Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1229–1240
https://doi.org/10.4134/BKMS.b160399
pISSN: 1015-8634 / eISSN: 2234-3016

ABSTRACT HARMONIC ANALYSIS OVER SPACES OF COMPLEX MEASURES ON HOMOGENEOUS SPACES OF COMPACT GROUPS

Arash Ghaani Farashahi

Abstract. This paper presents a systematic study of the abstract harmonic analysis over spaces of complex measures on homogeneous spaces of compact groups. Let G be a compact group and H be a closed subgroup of G. Then we study abstract harmonic analysis of complex measures over the left coset space G/H.

1. Introduction

The mathematical theory of complex measures plays significant and classical roles in abstract harmonic analysis, representation theory, functional analysis, operator theory, and C^*-algebras, see [16, 17, 23, 24] and references therein.

The following paper studies abstract harmonic analysis over spaces of complex measures on homogeneous spaces (coset spaces) of compact groups. In a nutshell, homogeneous spaces are group-like structures with many interesting applications in mathematical physics, differential geometry, geometric analysis, and coherent state (covariant) transforms, see [8, 10–12, 15, 18–22].

This article contains 4 sections. Section 2 is devoted to fix notations and a summary of complex measures, Banach space adjoint, classical harmonic analysis over compact groups and homogeneous spaces (left coset spaces) of compact groups. Let G be a compact group and H be a closed subgroup of G. Then the left coset space G/H is a homogeneous space of G via the left action of G on the coset space G/H. Let μ be the normalized G-invariant measure over the homogeneous space G/H associated to the Weil's formula and $1 \leq p \leq \infty$. In Section 3 we study abstract harmonic analysis on the Banach L^p-function spaces over homogeneous spaces of compact groups. Then we present a systematic study for abstract harmonic analysis over spaces of complex measures on homogeneous spaces of compact groups, in Section 4. Let $M(G/H)$ be the Banach space consists of all regular countably additive complex Borel measures on the left coset space G/H. We shall introduce

Received May 8, 2016; Accepted February 13, 2017.
2010 *Mathematics Subject Classification.* Primary 43A85, 47A67.
Key words and phrases. homogeneous space, compact group, complex measure.

an explicit construction for the Banach space adjoint of the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$. As the main application of this construction, it is shown that there exists a canonical isometric linear embedding of the Banach function space $L^1(G/H,\mu)$ into the Banach measure space $M(G/H)$.

2. Preliminaries and notations

For a locally compact Hausdorff space, $\mathcal{C}(X)$ denotes the space of all continuous complex-valued functions on X, $\mathcal{C}_0(X)$ stands for the subspace of $\mathcal{C}(X)$ consists of all continuous complex-valued functions on X which are vanishing at infinity, and $\mathcal{C}_c(X)$ is the subspace of $\mathcal{C}(X)$ which contains all continuous complex-valued functions on X with compact support. It is easy to see that

$$\mathcal{C}_c(X) \subseteq \mathcal{C}_0(X) \subseteq \mathcal{C}(X).$$

Let X be a compact Hausdorff space. Then we have

$$\mathcal{C}_c(X) = \mathcal{C}_0(X) = \mathcal{C}(X).$$

If μ is a positive Radon measure on X, for each $1 \le p < \infty$ the Banach space of equivalence classes of μ-measurable complex valued functions $f : X \to \mathbb{C}$ such that

$$\|f\|_{L^p(X,\mu)} = \left(\int_X |f(x)|^p d\mu(x)\right)^{1/p} < \infty,$$

is denoted by $L^p(X,\mu)$ which contains $\mathcal{C}(X)$ as a $\|\cdot\|_{L^p(X,\mu)}$-dense subspace. The linear space consists of all regular countably additive complex Borel measures on X is denoted by $M(X)$. It is a Banach space with respect to the total variation norm $\|\cdot\|_{M(X)}$, defined as

$$\|\lambda\|_{M(X)} := |\lambda|(X),$$

for all $\lambda \in M(X)$, where $|\lambda|$ is the absolute value of λ.

It is well-known as the Riesz-Markov theorem [16] that, (i) for any unique regular countably additive complex Borel measure λ on X, the mapping $f \mapsto \lambda(f)$ is a continuous functional on $\mathcal{C}(X)$, where

$$\lambda(f) := \int_X f(x) d\lambda(x).$$

(ii) for any continuous linear functional Γ on $\mathcal{C}(X)$, there is a unique regular countably additive complex Borel measure λ_Γ on X such that

$$\Gamma(f) = \lambda_\Gamma(f) = \int_X f(x) d\lambda_\Gamma(x)$$

for all $f \in \mathcal{C}(X)$. Also, we have

$$\|\Gamma\| = \|\lambda_\Gamma\|_{M(X)},$$

where $\|\Gamma\|$ is the operator norm of the functional Γ, that is

$$\|\Gamma\| := \sup_{\{f\in\mathcal{C}(X):\|f\|_{\sup}\le 1\}} |\Gamma(f)|.$$

Let $\mathcal{X},\mathcal{Y}$ be Banach spaces with the topological (linear) dual spaces $\mathcal{X}^*$ and $\mathcal{Y}^*$ respectively. Also, let $T:\mathcal{X}\to\mathcal{Y}$ be a bounded linear operator. The Banach space adjoint of the linear map T, is the bounded linear map $T^*:\mathcal{Y}^*\to\mathcal{X}^*$ given by

$$T^*(\beta)(a):=\beta(T(a))$$

for all $a\in X$ and $\beta\in\mathcal{Y}^*$.

Let G be a compact group with the probability Haar measure σ. For $p\geq 1$ the notation $L^p(G)$ stands for the Banach function space $L^p(G,\sigma)$. Each function $g\in L^1(G)$ defines a regular countably additive complex Borel measure $\sigma_g\in M(G)$ which satisfies

$$\sigma_g(f)=\int_G f(x)g(x)d\sigma(x)$$

for all $f\in\mathcal{C}(G)$. Then $g\mapsto\sigma_g$ is an isometric linear embedding of the Banach function space $L^1(G)$ into the Banach measure space $M(G)$, see [16].

Let H be a closed subgroup of G with the normalized Haar measure dh. The left coset space G/H is considered as a locally compact homogeneous space that G acts on it from the left, and $q:G\to G/H$ given by $x\mapsto q(x):=xH$ is the surjective canonical map. The classical aspects of abstract harmonic analysis on locally compact homogeneous spaces are quite well studied by several authors, see [5,16,17,24] and references therein. If G is compact, each transitive G-space can be considered as a left coset space G/H for some closed subgroup H of G. The function space $\mathcal{C}(G/H)$ consists of all functions $T_H(f)$, where $f\in\mathcal{C}(G)$ and

$$T_H(f)(xH)=\int_H f(xh)dh. \tag{2.1}$$

Let μ be a Radon measure on G/H and $x\in G$. The translation μ_x of μ is defined by $\mu_x(E)=\mu(xE)$, for all Borel subsets E of G/H. The measure μ is called G-invariant if $\mu_x=\mu$, for all $x\in G$. The homogeneous space G/H has a normalized G-invariant measure μ, which satisfies the Weil's formula

$$\int_{G/H} T_H(f)(xH)d\mu(xH)=\int_G f(x)dx, \tag{2.2}$$

and hence the linear map T_H is norm-decreasing, that is

$$\|T_H(f)\|_{L^1(G/H,\mu)}\leq\|f\|_{L^1(G)}$$

for all $f\in L^1(G)$, see [5,16,24].

3. Abstract harmonic analysis of L^p-function spaces over homogeneous spaces of compact groups

In this section, we present some classical results related to abstract harmonic analysis of function spaces over the homogeneous spaces of compact groups [6,7,9,13,14]. We assume that G is a compact group with the probability Haar measure dx, H is a closed subgroup of G with the probability Haar measure

dh, and μ is the normalized G-invariant measure on the compact homogeneous space G/H satisfying (2.2).

In this case, we have

$$\mathcal{C}_c(G) = \mathcal{C}_0(G) = \mathcal{C}(G),$$

and also

$$\mathcal{C}_c(G/H) = \mathcal{C}_0(G/H) = \mathcal{C}(G/H).$$

The following proposition shows that the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.

Proposition 3.1. *Let H be a closed subgroup of a compact group G. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ is uniformly continuous.*

Proof. Let $f \in \mathcal{C}(G)$ and $x \in G$. Then we have

$$\begin{aligned} |T_H(f)(xH)| &= \left| \int_H f(xh)dh \right| \\ &\leq \int_H |f(xh)|dh \\ &\leq \|f\|_{\sup} \left(\int_H dh \right) = \|f\|_{\sup}, \end{aligned}$$

which implies

$$\|T_H(f)\|_{\sup} = \sup_{xH \in G/H} |T_H(f)(xH)| \leq \|f\|_{\sup}.$$

□

For a function $\psi : G/H \to \mathbb{C}$, define $\psi_q : G \to \mathbb{C}$ by

$$\psi_q(x) := \psi \circ q(x) = \psi(xH)$$

for all $xH \in G/H$.

Then we can present the following results.

Corollary 3.2. *Let H be a closed subgroup of a compact group G and $\psi \in \mathcal{C}(G/H)$. Then*

(1) *$\psi_q \in \mathcal{C}(G)$ and $T_H(\psi_q) = \psi$.*
(2) *$\|\psi\|_{\sup} = \|\psi_q\|_{\sup}$.*

Proof. Let $\psi \in \mathcal{C}(G/H)$.

(1) It is easy to see that $\psi_q \in \mathcal{C}(G)$. Let $x \in G$. Then we can write

$$\begin{aligned} T_H(\psi_q)(xH) &= \int_H \psi_q(xh)dh \\ &= \int_H \psi(xhH)dh \\ &= \int_H \psi(xH)dh = \psi(xH). \end{aligned}$$

(2) Also, we can write

$$\begin{aligned}\|\psi\|_{\sup} &= \sup_{xH\in G/H} |\psi(xH)| \\ &= \sup_{x\in G} |\psi(xH)| \\ &= \sup_{x\in G} |\psi_q(x)| = \|\psi_q\|_{\sup}.\end{aligned}$$

□

Remark 3.3. Let G be a compact group and H be a closed subgroup of G. Let $T_H^* : \mathcal{C}(G/H)^* \to \mathcal{C}(G)^*$ be the Banach space adjoint of the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ given by (2.1). Then

$$T_H^*(\beta) = \beta_q,\ \forall \beta \in \mathcal{C}(G/H)^*,$$

where $\beta_q : \mathcal{C}(G) \to \mathbb{C}$ is given by

$$\beta_q(f) := \beta(T_H(f)),\ \forall f \in \mathcal{C}(G).$$

Next theorem proves that the linear map T_H is norm-decreasing in other L^p-spaces, when $p > 1$.

Theorem 3.4. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $p \geq 1$. The linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ has a unique extension to a bounded linear map from $L^p(G)$ onto $L^p(G/H,\mu)$.*

Proof. We shall show that, each $f \in \mathcal{C}(G)$ satisfies

$$\|T_H(f)\|_{L^p(G/H,\mu)} \leq \|f\|_{L^p(G)}. \tag{3.1}$$

Using compactness of H, and the Weil's formula, we can write

$$\begin{aligned}\|T_H(f)\|^p_{L^p(G/H,\mu)} &= \int_{G/H} |T_H(f)(xH)|^p d\mu(xH) \\ &= \int_{G/H} \left|\int_H f(xh)dh\right|^p d\mu(xH) \\ &\leq \int_{G/H} \left(\int_H |f(xh)|dh\right)^p d\mu(xH) \\ &\leq \int_{G/H}\int_H |f(xh)|^p dhd\mu(xH) \\ &= \int_{G/H}\int_H |f|^p(xh)dhd\mu(xH) \\ &= \int_{G/H} T_H(|f|^p)(xH)d\mu(xH) \\ &= \int_G |f(x)|^p dx = \|f\|^p_{L^p(G)},\end{aligned}$$

which implies (3.1). Thus, we can extend T_H to a bounded linear operator from $L^p(G)$ onto $L^p(G/H,\mu)$, which still will be denoted by T_H. □

As an immediate consequence of Theorem 3.4 we deduce the following corollary.

Corollary 3.5. *Let H be a closed subgroup of a compact group G, μ be the normalized G-invariant measure on G/H, and $p \geq 1$. Let $\varphi \in L^p(G/H,\mu)$ and $\varphi_q := \varphi \circ q$. Then $\varphi_q \in L^p(G)$ with*

$$\|\varphi_q\|_{L^p(G)} = \|\varphi\|_{L^p(G/H,\mu)}. \tag{3.2}$$

Proof. Indeed, using the Weil's formula, we can write

$$\begin{aligned}\|\varphi_q\|^p_{L^p(G)} &= \int_G |\varphi_q(x)|^p dx \\ &= \int_{G/H} T_H\left(|\varphi_q|^p\right)(xH)d\mu(xH) = \int_{G/H}\left(\int_H |\varphi_q(xh)|^p dh\right)d\mu(xH),\end{aligned}$$

and since H is compact and dh is normalized, we get

$$\begin{aligned}\int_{G/H}\left(\int_H |\varphi_q(xh)|^p dh\right)d\mu(xH) &= \int_{G/H}\left(\int_H |\varphi(xhH)|^p dh\right)d\mu(xH) \\ &= \int_{G/H}\left(\int_H |\varphi(xH)|^p dh\right)d\mu(xH) \\ &= \int_{G/H} |\varphi(xH)|^p\left(\int_H dh\right)d\mu(xH) \\ &= \int_{G/H} |\varphi(xH)|^p d\mu(xH) = \|\varphi\|^p_{L^p(G/H,\mu)},\end{aligned}$$

which implies (3.2). □

Let $\mathcal{J}^2(G,H) := \{f \in L^2(G) : T_H(f) = 0\}$ and $\mathcal{J}^2(G,H)^\perp$ be the orthogonal complement of the closed subspace $\mathcal{J}^2(G,H)$ in $L^2(G)$.

Proposition 3.6. *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. Then $T_H : L^2(G) \to L^2(G/H,\mu)$ is a partial isometric linear map.*

Proof. Let $\varphi \in L^2(G/H,\mu)$. Invoking Corollary 3.5 we claim that $T_H^*(\varphi) = \varphi_q$, and hence $T_H T_H^*(\varphi) = \varphi$. Indeed, using the Weil's formula, we can write

$$\begin{aligned}\langle T_H^*(\varphi), f\rangle_{L^2(G)} &= \langle \varphi, T_H(f)\rangle_{L^2(G/H,\mu)} \\ &= \int_{G/H} \varphi(xH)\overline{T_H(f)(xH)}d\mu(xH) \\ &= \int_{G/H} \varphi(xH)T_H(\overline{f})(xH)d\mu(xH) \\ &= \int_{G/H} T_H(\varphi_q.\overline{f})(xH)d\mu(xH)\end{aligned}$$

$$= \int_G \varphi_q(x)\overline{f(x)}dx = \langle \varphi_q, f\rangle_{L^2(G)}$$

for all $f \in L^2(G)$, which implies that $T_H^*(\varphi) = \varphi_q$. Now a straightforward calculation shows that $T_H = T_H T_H^* T_H$. Then, by Theorem 2.3.3 of [23], T_H is a partial isometric operator. □

The following corollaries are straightforward consequences of Proposition 3.6.

Corollary 3.7. *Let H be a closed subgroup of a compact group G. Let $P_{\mathcal{J}^2(G,H)}$ and $P_{\mathcal{J}^2(G,H)^\perp}$ be the orthogonal projections onto the closed subspaces $\mathcal{J}^2(G,H)$ and $\mathcal{J}^2(G,H)^\perp$ respectively. Then for each $f \in L^2(G)$ and for a.e. $x \in G$ we have*

(1) $P_{\mathcal{J}^2(G,H)^\perp}(f)(x) = T_H(f)(xH)$.
(2) $P_{\mathcal{J}^2(G,H)}(f)(x) = f(x) - T_H(f)(xH)$.

Corollary 3.8. *Let H be a compact subgroup of a locally compact group G and μ be the normalized G-invariant measure on G/H. Then*

(1) $\mathcal{J}^2(G,H)^\perp = \{\psi_q = \psi \circ q : \psi \in L^2(G/H,\mu)\}$.
(2) *For $f \in \mathcal{J}^2(G,H)^\perp$ and $h \in H$, we have $R_h f = f$.*
(3) *For $f, g \in \mathcal{J}^2(G,H)^\perp$, we have $\langle T_H(f), T_H(g)\rangle_{L^2(G/H,\mu)} = \langle f, g\rangle_{L^2(G)}$.*

Remark 3.9. Invoking Corollary 3.8, one can regard the Hilbert space $L^2(G/H, \mu)$ as a closed subspace of $L^2(G)$, that is the closed subspace consists of all $f \in L^2(G)$ which satisfies $R_h f = f$ for all $h \in H$. Then Theorem 3.4 and Proposition 3.6 guarantee that the linear map

$$T_H : L^2(G) \to L^2(G/H,\mu) \subset L^2(G)$$

is an orthogonal projection onto $L^2(G/H,\mu)$.

4. Abstract harmonic analysis over spaces of complex measures on homogeneous spaces of compact groups

Throughout this section we present some basic results concerning abstract harmonic analysis over spaces of complex measures on homogeneous spaces of compact groups. It should be mentioned that, from now on by a complex measure we mean a regular countably additive complex Borel measure. Also, it is still assumed that G is a compact groups, H is a closed subgroup of G, and μ is the normalized G-invariant measure over the left coset space G/H associated to (2.1) with respect to the probability measures of G and H.

For a complex measure $\nu \in M(G)$, let $T_H(\nu) \in M(G/H)$ be the complex measure which satisfies [24]

$$\int_{G/H} \psi(xH)dT_H(\nu)(xH) = \int_G \psi_q(x)d\nu(x)$$

for all $\psi \in \mathcal{C}(G/H)$. Then, $T_H : M(G) \to M(G/H)$ given by $\nu \mapsto T_H(\nu)$ is a surjective linear map.

Let $\lambda \in M(G/H)$ be a complex measure. Then, $\Gamma_\lambda : \mathcal{C}(G) \to \mathbb{C}$ given by

$$f \mapsto \Gamma_\lambda(f) := \int_{G/H} T_H(f)(xH)d\lambda(xH),$$

is a linear functional. Also, it is continuous. Because, we can write

$$\begin{aligned}
|\Gamma_\lambda(f)| &= \left|\int_{G/H} T_H(f)(xH)d\lambda(xH)\right| \\
&\le \int_{G/H} |T_H(f)(xH)|d|\lambda|(xH) \\
&\le \|T_H(f)\|_{\sup} \cdot \left(\int_{G/H} d|\lambda|(xH)\right) \\
&= \|T_H(f)\|_{\sup} \cdot |\lambda|(G/H) \\
&= \|T_H(f)\|_{\sup} \cdot \|\lambda\|_{M(G/H)} \le \|f\|_{\sup}\|\lambda\|_{M(G/H)}.
\end{aligned}$$

Thus, invoking Riesz-Markov theorem, there exists a unique complex measure, denoted by $\lambda_q \in M(G)$, satisfying

$$\int_G f(x)d\lambda_q(x) = \int_{G/H} T_H(f)(xH)d\lambda(xH) \tag{4.1}$$

for all $f \in \mathcal{C}(G)$.

Then we can summarize some interesting observations as follows.

Proposition 4.1. *Let G be a compact group and H a closed subgroup of G. Let $\lambda \in M(G/H)$. Then we have*

(1) $T_H(\lambda_q) = \lambda$.

(2) *For each $f \in \mathcal{C}(G)$ and $h \in H$ we have*
$$\int_G f(xh)d\lambda_q(x) = \int_G f(x)d\lambda_q(x).$$

(3) *For each Borel subset $S \subseteq G$ and $h \in H$ we have*
$$\lambda_q(Sh) = \lambda_q(S),$$
where $Sh = \{gh : g \in S\}$.

Proof. (1) Let $\psi \in \mathcal{C}(G/H)$ be given. Then we have $\psi_q \in \mathcal{C}(G)$ with $T_H(\psi_q) = \psi$. Hence, we can write

$$\begin{aligned}
\int_{G/H} \psi(xH)dT_H(\lambda_q)(xH) &= \int_G \psi_q(x)d\lambda_q(x) \\
&= \int_{G/H} T_H(\psi_q)(xH)d\lambda(xH)
\end{aligned}$$

$$= \int_{G/H} \psi(xH)d\lambda(xH).$$

(2) Let $f \in \mathcal{C}(G)$ and $h \in H$. Then we have

$$\begin{aligned}\int_G f(xh)d\lambda_q(x) &= \int_G R_h f(x)d\lambda_q(x)\\ &= \int_{G/H} T_H(R_h f)(xH)d\lambda(xH)\\ &= \int_{G/H} T_H(f)(xH)d\lambda(xH) = \int_G f(x)d\lambda_q(x).\end{aligned}$$

(3) It is a straightforward consequence of (2). □

Next theorem presents the connection of the map $\lambda \mapsto \lambda_q$ with the Banach space adjoint of the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ in the framework of complex measures.

Theorem 4.2. *Let G be a compact group and H be a closed subgroup of G. Let $T_H^* : M(G/H) \to M(G)$ be the Banach space adjoint of the linear map $T_H : \mathcal{C}(G) \to \mathcal{C}(G/H)$ given by* (2.1). *Then*

$$T_H^*(\lambda) = \lambda_q, \ \forall \lambda \in M(G/H).$$

Proof. Let $\lambda \in M(G/H)$ and $f \in \mathcal{C}(G)$. Then, we can write

$$\begin{aligned}T_H^*(\lambda)(f) &= \lambda(T_H(f))\\ &= \int_{G/H} T_H(f)(xH)d\lambda(xH)\\ &= \int_G f(x)d\lambda_q(x) = \lambda_q(f),\end{aligned}$$

which implies that $T_H^*(\lambda) = \lambda_q$. □

For a G-invariant measure μ and a function $\varphi \in L^1(G/H, \mu)$, one can define the continuous linear functional given by

$$\psi \mapsto \int_{G/H} \psi(xH)\varphi(xH)d\mu(xH) \tag{4.2}$$

for all $\psi \in \mathcal{C}(G/H)$. Indeed,

$$\begin{aligned}\left|\int_{G/H} \psi(xH)\varphi(xH)d\mu(xH)\right| &\leq \int_{G/H} |\psi(xH)| \cdot |\varphi(xH)|d\mu(xH)\\ &= \|\psi\|_{\sup} \cdot \left(\int_{G/H} |\varphi(xH)|d\mu(xH)\right)\\ &= \|\psi\|_{\sup} \cdot \|\varphi\|_{L^1(G/H,\mu)}.\end{aligned}$$

Let μ_φ be the complex Radon measure on the coset space G/H associated to the continuous linear functional given by (4.2). Thus, we get

$$\int_{G/H}\psi(xH)d\mu_\varphi(xH)=\int_{G/H}\psi(xH)\varphi(xH)d\mu(xH) \tag{4.3}$$

for all $\psi\in\mathcal{C}(G/H)$.

We then can also present the following result.

Proposition 4.3. *Let G be a compact group and H be a closed subgroup of G. Let μ be the normalized G-invariant measure over G/H associated to* (2.2) *with respect to probability measures of G and H. Then, for all $\varphi\in L^1(G/H,\mu)$, we have*

$$(\mu_\varphi)_q=\sigma_{\varphi_q}.$$

Proof. Let $f\in\mathcal{C}(G)$. Then, using the Weil's formula, we have

$$\begin{aligned}\int_G f(x)d(\mu_\varphi)_q(x)&=\int_{G/H}T_H(f)(xH)d\mu_\varphi(xH)\\&=\int_{G/H}T_H(f)(xH)\varphi(xH)d\mu(xH)\\&=\int_{G/H}T_H(f\cdot\varphi_q)(xH)d\mu(xH)\\&=\int_G f(x)\varphi_q(x)dx=\int_G f(x)d\sigma_{\varphi_q}(x),\end{aligned}$$

which completes the proof. □

Also, we can prove the following isometric relation for the linear map T_H^*.

Proposition 4.4. *Let G be a compact group, H be a closed subgroup of G and $\lambda\in M(G/H)$. Then we have*

$$\|\lambda_q\|_{M(G)}=\|\lambda\|_{M(G/H)}.$$

Proof. Let $\lambda\in M(G/H)$. Then we can write

$$\begin{aligned}\|\lambda_q\|_{M(G)}&=\sup_{\{f\in\mathcal{C}(G):\|f\|_{\sup}\le1\}}|\lambda_q(f)|\\&=\sup_{\{f\in\mathcal{C}(G):\|f\|_{\sup}\le1\}}|\lambda(T_H(f))|\\&=\sup_{\{\psi\in\mathcal{C}(G/H):\|\psi\|_{\sup}\le1\}}|\lambda(\psi)|=\|\lambda\|_{M(G/H)}.\end{aligned}$$

□

We hereby finish the paper by the following theorem which states that the mapping $\varphi\mapsto\mu_\varphi$ is an isometric linear embedding of the Banach function space $L^1(G/H,\mu)$ into the Banach measure space $M(G/H)$.

Theorem 4.5. *Let G be a compact group and H be a closed subgroup of G. Let μ be the normalized G-invariant measure over G/H associated to* (2.2) *with respect to probability measures of G and H. Then, $\varphi \mapsto \mu_\varphi$ defines an isometric linear embedding of the Banach function space $L^1(G/H,\mu)$ into the Banach measure space $M(G/H)$.*

Proof. Let σ be the probability measure on G. Let μ be the normalized G-invariant measure over G/H associated to (2.2) with respect to probability measures of G and H, $c \in \mathbb{C}$, and $\varphi, \varphi' \in L^1(G/H,\mu)$. It is easy to check that $\mu_{c\varphi+\varphi'} = c\mu_\varphi + \mu_{\varphi'}$. Then, using (3.2), Proposition 4.3 and also Proposition 4.4 we can write

$$\begin{aligned}\|\mu_\varphi\|_{M(G/H)} &= \|(\mu_\varphi)_q\|_{M(G)}\\ &= \|\sigma_{\varphi_q}\|_{M(G)}\\ &= \|\varphi_q\|_{L^1(G,\sigma)} = \|\varphi\|_{L^1(G/H,\mu)}.\end{aligned}$$

Thus $\varphi \mapsto \mu_\varphi$ is an isometric linear map. Hence, it has closed range. So we deduce that $\varphi \mapsto \mu_\varphi$ is an isometric embedding of the Banach function space $L^1(G/H,\mu)$ into the Banach measure space $M(G/H)$. □

Acknowledgement. The author would like to express his deepest gratitude to Prof. Hans G. Feichtinger for his valuable comments.

References

[1] A. Arefijamaal, *On construction of coherent state associated with homogeneous spaces*, Turkish J. Math. **34** (2010), no. 4, 515–521.

[2] A. Derighetti, *Convolution operators on groups*, Lecture Notes of the Unione Matematica Italiana, 11. Springer, Heidelberg; UMI, Bologna, 2011.

[3] ———, *On the multipliers of a quotient group*, Bull. Sci. Math. (2) **107** (1983), no. 1, 3–23.

[4] J. Dixmier, *C^*-Algebras*, North-Holland and Publishing company, 1977.

[5] G. B. Folland, *A course in Abstract Harmonic Analysis*, CRC press, 1995.

[6] A. Ghaani Farashahi, *Abstract non-commutative harmonic analysis of coherent state transforms*, Ph.D. thesis, Ferdowsi University of Mashhad (FUM), Mashhad, 2012.

[7] ———, *Convolution and involution on function spaces of homogeneous spaces*, Bull. Malays. Math. Sci. Soc. (2) **36** (2013) no. 4, 1109–1122.

[8] ———, *Abstract harmonic analysis of relative convolutions over canonical homogeneous spaces of semidirect product groups*, J. Aust. Math. Soc. **101** (2016), no. 2, 171–187.

[9] ———, *Trigonometric polynomials over homogeneous spaces of compact groups*, Adv. Oper. Theory **2** (2017), no. 1, 87–97.

[10] ———, *Abstract harmonic analysis of wave-packet transforms over locally compact abelian groups*, Banach J. Math. Anal. **11** (2017), no. 1, 50–71.

[11] ———, *Abstract Plancherel* (*trace*) *formulas over homogeneous spaces of compact groups*, Canadian Math. Bulletin: doi:10.4153/CMB-2016-037-6.

[12] ———, *Abstract relative Fourier transforms over canonical homogeneous spaces of semi-direct product groups with Abelian normal factor*, J. Korean Math. Soc. **54** (2017), no. 1, 117–139.

[13] ———, *A class of abstract linear representations for convolution function algebras over homogeneous spaces of compact groups*, Canadian J. Math.: http://dx.doi.org/10.4153/CJM-2016-043-9.

[14] ______, *Abstract convolution function algebras over homogeneous spaces of compact groups*, Illinois J. Math., to appear.
[15] ______, *Abstract operator-valued Fourier transforms over homogeneous spaces of compact groups*, Groups, Geometry, Dynamics, to appear.
[16] E. Hewitt and K. A. Ross, *Absrtact Harmonic Analysis. Vol 1*, 1963.
[17] ______, *Absrtact Harmonic Analysis. Vol 2*, 1970.
[18] V. Kisil, *Relative convolutions. I. Properties and applications*, Adv. Math. **147** (1999), no. 1, 35–73.
[19] ______, *Operator covariant transform and local principle*, J. Phys. A **45** (2012), no. 24, 244022, 10 pp.
[20] ______, *Geometry of Möbius Transformations. Elliptic, Parabolic and Hyperbolic Actions of $SL_2(\mathbb{R})$*, Imperial College Press, London, 2012.
[21] ______, *Erlangen program at large: an overview. Advances in applied analysis*, 1–94, Trends Math., Birkhäuser/Springer Basel AG, Basel, 2012.
[22] ______, *Calculus of operators: covariant transform and relative convolutions*, Banach J. Math. Anal. **8** (2014), no. 2, 156–184.
[23] G. J. Murphy, *C*-Algebras and Operator theory*, Academic Press, INC.
[24] H. Reiter and J. D. Stegeman, *Classical Harmonic Analysis*, 2nd Ed, Oxford University Press, New York, 2000.

Arash Ghaani Farashahi
Numerical Harmonic Analysis Group (NuHAG)
Faculty of Mathematics
University of Vienna
Oskar-Morgenstern-Platz 1, A-1090 Wien, Vienna, Austria
E-mail address: `arash.ghaani.farashahi@univie.ac.at`
`ghaanifarashahi@hotmail.com`