2-a^2}+\sum_{n=1}^{\infty}\frac{H_n^{(2m+1)}}{n^2-a^2}\\&=\zeta(2m+1)\sum_{n=1}^{\infty}\frac{1}{n^2-a^2}+\sum_{n=1}^{\infty}\frac{1}{n^{2m+1}(n^2-a^2)}.\end{aligned}\tag{5.8}$$

From Theorem 5.2 and formula (5.8), we can get the following corollary.

Corollary 5.3. *For integer $m \in \mathbb{N}_0$ and real $a \notin \mathbb{N}_0^-$, the parametric Euler sums*

$$S\left(\begin{array}{c} 2m+1 \\ a,-a \end{array}\right)$$

are reducible to Hurwitz zeta values and rational series.

At the end of this section, we establish some connection between linear and quadratic parametric Euler sums.

Theorem 5.4. *For integers $m, p \in \mathbb{N} \setminus \{1\}$ and real $a, b \notin \mathbb{N}_0^-$ with $x \in [-1, 1)$ and $a + b \notin \mathbb{N}^-$. Then the following identity holds*:

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left\{(-1)^{p-1}\frac{\zeta_n(m,b+1)}{(n+b)^p}-(-1)^{m-1}\frac{\zeta_n(p,a+1)}{(n+a)^m}\right\}\left(\sum_{k=1}^{n}\frac{x^{k+a+b}}{k+a+b}\right)\\
&=\sum_{k=1}^{m-1}(-1)^{k-1}{}_bH_{m+1-k}(x)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(p,a+1)}{(n+a)^k}x^{n+a}\right)\\
&\quad-\sum_{k=1}^{p-1}(-1)^{k-1}{}_aH_{p+1-k}(x)\left(\sum_{n=1}^{\infty}\frac{\zeta_n(m,b+1)}{(n+b)^k}x^{n+b}\right)\\
&\quad+(-1)^{m-1}\sum_{n=1}^{\infty}\frac{\zeta_n(p,a+1)}{(n+a)^m}\left\{x^{n+a}{}_bH_1(x)-{}_{a+b}H_1(x)\right\}\\
&\quad-(-1)^{p-1}\sum_{n=1}^{\infty}\frac{\zeta_n(m,b+1)}{(n+b)^k}\left\{x^{n+b}{}_aH_1(x)-{}_{a+b}H_1(x)\right\}. \qquad (5.9)
\end{aligned}$$

Proof. To prove the identity, we consider the following integral

$$\int_0^x \frac{{}_aH_p(t)\,{}_bH_m(t)}{t(1-t)}dt,\ x\in(-1,1).$$

By the definition of ${}_aH_m(x)$, the following identity is easily derived

$$\frac{{}_aH_p(t)}{1-t}=\sum_{n=1}^{\infty}\zeta_n(p,a+1)\,t^{n+a}, t\in(-1,1).$$

Hence, we have the result

$$\begin{aligned}
\int_0^x \frac{{}_aH_p(t){}_bH_m(t)}{t(1-t)}dt &= \sum_{n=1}^{\infty}\zeta_n(p,a+1)\int_0^x t^{n+a-1}{}_bH_m(t)\,dt\\
&= \sum_{n=1}^{\infty}\zeta_n(m,b+1)\int_0^x t^{n+b-1}{}_aH_p(t)\,dt.
\end{aligned}$$

Then with the help of formula (5.3), and by a direct calculation we can easily deduce the desired result. □

Hence, letting $x \to 1$ in Theorem 5.4 and combining Lemma 2.2, and using the definition of digamma function, we can give the following corollary.

Corollary 5.5. *For integers $m, p \in \mathbb{N} \setminus \{1\}$ and real $a, b \notin \mathbb{N}_0^-$, we have*

$$\begin{aligned}
&\sum_{n=1}^{\infty} \left\{ (-1)^{p-1} \frac{\zeta_n(m, b+1)}{(n+b)^p} - (-1)^{m-1} \frac{\zeta_n(p, a+1)}{(n+a)^m} \right\} (\zeta_n(1, a+b+1)) \\
= &\sum_{k=2}^{m-1} (-1)^{k-1} \zeta(m+1-k, b+1) \left(\sum_{n=1}^{\infty} \frac{\zeta_n(p, a+1)}{(n+a)^k} \right) \\
&- \sum_{k=2}^{p-1} (-1)^{k-1} \zeta(p+1-k, a+1) \left(\sum_{n=1}^{\infty} \frac{\zeta_n(m, b+1)}{(n+b)^k} \right) \\
&+ \zeta(m, b+1) \zeta(p, a+1) (\psi(b+1) - \psi(a+1)) \\
&+ \zeta(p, a+1) \left(\sum_{n=1}^{\infty} \frac{\zeta_n(1, b+1)}{(n+b)^m} \right) - \zeta(p, a+1) \zeta(m+1, b+1) \\
&- \zeta(m, b+1) \left(\sum_{n=1}^{\infty} \frac{\zeta_n(1, a+1)}{(n+b)^p} \right) + \zeta(m, b+1) \zeta(p+1, a+1) \\
&+ (-1)^{m-1} (\psi(a+b+1) - \psi(b+1)) \left(\sum_{n=1}^{\infty} \frac{\zeta_n(p, a+1)}{(n+a)^m} \right) \\
&- (-1)^{p-1} (\psi(a+b+1) - \psi(a+1)) \left(\sum_{n=1}^{\infty} \frac{\zeta_n(m, b+1)}{(n+b)^p} \right).
\end{aligned}$$

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References

[1] D. H. Bailey, J. M. Borwein, and R. Girgensohn, *Experimental evaluation of Euler sums*, Experiment. Math. **3** (1994), no. 1, 17–30.

[2] D. Borwein, J. M. Borwein, and D. M. Bradley, *Parametric Euler sum identities*, J. Math. Anal. Appl. **316** (2006), no. 1, 328–338.

[3] D. Borwein, J. M. Borwein, and R. Girgensohn, *Explicit evaluation of Euler sums*, Proc. Edinburgh Math. **38** (1995), no. 2, 277–294.

[4] J. M. Borwein and D. M. Bradley, *Thirty-two goldbach variations*, Int. J. Number Theory **2** (2006), no. 1, 65–103.

[5] J. M. Borwein, D. M. Bradley, and D. J. Brodhurst, *Evaluations of k-fold Euler/Zagier sums: a compendium of results for arbitrary k*, Electron. J. Combin. **4** (1997), no. 2, Research Paper 5, 21 pp.

[6] J. M. Borwein, D. M. Bradley, D. J. Broadhurst, and P. Lisonek, *Special values of multiple polylogarithms*, Trans. Amer. Math. Soc. **353** (2001), no. 3, 907–941.

[7] J. M. Borwein and R. Girgensohn, *Evaluation of triple Euler sums*, Electron. J. Combin. **3** (1996), no. 1, Research Paper 23, 27 pp.

[8] K. Boyadzhiev, *Evaluation of Euler-Zagier sums*, Internat. J. Math. Sci. **27** (2001), no. 7, 407–412.

[9] L. Comtet, *Advanced Combinatorics*, D. Reidel Publishing Company, Boston, 1974.

[10] M. Eie and C.-S. Wei, *Evaluations of some quadruple Euler sums of even weight*, Funct. Approx. Comment. Math. **46** (2012), no. 1, 63–67.

[11] P. Flajolet and B. Salvy, *Euler sums and contour integral representations*, Experiment. Math. **7** (1998), no. 1, 15–35.
[12] P. Freitas, *Integrals of Polylogarithmic Functions, Recurrence Relations, and Associated Euler Sums*, Math. Compu. **74** (2005), no. 251, 1425–1440.
[13] M. E. Hoffman, *Multiple harmonic series*, Pacific J. Math. **152** (1992), 275–290.
[14] I. Mezö, *Nonlinear Euler sums*, Pacific J. Math. **272** (2014), no. 1, 201–226.
[15] N. Nielsen, *Handbuch der Theorie der Gammafunction and Theorie des Integrallogarithmus and verwandter Transzendenten*, Reprinted together as Die Gammafunction, Chelsea, New York, 1965.
[16] A. Sofo, *Harmonic sums and integral representations*, J. Appl. Anal. **16** (2010), no. 2, 265–277.
[17] ———, *Quadratic alternating harmonic number sums*, J. Number Theory **154** (2015), 144–159.
[18] A. Sofo and H. M. Srivastava, *Identities for the harmonic numbers and binomial coefficients*, Ramanujan J. **25** (2011), no. 1, 93–113.
[19] L. Tornheim, *Harmonic double series*, Amer. J. Math. **72** (1950), 303–314.
[20] H. Tsumura, *On some combinatorial relations for Tornheim's double series*, Acta Arith. **105** (2002), no. 3, 239–252.
[21] ———, *Evaluation formulas for Tornheim's type of alternating double series*, Math. Comp. **73** (2004), no. 245, 251–258.
[22] C. Xu and J. Cheng, *Some results on Euler sums*, Funct. Approx. Comment. Math. **54** (2016), no. 1, 25–37.
[23] C. Xu, Y. Yan, and Z. Shi, *Euler sums and integrals of polylogarithm functions*, J. Number Theory **165** (2016), 84–108.
[24] C. Xu, Y. Yang, and J. Zhang, *Explicit evaluation of quadratic Euler sums*, Int. J. Number Theory **13** (2017), no. 3, 655–672.
[25] C. Xu, M. Zhang, and W. Zhu, *Some evaluation of q-analogues of Euler sums*, Monatshefte Für Mathematik **182** (2017), no. 4, 957–975.

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