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TRANSLATION SURFACES IN THE 3-DIMENSIONAL GALILEAN SPACE SATISFYING $\boldsymbol{\Delta^{II}x_i = \lambda_i x_i}$

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Abstract. In this paper, we classify translation surfaces in the three dimensional Galilean space $\mathbb{G}_3$ satisfying some algebraic equations in terms of the coordinate functions and the Laplacian operators with respect to the second fundamental form of the surface. We also give explicit forms of these surfaces.

1. Introduction

Let $\mathbf{x} : \mathbf{M} \to \mathbb{E}^m$ be an isometric immersion of a connected n-dimensional manifold in the m-dimensional Euclidean space $\mathbb{E}^m$. Denote by $\mathbf{H}$ and Δ the mean curvature and the Laplacian of $\mathbf{M}$ with respect to the Riemannian metric on $\mathbf{M}$ induced from that of $\mathbb{E}^m$, respectively. Takahashi ([14]) proved that the submanifolds in $\mathbb{E}^m$ satisfying $\Delta\mathbf{x} = \lambda\mathbf{x}$, that is, all coordinate functions are eigenfunctions of the Laplacian with the same eigenvalue $\lambda \in \mathbb{R}$, are either the minimal submanifolds of $\mathbb{E}^m$ or the minimal submanifolds of hypersphere $\mathbb{S}^{m-1}$ in $\mathbb{E}^m$.

As an extension of Takahashi theorem, in [9] Garay studied hypersurfaces in $\mathbb{E}^m$ whose coordinate functions are eigenfunctions of the Laplacian, but not necessarily associated to the same eigenvalue. He considered hypersurfaces in $\mathbb{E}^m$ satisfying the condition

$$\Delta\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{1.1}$$

where $\mathbf{A} \in \mathrm{Mat}(m, \mathbb{R})$ is an $m \times m$-diagonal matrix, and proved that such hypersurfaces are minimal ($\mathbf{H} = 0$) in $\mathbb{E}^m$ and open pieces of either round hyperspheres or generalized right spherical cylinders.

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Related to this, Dillen, Pas and Verstraelen ([7]) investigated surfaces in $\mathbb{E}^3$ whose immersions satisfy the condition

$$\Delta \mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{B}, \tag{1.2}$$

where $\mathbf{A} \in \mathrm{Mat}(3, \mathbb{R})$ is a 3×3-real matrix and $\mathbf{B} \in \mathbb{R}^3$. In other words, each coordinate function is of 1-type in the sense of Chen ([4]). For the Lorentzian version of surfaces satisfying (1.2), Alias, Ferrandez and Lucas ([1]) proved that the only such surfaces are minimal surfaces and open pieces of Lorentz circular cylinders, hyperbolic cylinders, Lorentz hyperbolic cylinders, hyperbolic spaces or pseudo-spheres.

The notion of an isometric immersion $\mathbf{x}$ is naturally extended to smooth functions on submanifolds of Euclidean space or pseudo-Euclidean space. The most natural one of them is the Gauss map of the submanifold. In particular, if the submanifold is a hypersurface, the Gauss map can be identified with the unit normal vector field to it. Dillen, Pas and Verstraelen ([8]) studied surfaces of revolution in the three dimensional Euclidean space $\mathbb{E}^3$ such that its Gauss map $\mathbf{G}$ satisfies the condition

$$\Delta \mathbf{G} = \mathbf{A}\mathbf{G}, \tag{1.3}$$

where $\mathbf{A} \in \mathrm{Mat}(3, \mathbb{R})$. Baikoussis and Verstraelen ([2]) studied the helicoidal surfaces in $\mathbb{E}^3$. Choi ([5]) completely classified the surfaces of revolution satisfying the condition (1.3) in the three dimensional Minkowski space $\mathbb{E}_1^3$. The authors ([5, 16]) classified surfaces of revolution satisfying (1.2) and (1.3) in the three dimensional Minkowski space and pseudo-Galilean space. Yoon ([15]) classified the translation surfaces in the 3-dimensional Galilean space under the condition $\Delta \mathbf{x}^i = \lambda^i x^i$,where $\lambda^i \in \mathbb{R}$. The authors ([3, 10]) classified translation surfaces and surfaces of revolution satisfying $\Delta^{III}\mathbf{r}_i = \mu_i \mathbf{r}_i$ in the 3-dimensional space. Karacan, Yoon and Bukcu ([11]) classified translation surfaces of Type 1 satisfying $\Delta^J \mathbf{x}_i = \lambda_i \mathbf{x}_i$, $j = 1, 2$ and $\Delta^{III}\mathbf{x}_i = \lambda_i \mathbf{x}_i$. Sipus and Divjak ([13]) described translation surfaces in the Galilean space having constant Gaussian and mean curvatures as well as translation Weingarten surfaces. The main purpose of this paper is to complete classification of translation surfaces in the three dimensional Galilean space $\mathbb{G}_3$ in terms of the position vector field and the Laplacian operator.

2. Preliminaries

The Galilean space $\mathbb{G}_3$ is a Cayley-Klein space defined from a 3-dimensional projective space $\mathcal{P}(\mathbb{R}^3)$ with the absolute figure that consists of an ordered triple $\{w, f, I\}$, where w is the ideal (absolute) plane, f the line (absolute line) in w and I the fixed elliptic involution of points of f. We introduce homogeneous coordinates in $\mathbb{G}_3$ in such a way that the absolute plane w is given by $x_0 = 0$, the absolute line f by $x_0 = x_1 = 0$ and the elliptic involution by $(0 : 0 : x_2 : x_3) \to (0 : 0 : x_3 : -x_2)$. In affine coordinates defined

by $(0 : x_1 : x_2 : x_3) \to (1 : x : y : z)$, distance between points $P_i = (x_i,\, y_i,\, z_i)$, $i = 1, 2$, is defined by

$$d(P_1, P_2) = \begin{cases} |x_2 - x_1| \quad , & \text{if} \quad x_1 \neq x_2 \\ \sqrt{(y_2 - y_1)^2 + (z_2 - z_1)^2}\ , & \text{if} \quad x_1 = x_2. \end{cases} \tag{2.1}$$

The group of motions of G_3 is a six-parameter group given (in affine coordinates) by

$$\begin{aligned} \overline{x} &= a + x, \\ \overline{y} &= b + cx + y\cos\theta + z\sin\theta, \\ \overline{z} &= d + ex - y\sin\theta + z\cos\theta. \end{aligned}$$

With respect to the absolute figure, there are two types of lines in the Galilean space − isotropic lines which intersect the absolute line f and nonisotropic lines which do not. A plane is called Euclidean if it contains f, otherwise it is called isotropic. In the given affine coordinates, isotropic vectors are of the form $(0, y, z)$, whereas Euclidean planes are of the form $x = k, k \in \mathbb{R}$. The induced geometry of a Euclidean plane is Euclidean and of an isotropic plane isotropic (i.e., 2-dimensional Galilean or flag-geometry).

A C^r-surface S, $r \geq 1$, immersed in the Galilean space, $\mathbf{x} : U \to S$, $U \subset \mathbb{R}^2$, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$, has the following first fundamental form

$$\mathbf{I} = (g_1 du + g_2 dv)^2 + \epsilon \left(h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2\right),$$

where the symbols $g_i = x_i$, $h_{ij} = \widetilde{\mathbf{x}}_i.\widetilde{\mathbf{x}}_j$ stand for derivatives of the first coordinate function $x(u, v)$ with respect to u, v and for the Euclidean scalar product of the projections $\widetilde{\mathbf{x}}_k$ of vectors $\mathbf{x}_k$ onto the yz-plane, respectively. Furthermore,

$$\epsilon = \begin{cases} 0\ , & \text{if direction } du : dv \text{ is non-isotropic,} \\ 1\ , & \text{if direction } du : dv \text{ is isotropic.} \end{cases}$$

In every point of a surface there exists a unique isotropic direction defined by $g_1 du + g_2 dv = 0$. In that direction, the arc length is measured by

$$\begin{aligned} ds^2 &= h_{11} du^2 + 2h_{12} du dv + h_{22} dv^2 \\ &= \frac{h_{11} g_2^2 - 2h_{12} g_1 g_2 + h_{22} g_1^2}{g_1^2} \\ &= \frac{W^2}{g_1^2} dv^2, \end{aligned}$$

where $g_1 \neq 0$.

A surface is called admissible if it has no Euclidean tangent planes. Therefore, for an admissible surface either $g_1 \neq 0$ or $g_2 \neq 0$ holds. An admissible surface can always locally be expressed as

$$z = f(u, v).$$

The Gaussian $\mathbf{K}$ and mean curvature $\mathbf{H}$ are C^{r-2} functions, $r \geq 2$, defined by

$$\mathbf{K} = \frac{LN - M^2}{W^2},\ \mathbf{H} = \frac{g_2^2 L - 2g_1 g_2 M + g_1^2 N}{2W^2},$$

where

$$L_{ij} = \frac{x_1 \mathbf{x}_{ij} - x_{ij}\mathbf{x}_1}{x_1} \cdot \mathbf{N},\ \ x_1 = g_1 \neq 0.$$

We will use L_{ij}, $i, j = 1, 2$, for L, M, N if more convenient. The vector $\mathbf{N}$ defines a normal vector to a surface

$$\mathbf{N} = \frac{1}{W}\left(0, -x_2 z_1 + x_1 z_2, x_2 y_1 - x_1 y_2\right),$$

where $W^2 = \left(x_2 \mathbf{x}_1 - x_1 \mathbf{x}_2\right)^2$ ([13]).

It is well known in terms of local coordinates $\{u, v\}$ of $\mathbf{M}$ the Laplacian operator $\mathbf{\Delta^{II}}$ the second fundamental form on $\mathbf{M}$ are defined by ([11, 12])

$$\mathbf{\Delta^{II}}\mathbf{x} = -\frac{1}{\sqrt{|LN - M^2|}}\left[\frac{\partial}{\partial u}\left(\frac{N\mathbf{x}_u - M\mathbf{x}_v}{\sqrt{|LN - M^2|}}\right) - \frac{\partial}{\partial v}\left(\frac{M\mathbf{x}_u - L\mathbf{x}_v}{\sqrt{|LN - M^2|}}\right)\right]. \tag{2.2}$$

3. Translation surfaces in $\mathbb{G}_3$

For counter parts of Euclidean results, we will consider translation surfaces that are obtained by translating two planar curves. In order to obtain an admissible surface, translated curves can be, with respect to the absolute figure, either

Type 1: a non-isotropic curve (having its tangents non-isotropic) and an isotropic curve or,

Type 2: non-isotropic curves.

There are no motions of the Galilean space that carry one type of a curve into another, so we will treat them separately. Translation surfaces of the Type 1 in the Galilean space can be locally represented by

$$z = f(u) + g(v),$$

which yields the parametrization

$$\mathbf{x}(u, v) = \left(u, v, f(u) + g(v)\right). \tag{3.1}$$

One translated curve is a non-isotropic curve in the plane $y = 0$

$$\alpha(u) = (u, 0, f(u))$$

and the other is an isotropic curve in the plane $x = 0$

$$\beta(v) = (0, v, g(v)).$$

Translation surfaces of the Type 2 in the Galilean space $\mathbb{G}_3$, a surface having both translated curves non-isotropic

$$\mathbf{x}(u, v) = \left(u + v, g(v), f(u)\right), \tag{3.2}$$

where $\alpha(u) = (u, 0, f(u))$ is a curve in the isotropic plane $y = 0$, and $\beta(v) = (v, g(v), 0)$ is a curve in the isotropic plane $z = 0$ ([13]).

In this paper, we will investigate the translation surfaces of Type 1 and Type 2 in the three dimensional Galilean space $\mathbb{G}_3$.

4. Translation surfaces of type 1 satisfying $\mathbf{\Delta^{II}x_i = \lambda_i x_i}$

In this section, we classify translation surfaces with non-degenerate second fundamental form in $\mathbb{G}_3$ satisfying the equation

$$\mathbf{\Delta^{II}}\mathbf{x}_i = \lambda_i \mathbf{x}_i, \tag{4.1}$$

where $\lambda_i \in \mathbb{R}$, $i=1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x} = \left(\Delta^{\mathbf{II}}\mathbf{x}_1, \Delta^{\mathbf{II}}\mathbf{x}_2, \Delta^{\mathbf{II}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = u,\ \mathbf{x}_2 = v,\ \mathbf{x}_3 = f(u) + g(v).$$

For the translation surface given by (3.1), the coefficients of the second fundamental form are given by

$$L_{11} = L = -\frac{f''}{\sqrt{\left(1+g'^2\right)}},\quad L_{22} = \ N = -\frac{g''}{\sqrt{\left(1+g'^2\right)}},\ L_{12} = M = 0. \tag{4.2}$$

The Gaussian curvature $\mathbf{K}$ is

$$\mathbf{K} = \frac{f''(u)g''(v)}{\left(1+g'^2\right)^2}.$$

Suppose that the surface has non zero Gaussian curvature, so

$$f''(u)g''(v) \neq 0, \forall\ u, v \in I.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (4.2) and (2.2) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x} = \begin{pmatrix} \frac{\sqrt{1+g'^2}f'''}{2f''^2}, \\ \frac{\sqrt{1+g'^2}g'''}{2g''^2}, \\ \frac{\sqrt{1+g'^2}}{2f''^2g''^2}\left(-4f''^2g''^2 + f'g''^2f''' + g'f''^2g'''\right) \end{pmatrix}. \tag{4.3}$$

The equation (4.1) by means of (4.2) gives rise to the following system of ordinary differential equations

$$\frac{\sqrt{1+g'^2}f'''}{2f''^2} = \lambda_1 u, \tag{4.4}$$

$$\frac{\sqrt{1+g'^2}g'''}{2g''^2} = \lambda_2 v, \tag{4.5}$$

$$\frac{\sqrt{1+g'^2}}{2f''^2g''^2}\left(-4f''^2g''^2 + f'g''^2f''' + g'f''^2g'''\right) = \lambda_3\left(f(u)+g(v)\right), \tag{4.6}$$

where $\lambda_i \in \mathbb{R}$. This means that $\mathbf{M}$ is at most of 3- types. Combining equations (4.4), (4.5) and (4.6), we have

$$f'\lambda_1 u - \lambda_3 f = -g'\lambda_2 v + \lambda_3 g + 2\sqrt{1+g'^2}. \tag{4.7}$$

We discuss six cases according to constants $\lambda_1, \lambda_2, \lambda_3$.

Case 1: Let $\lambda_1 = 0, \lambda_2 \neq 0, \lambda_3 \neq 0$, from (4.7), we obtain

$$-\lambda_3 f = -g'\lambda_2 v + \lambda_3 g + 2\sqrt{1+g'^2}. \tag{4.8}$$

Here u and v are independent variables, so each side of (4.8) is equal to a constant, call it p. Hence, the two equations

$$-\lambda_3 f = p = -g'\lambda_2 v + \lambda_3 g + 2\sqrt{1+g'^2}. \tag{4.9}$$

This differential equation for the function $f(u)$ admits the solution

$$f(u) = -\frac{p}{\lambda_3}, \tag{4.10}$$

where for some constants $p \neq 0$ and $\lambda_3 \neq 0$. But, there is no suitable solution for the function $g(v)$. In particular, if $p = 0$, then we have

$$f(u) = 0, \tag{4.11}$$

there is no suitable solution for the function $g(v)$.

Case 2: Let $\lambda_1 = \lambda_2 = 0$, $\lambda_3 \neq 0$, from (4.7), we obtain

$$-\lambda_3 f = \lambda_3 g + 2\sqrt{1+g'^2}. \tag{4.12}$$

This differential equation admits the solutions

$$\begin{aligned} f(u) &= -\frac{p}{\lambda_3}, \\ g(v) &= \frac{4e^{\frac{\lambda_3 v}{2} - \lambda_3 c_1} + e^{-\frac{\lambda_3 v}{2} + \lambda_3 c_1} + 2p}{2\lambda_3}, \\ g(v) &= \frac{4e^{-\frac{\lambda_3 v}{2} - \lambda_3 c_1} + e^{\frac{\lambda_3 v}{2} + \lambda_3 c_1} + 2p}{2\lambda_3}, \end{aligned} \tag{4.13}$$

where $c_1, p \in \mathbb{R}$ with $\lambda_3 \neq 0$. In this case, $\mathbf{M}$ is parametrized by

$$\begin{aligned} \mathbf{x}(u,v) &= \left(u, v, \left(-\frac{p}{\lambda_3}\right) + \left(\frac{4e^{\frac{\lambda_3 v}{2} - \lambda_3 c_1} + e^{-\frac{\lambda_3 v}{2} + \lambda_3 c_1} + 2p}{2\lambda_3}\right)\right), \\ \mathbf{x}(u,v) &= \left(u, v, \left(-\frac{p}{\lambda_3}\right) + \left(\frac{4e^{-\frac{\lambda_3 v}{2} - \lambda_3 c_1} + e^{\frac{\lambda_3 v}{2} + \lambda_3 c_1} + 2p}{2\lambda_3}\right)\right). \end{aligned} \tag{4.14}$$

In particular, if $p=0$, then we have

$$\begin{aligned} f(u) &= 0, \\ g(v) &= \frac{e^{-\frac{\lambda_3 v}{2}-\lambda_3 c_1}\left(e^{\lambda_3 v+2c_1\lambda_3}+4\lambda_3^2\right)}{2\lambda_3^2}, \\ g(v) &= \frac{e^{-\frac{\lambda_3 v}{2}-\lambda_3 c_1}\left(e^{2\lambda_3 c_1}+4e^{\lambda_3 v}\lambda_3^2\right)}{2\lambda_3^2}, \end{aligned} \tag{4.15}$$

where $c_1, p \in \mathbb{R}$ with $\lambda_3 \neq 0$. In this case, $\mathbf{M}$ is parametrized by

$$\begin{aligned} \mathbf{x}(u,v) &= \left(u, v, \frac{e^{-\frac{\lambda_3 v}{2}-\lambda_3 c_1}\left(e^{\lambda_3 v+2c_1\lambda_3}+4\lambda_3^2\right)}{2\lambda_3^2}\right), \\ \mathbf{x}(u,v) &= \left(u, v, \frac{e^{-\frac{\lambda_3 v}{2}-\lambda_3 c_1}\left(e^{2\lambda_3 c_1}+4e^{\lambda_3 v}\lambda_3^2\right)}{2\lambda_3^2}\right). \end{aligned} \tag{4.16}$$

Case 3: Let $\lambda_1 \neq 0, \lambda_2 = 0$, $\lambda_3 = 0$, from (4.7), we obtain

$$f'\lambda_1 u = 2\sqrt{1+g'^2}. \tag{4.17}$$

This differential equation admits the solutions

$$\begin{aligned} f(u) &= c_1 + \frac{p\ln u}{\lambda_1}, \\ g(v) &= \pm\frac{v\sqrt{p^2-4}}{2} + c_2, \end{aligned} \tag{4.18}$$

where $c_1, c_2, p \in \mathbb{R}$ with $p^2-4>0$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \left(u, v, \left(c_1 + \frac{p\ln u}{\lambda_1}\right) + \left(\pm\frac{v\sqrt{p^2-4}}{2} + c_2\right)\right). \tag{4.19}$$

In particular, if $p=0$, then we have

$$f(u) = c_1. \tag{4.20}$$

But, there is no suitable solution for the function $g(v)$.

Case 4: Let $\lambda_1 = 0, \lambda_2 \neq 0$, $\lambda_3 = 0$, from (4.7), we obtain

$$-g'\lambda_2 v + 2\sqrt{1+g'^2} = 0. \tag{4.21}$$

This differential equation admits the solution

$$g(v) = c_1 \pm \frac{2\ln\left(\lambda_2^2 v + \lambda_2\sqrt{\lambda_2^2 v^2 - 4}\right)}{2\lambda_2}, \tag{4.22}$$

where for some constant $c_1 \in \mathbb{R}$ with $\lambda_2^2 v^2 - p^2 > 0$. Here, the function $f(u)$ independent of selection of the function $g(v)$. In this case, $\mathbf{M}$ is parametrized

by

$$\mathbf{x}(u,v)=\left(u,v,f(u)+c_1\pm\frac{2\ln\left(\lambda_2^2v+\lambda_2\sqrt{\lambda_2^2v^2-4}\right)}{2\lambda_2}\right), \tag{4.23}$$

where $f''\neq 0$.

Case 5: Let $\lambda_1=\lambda_2=\lambda_3=0$, from (4.7), we obtain

$$0=2\sqrt{1+g'^2}. \tag{4.24}$$

There is no suitable solutions. Here, the function $f(u)$ independent of selection of the function $g(v)$.

Case 6: Let $\lambda_1\neq 0,\lambda_2\neq 0$, $\lambda_3=0$, from (4.7), we obtain

$$f'\lambda_1u=-g'\lambda_2v+2\sqrt{1+g'^2}. \tag{4.25}$$

Their general solutions are

$$\begin{cases} f(u)=c_1+\frac{p\ln u}{\lambda_1}, \\ g(v)=c_2\mp\frac{p\ln(2-\lambda_2v)}{2\lambda_2}\pm\frac{p\ln(2+\lambda_2v)}{2\lambda_2}+\frac{p\ln\left(4-\lambda_2^2v^2\right)}{2\lambda_2} \\ \mp\frac{2\ln\left(\lambda_2v+\sqrt{-4+p^2+\lambda_2^2v^2}\right)}{\lambda_2}\pm\frac{p\ln\left(4-p^2-2\lambda_2v-p\sqrt{-4+p^2+\lambda_2^2v^2}\right)}{2\lambda_2} \\ \mp\frac{p\ln\left(4-p^2+2\lambda_2v-p\sqrt{-4+p^2+\lambda_2^2v^2}\right)}{2\lambda_2}, \end{cases} \tag{4.26}$$

where $c_1,c_2,p\in\mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\left(\begin{array}{c} u, \\ v, \\ \left(c_1+\frac{p\ln u}{\lambda_1}\right) \\ +\left(\begin{array}{c} c_2\mp\frac{p\ln(2-\lambda_2v)}{2\lambda_2}\pm\frac{p\ln(2+\lambda_2v)}{2\lambda_2}+\frac{p\ln\left(4-\lambda_2^2v^2\right)}{2\lambda_2} \\ \mp\frac{2\ln\left(\lambda_2v+\sqrt{-4+p^2+\lambda_2^2v^2}\right)}{\lambda_2} \\ \pm\frac{p\ln\left(4-p^2-2\lambda_2v-p\sqrt{-4+p^2+\lambda_2^2v^2}\right)}{2\lambda_2} \\ \mp\frac{p\ln\left(4-p^2+2\lambda_2v-p\sqrt{-4+p^2+\lambda_2^2v^2}\right)}{2\lambda_2} \end{array}\right) \end{array}\right). \tag{4.27}$$

In particular, if $p=0$, then we have

$$\begin{aligned} f(u)&=c_1, \\ g(v)&=c_2\pm\frac{2\ln\left(\lambda_2^2v+\lambda_2\sqrt{-4+\lambda_2^2v^2}\right)}{2\lambda_2}, \end{aligned} \tag{4.28}$$

where $c_1,c_2\in\mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\left(u,v,(c_1)+\left(c_2\pm\frac{2\ln\left(\lambda_2^2v+\lambda_2\sqrt{-4+\lambda_2^2v^2}\right)}{2\lambda_2}\right)\right). \tag{4.29}$$

Using the solutions (4.10), (4.11), (4.13), (4.15), (4.18), (4.20) and (4.28) give rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Definition. A surface of in the three dimensional Galilean space is said to be **II**-harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 4.1. *Let* $\mathbf{M}$ *be a translation surface given by* (3.1) *in the three dimensional Galilean space* $\mathbb{G}_3$. *Then there is no* **II**-*harmonic the surface* $\mathbf{M}$.

Theorem 4.2. *Let* $\mathbf{M}$ *be a non* **II**-*harmonic translation surface with non-degenerate second fundamental form given by* (3.1) *in the three dimensional Galilean space* $\mathbb{G}_3$. *If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x}_i{=}\lambda_i\mathbf{x}_i$, *where* $\lambda_i{\in}\mathbb{R}$, $i{=}1,2,3$, *then it is congruent to an open part of the surfaces* (4.23) *and* (4.27).

5. Translation surfaces of type 2 satisfying $\boldsymbol{\Delta^{\mathbf{II}}\mathbf{x}_i = \lambda_i\mathbf{x}_i}$

In this section, we classify translation surfaces with non-degenerate second fundamental form in $\mathbb{G}_3$ satisfying the equation

$$\boldsymbol{\Delta}^{\mathbf{II}}\mathbf{x}_i{=}\,\lambda_i\mathbf{x}_i, \tag{5.1}$$

where $\lambda_i{\in}\mathbb{R}$, $i{=}1,2,3$ and

$$\Delta^{\mathbf{II}}\mathbf{x} = \left(\Delta^{\mathbf{II}}\mathbf{x}_1,\, \Delta^{\mathbf{II}}\mathbf{x}_2,\, \Delta^{\mathbf{II}}\mathbf{x}_3\right),$$

where

$$\mathbf{x}_1 = u + v,\ \mathbf{x}_2 = g(v),\ \mathbf{x}_3 = f(u).$$

For the translation surface given by (3.2), the coefficient of the second fundamental form is given by

$$L_{11} = L = \frac{f''g'}{\sqrt{f'^2+g'^2}},\quad L_{22} = \ N = \frac{f'g''}{\sqrt{f'^2+g'^2}},\ L_{12} = M = 0. \tag{5.2}$$

The Gaussian curvature $\mathbf{K}$ is

$$\mathbf{K} = \frac{f'f''g'g''}{\left(f'^2+g'^2\right)^2}.$$

Suppose that the surface has non zero Gaussian curvature, so

$$f'(u)f''(u)g'(v)g''(v) \neq 0, \forall\ u,v \in I.$$

By a straightforward computation, the Laplacian operator on $\mathbf{M}$ with the help of (5.2) and (2.2) turns out to be

$$\Delta^{\mathbf{II}}\mathbf{x} = \begin{pmatrix} \frac{\sqrt{f'^2+g'^2}}{2f'g'f''^2g''^2}\left(-2f''^2g''^2 + f'g''^2f''' + g'f''^2g'''\right), \\ \frac{\sqrt{f'^2+g'^2}}{2f'g''^2}\left(-3g''^2 + g'g'''\right), \\ \frac{\sqrt{f'^2+g'^2}}{2g'f''^2}\left(-3f''^2 + f'f'''\right) \end{pmatrix}. \tag{5.3}$$

The equation (5.1) by means of (5.2) gives rise to the following system of ordinary differential equations

$$\frac{\sqrt{f'^2+g'^2}}{2f'g'f''^2g''^2}\left(-2f''^2g''^2+f'g''^2f'''+g'f''^2g'''\right)=\lambda_1\left(u+v\right), \tag{5.4}$$

$$\frac{\sqrt{f'^2+g'^2}}{2f'g''^2}\left(-3g''^2+g'g'''\right)=\lambda_2 g(v), \tag{5.5}$$

$$\frac{\sqrt{f'^2+g'^2}}{2g'f''^2}\left(-3f''^2+f'f'''\right)=\lambda_3 f(u), \tag{5.6}$$

where $\lambda_i\in\mathbb{R}$. This means that $\mathbf{M}$ is at most of 3-types. Combining equations (5.4), (5.5) and (5.6), we have

$$\frac{2\sqrt{f'^2+g'^2}}{f'g'}+\lambda_3\frac{f}{f'}+\lambda_2\frac{g}{g'}=\lambda_1u+\lambda_1v. \tag{5.7}$$

We discuss six cases according to constants $\lambda_1,\lambda_2,\lambda_3$.

Case 1: Let $\lambda_1=0,\lambda_2\neq0,\lambda_3\neq0$, from (5.7), we obtain

$$\frac{2\sqrt{f'^2+g'^2}}{f'g'}+\lambda_3\frac{f}{f'}+\lambda_2\frac{g}{g'}=0. \tag{5.8}$$

There is no suitable solutions. Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$.

Case 2: Let $\lambda_1=\lambda_2=0$, $\lambda_3\neq0$, from (5.7), we obtain

$$\frac{2\sqrt{f'^2+g'^2}}{f'g'}+\lambda_3\frac{f}{f'}=0. \tag{5.9}$$

This differential equation admits the solution

$$f(u)=c_1u+c_2,g(v)=c_3\pm\frac{2c_1v}{\sqrt{-4+c_2\lambda_3^2u+c_1^2\lambda_3^2u^2}}, \tag{5.10}$$

where $c_i\in\mathbb{R}$ with $\lambda_3\neq0$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v)=\left(u+v,\left(c_3\pm\frac{2c_1v}{\sqrt{-4+c_2\lambda_3^2u+c_1^2\lambda_3^2u^2}}\right),(c_1u+c_2)\right). \tag{5.11}$$

Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$.

Case 3: Let $\lambda_1\neq0,\lambda_2=0$, $\lambda_3=0$, from (5.7), we obtain

$$\frac{2\sqrt{f'^2+g'^2}}{f'g'}=\lambda_1u+\lambda_1v. \tag{5.12}$$

This differential equation admits the solution

$$\begin{aligned} f(u) &= c_1 u + c_2, \\ g(v) &= c_3 \pm \frac{2\ln\left(c_4^2\lambda_1^2(u+v) + c_4\lambda_1\sqrt{-4+c_4^2\lambda_1^2(u+v)^2}\right)}{\lambda_1}, \end{aligned} \tag{5.13}$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \begin{pmatrix} u+v, \\ \left(c_3 \pm \frac{2\ln\left(c_4^2\lambda_1^2(u+v)+c_4\lambda_1\sqrt{-4+c_4^2\lambda_1^2(u+v)^2}\right)}{\lambda_1}\right), \\ (c_1u+c_2) \end{pmatrix}. \tag{5.14}$$

Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$. For example, if we choose $f = e^u$, (5.12) admits the solution

$$\begin{aligned} f(u) &= e^u, \\ g(v) &= c_1 \pm \frac{2\ln\left(\lambda_1 e^u\left(\lambda_1 e^u(u+v) + \sqrt{-4+\lambda_1^2 e^{2u}(u+v)^2}\right)\right)}{\lambda_1}, \end{aligned} \tag{5.15}$$

where $c_1 \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\mathbf{x}(u,v) = \begin{pmatrix} u+v, \\ \left(c_1 \pm \frac{2\ln\left(\lambda_1 e^u\left(\lambda_1 e^u(u+v)+\sqrt{-4+\lambda_1^2e^{2u}(u+v)^2}\right)\right)}{\lambda_1}\right), \\ e^u \end{pmatrix}. \tag{5.16}$$

Case 4: Let $\lambda_1 = 0, \lambda_2 \neq 0$, $\lambda_3 = 0$, from (5.7), we obtain

$$\frac{2\sqrt{f'^2+g'^2}}{f'g'} + \lambda_2\frac{g}{g'} = 0. \tag{5.17}$$

This differential equation admits the solution

$$\begin{aligned} f(u) &= c_1 u + c_2, \\ g(v) &= \frac{e^{-\frac{1}{2}c_1\lambda_2 v - c_3\lambda_2}\left(4\lambda_2^2 e^{c_1\lambda_2 v} + e^{2c_3\lambda_2}\right)}{2\lambda_2^2}, \\ g(v) &= \frac{e^{-\frac{1}{2}c_1\lambda_2 v - c_3\lambda_2}\left(4\lambda_2^2 + e^{c_1\lambda_2 v + 2c_3\lambda_2}\right)}{2\lambda_2^2}, \end{aligned} \tag{5.18}$$

where $c_i \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$\begin{aligned} \mathbf{x}(u,v) &= \left(u+v, \left(\frac{e^{-\frac{1}{2}c_1\lambda_2 v - c_3\lambda_2}\left(4\lambda_2^2 e^{c_1\lambda_2 v} + e^{2c_3\lambda_2}\right)}{2\lambda_2^2}\right), (c_1u+c_2)\right), \\ \mathbf{x}(u,v) &= \left(u+v, \left(\frac{e^{-\frac{1}{2}c_1\lambda_2 v - c_3\lambda_2}\left(4\lambda_2^2 + e^{c_1\lambda_2 v + 2c_3\lambda_2}\right)}{2\lambda_2^2}\right), (c_1u+c_2)\right). \end{aligned} \tag{5.19}$$

Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$. For example, if we choose $f = e^u$, (5.17) admits the

solution

$$
\begin{aligned}
f(u) &= e^u, \\
g(v) &= \frac{e^{-\frac{1}{2}e^u\lambda_2 v - c_1\lambda_2}\left(4\lambda_2^2 e^{e^u\lambda_2 v} + e^{2c_1\lambda_2}\right)}{2\lambda_2^2}, \\
g(v) &= \frac{e^{-\frac{1}{2}e^u\lambda_2 v - c_1\lambda_2}\left(4\lambda_2^2 + e^{e^u\lambda_2 v + 2c_1\lambda_2}\right)}{2\lambda_2^2},
\end{aligned} \tag{5.20}
$$

where $c_1 \in \mathbb{R}$. In this case, $\mathbf{M}$ is parametrized by

$$
\begin{aligned}
\mathbf{x}(u,v) &= \left(u+v, \left(\frac{e^{-\frac{1}{2}e^u\lambda_2 v - c_1\lambda_2}\left(4\lambda_2^2 e^{e^u\lambda_2 v} + e^{2c_1\lambda_2}\right)}{2\lambda_2^2}\right), e^u\right), \\
\mathbf{x}(u,v) &= \left(u+v, \left(\frac{e^{-\frac{1}{2}e^u\lambda_2 v - c_1\lambda_2}\left(4\lambda_2^2 + e^{e^u\lambda_2 v + 2c_1\lambda_2}\right)}{2\lambda_2^2}\right), e^u\right).
\end{aligned} \tag{5.21}
$$

Case 5: Let $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from (5.7), we obtain

$$
\frac{2\sqrt{f'^2 + g'^2}}{f'g'} = 0. \tag{5.22}
$$

There is no suitable solutions.

Case 6: Let $\lambda_1 \neq 0, \lambda_2 \neq 0$, $\lambda_3 = 0$, from (4.7), we obtain

$$
\frac{2\sqrt{f'^2 + g'^2}}{f'g'} + \lambda_2\frac{g}{g'} = \lambda_1 u + \lambda_1 v. \tag{5.23}
$$

There is no suitable solutions. Based on the selection of the function $f(u)$, it is possible to obtain other form of the function $g(v)$.

Using the solutions (5.10), (5.11), (5.13) and (5.18) give rise a contradiction with our assumption saying that the solution must be non-degenerate second fundamental form.

Definition. A surface of in the three dimensional Galilean space is said to be $\mathbf{II}-$harmonic if it satisfies the condition $\Delta^{\mathbf{II}}\mathbf{x} = \mathbf{0}$.

Theorem 5.1. *Let* $\mathbf{M}$ *be a translation surface given by* (3.2) *in the three dimensional Galilean space* $\mathbb{G}_3$*. Then there is no* $\mathbf{II}$*-harmonic the surface* $\mathbf{M}$*.*

Theorem 5.2. *Let* $\mathbf{M}$ *be a non* $\mathbf{II}$*-harmonic translation surface with non-degenerate second fundamental form given by* (3.2) *in the three dimensional Galilean space* $\mathbb{G}_3$*. If the surface* $\mathbf{M}$ *satisfies the condition* $\Delta^{\mathbf{II}}\mathbf{x}_i{=}\lambda_i\mathbf{x}_i$*, where* $\lambda_i{\in}\mathbb{R}$*,* $i{=}1,2,3$*, then it is congruent to an open part of the surfaces* (5.16) *and* (5.21)*.*

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