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REDUCING SUBSPACES FOR A CLASS OF TOEPLITZ OPERATORS ON WEIGHTED HARDY SPACES OVER BIDISK

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Abstract. We consider weighted Hardy spaces on bidisk $\mathbb{D}^2$ which generalize the weighted Bergman spaces $A^2_\alpha(\mathbb{D}^2)$. Let z, w be coordinate functions and $T_{\overline{z}^N w}$ Toeplitz operator with symbol $\overline{z}^N w$. In this paper, we study the reducing subspaces of $T_{\overline{z}^N w}$ on the weighted Hardy spaces.

1. Introduction

Let X be a closed subspace in a Hilbert space. Then X is an invariant subspace of an operator A if $AX \subset X$. In addition, X is a reducing subspace of an operator A if X is an invariant subspace of both A and its adjoint A^*. The reducing subspace X is called minimal if $\{0\}$ and X are the only reducing subspaces contained in X.

Many mathematicians study the reducing subspaces of operators on Hilbert spaces. For instance, M. Albaseer, Y. Lu and Y. Shi [1] determined the reducing subspaces of Toeplitz operator $T_{z^N\overline{w}^M}$ on the Bergman space $A^2(\mathbb{D}^2)$, where N and M are positive integers. In this paper, we will study the reducing subspaces of the operator $T_{\overline{z}^N w}$ on weighted Hardy spaces over bidisk. The weighted Hardy spaces over bidisk is a generalization of the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$. The definitions and notations in this paper are as follows.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. For $j = 1, 2$, let $d\mu_j = d\sigma_j(r)d\theta_j/2\pi$ be the probably measures on the unit disk $\overline{\mathbb{D}}$. We consider weighted Hardy space $H^2(\mathbb{D}^2, d\mu)$ which is the closure of all analytic polynomials in $L^2(\mathbb{D}^2, d\mu)$, where $d\mu(z, w) = d\mu_1(z)d\mu_2(w)$. Here $L^2(\mathbb{D}^2, d\mu)$ is the Hilbert space of square integrable functions on $\mathbb{D}^2$ with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}^2} f(z, w)\overline{g(z, w)}d\mu_1(z)d\mu_2(w).$$

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If $d\mu_1(z) = d\mu_2(z) = (\alpha+1)(1-|z|^2)^\alpha dA(z)$ where $\alpha > -1$ and dA is the normalized area measure on $\mathbb{D}$, then the weighted Hardy space $H^2(\mathbb{D}^2, d\mu_1 d\mu_2)$ is the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$ over bidisk $\mathbb{D}^2$.

Next we will introduce notions of weight sequences. Put

$$\omega_1(n) = \int_{\mathbb{D}} |z|^{2n} d\mu_1(z), \omega_2(n) = \int_{\mathbb{D}} |w|^{2n} d\mu_2(w).$$

Throughout this paper, we assume that

$$\sup_n \frac{\omega_1(n+1)}{\omega_1(n)} < \infty \text{ and } \sup_n \frac{\omega_2(n+1)}{\omega_2(n)} < \infty$$

so that the multiplication operators defined by z and w are bounded. Let P be the orthogonal projection from $L^2(\mathbb{D}^2, d\mu)$ onto $H^2(\mathbb{D}^2, d\mu)$. For φ in L^∞, put

$$T_\varphi f = P(\varphi f) \ (f \in H^2(\mathbb{D}^2, d\mu))$$

and then T_φ is called a Toeplitz operator. By calculation we have the following lemma.

Lemma 1.1. *Let N_1 and N_2 be natural numbers. The following equalities hold*:

$$T_{\overline{z}^{N_1} w^{N_2}}(z^k w^l) = \begin{cases} \frac{\omega_1(k)}{\omega_1(k-N_1)} z^{k-N_1} w^{l+N_2} & (k \geq N_1) \\ 0 & (k < N_1) \end{cases}$$

and

$$T^*_{\overline{z}^{N_1} w^{N_2}}(z^k w^l) = \begin{cases} \frac{\omega_2(l)}{\omega_2(l-N_2)} z^{k+N_1} w^{l-N_2} & (l \geq N_2) \\ 0 & (l < N_2) \end{cases}$$

for nonnegative integers k, l.

From Lemma 1.1, we show an example of the reducing subspaces of $T_{\overline{z}^N w}$.

Proposition 1.2. *Let $I_0 = \{(n,0); 0 \leq n < N\}$ as a subset of multi-indices. A subspace*

$$X_0 = \mathrm{Span}\{z^n; (n,0) \in I_0\}$$

is contained in the kernel of $T_{\overline{z}^N w}$ and $T^*_{\overline{z}^N w}$. Moreover X_0 is the reducing subspace of $T_{\overline{z}^N w}$, where we denote by Span X *the closed linear span of a subset X in $H^2(\mathbb{D}^2, d\mu)$.*

In this paper, we study the reducing subspaces contained in $X_0^\perp$. Fix a natural number N. Our main theorems are as follows. For the definition of transparent polynomials, see Section 2.

Theorem 1.3. *Let $X \subset X_0^\perp$ be a reducing subspace of $T_{\overline{z}^N w}$ on $H^2(\mathbb{D}^2, d\mu)$. Then the reducing subspace X contains the minimal reducing subspace X_p where p is a transparent polynomial. Moreover X is the minimal reducing subspace of $T_{\overline{z}^N w}$ if and only if there exists a transparent polynomial p such that $X = X_p$.*

In Section 2, we will prepare for considering the reducing subspaces of the operator $T_{\overline{z}^N w}$ contained in $X_0^{\perp}$. We note that the statements in Section 2 is valid even if $N = 1$. In Section 3, we will state our main theorem and study examples of the reducing subspaces on concrete function spaces. In this paper, we use the technique in [4, 5, 7] with the similar way.

2. Preliminaries

Let I be a subset of multi-indices such that $I = \{(n, 0); n \geq N\}$. We put the order on the set I induced by the set of nonnegative integers; $(m, 0) < (n, 0)$ if $m < n$.

We say that $(m, 0) \in I$ and $(n, 0) \in I$ are equivalent if

$$\frac{\omega_1(m)}{\omega_1(m - lN)} = \frac{\omega_1(n)}{\omega_1(n - lN)}$$

for all l satisfying $0 < lN \leq m, 0 < lN \leq n$. In this case, we write $(m, 0) \sim (n, 0)$.

For a natural number k, let I_k be a subset of I such that

$$I_k = \{(n, 0); kN \leq n \leq (k + 1)N - 1\}.$$

If a polynomial $p(z)$ is in a form of

$$p(z) = \sum_{(n,0)\in I_k} b_n z^n,$$

then we say that p is a transparent polynomial when we have $\alpha \sim \beta$ for any two nonzero coefficient b_α, b_β of p.

We partition the set I_k into equivalent classes and sort them in the order of the minimal multi-index. We denote the sorted equivalent classes by $\Omega_1, \Omega_2, \ldots, \Omega_{\tilde{N}}$ with $\tilde{N} \leq N$.

For a function $p(z) = \sum_{(n,0)\in I_k} a_n z^n$, the decomposition

$$p(z) = \sum_m p_m(z)$$

is called the canonical decomposition of p, where

$$p_m(z) = \sum_{(n,0)\in\Omega_m} a_n z^n.$$

We note that each g_i is orthogonal to g_j if $i \neq j$.

Let $\mathbb{S}$ be the vector space consisting of all finite linear combinations of finite products of the operators $T_{\overline{z}^N w}$ and its adjoint $T^*_{\overline{z}^N w}$. For any $f \in H^2(\mathbb{D}^2, d\mu)$, we put $\mathbb{S}f = \{Tf; T \in \mathbb{S}\}$. Denote the closure of $\mathbb{S}f$ in $H^2(\mathbb{D}^2, d\mu)$ by X_f which we call the reducing subspace generated by f. It is easy to see that X_f is the smallest reducing subspace containing f. Lemma 2.1 shows the relation between reducing subspaces and transparent polynomials.

Lemma 2.1. *If*

$$p(z,w)=\sum_{(n,0)\in I_k} b_n z^n$$

is a transparent polynomial, then

$$X_p=\mathrm{Span}\{\sum_{(n,0)\in I_k} b_n z^{n-lN}w^l; 0\le l\le k\}.$$

Proof. Let

$$X=\mathrm{Span}\{\sum_{(n,0)\in I_k} b_n z^{n-lN}w^l; 0\le l\le k\}.$$

It is obvious that $p\in X\subset X_p$. From the definition of X_p, it is enough to show that X is a reducing subspace of $T_{\overline{z}^N w}$. Let $(M,0)$ be the minimal multi-index of nonzero coefficients of p. For $0\le l<k$, we compute

$$\begin{aligned}
&T_{\overline{z}^N w}\sum_{(n,0)\in I_k} b_n z^{n-lN}w^l\\
&=\sum_{(n,0)\in I_k} b_n\frac{\omega_1(n-lN)}{\omega_1(n-(l+1)N)}z^{n-(l+1)N}w^{l+1}\\
&=\sum_{(n,0)\in I_k} b_n\frac{\omega_1(M-lN)}{\omega_1(M-(l+1)N)}z^{n-(l+1)N}w^{l+1}\\
&=\frac{\omega_1(M-lN)}{\omega_1(M-(l+1)N)}\sum_{(n,0)\in I_k} b_n z^{n-(l+1)N}w^{l+1}\in X.
\end{aligned}$$

If $l=k$, then it is clear that

$$T_{\overline{z}^N w}\sum_{(n,0)\in I_k} b_n z^{n-lN}w^l=0.$$

Moreover, for $l>0$, we obtain

$$\begin{aligned}
T^*_{\overline{z}^N w}\sum_{(n,0)\in I_k} b_n z^{n-lN}w^l&=\sum_{(n,0)\in I_k} b_n\frac{\omega_2(l)}{\omega_2(l-1)}z^{n-(l-1)N}w^{l-1}\\
&=\frac{\omega_2(l)}{\omega_2(l-1)}\sum_{(n,0)\in I_k} b_n z^{n-(l-1)N}w^{l-1}\in X.
\end{aligned}$$

If $l=0$, then it is easy to see that

$$T^*_{\overline{z}^N w}\sum_{(n,0)\in I_k} b_n z^n=0.$$

By these computation, we can show that X is a reducing subspace of $T_{\overline{z}^N w}$. □

Proposition 2.2. *If p is a transparent polynomial, then X_p is the minimal reducing subspace.*

Proof. It is clear from Lemma 2.1. □

For $f \in \mathrm{Hol}(\mathbb{D}^2)$, we denote $f^{(k)}(0,0) = \frac{\partial^k}{\partial^k z} f(0,0)$. For any subspace X with $X \neq \{0\}$, let $(M,0)$ be minimal multi-index such that there exists some $f \in X$ with $f^{(M)}(0,0) \neq 0$ but $g^{(k)}(0,0) = 0$ for all $g \in X$ and $(k,0) < (M,0)$. We call $(M,0)$ the order of X at the origin.

Proposition 2.3. *Let $X \subset X_0^\perp$ be a nonzero reducing subspace of $T_{\overline{z}^N w}$ and $(M,0)$ the order of X at the origin. Then X has a transparent polynomial containing the term z^M.*

Proof. Throughout the proof of Proposition 2.3, we denote $T = T_{\overline{z}^N w}$.

If f is a function in X with Taylor expansion

$$f(z,w) = \sum_{(n_1,n_2)} a(n_1,n_2) z^{n_1} w^{n_2},$$

then the mapping from f to $f^{(M)}(0,0)$ is a bounded linear functional on $H^2(\mathbb{D}^2, d\mu)$. By Riesz representation theorem, the extremal problem

$$\sup\{\mathrm{Re} f^{(M)}(0,0); f \in X, \|f\| \leq 1\}$$

has a unique solution G with $\|G\| = 1$ and $G^{(M)}(0,0) > 0$. At first we prove $T^*G = 0$. Put $g_f = \frac{G+Tf}{\|G+Tf\|}$ for $f \in X$. Since $\mathrm{Re} g_f^{(M)}(0,0) \leq G^{(M)}(0,0)$, it is easy to see that $\|G + Tf\| \geq 1$ for all $f \in X$. From this inequality we obtain $G \perp Tf$. Since $T^*G \in X$, we obtain $T^*G = 0$. Let k' be a natural number such that $(M,0) \in I_{k'}$. The same argument shows that $T^{k'+1}G = 0$. Therefore the function G is in the form of $G(z) = \sum_{(n,0)\in I_{k'}} b_n z^n$.

Let $G(z) = \sum_{i=1}^{\tilde{N}} g_i(z)$ be the canonical decomposition of G with $g_i \neq 0$. It is trivial that g_1 contains the term z^M. Put $(M^{(i)},0)$ the minimal multi-index of g_i. We note that if $i < j$, then $(M^{(i)},0)$ and $(M^{(j)},0)$ are not equivalent, and $(M,0) \leq (M^{(i)},0) < (M^{(j)},0)$.

Now we will show that g_1 is in X. Put $g_j(z) = \sum_{n \geq M^{(j)}} b_n z^n$. Here we see that for $k \leq \frac{M^{(j)}}{N}$,

$$\begin{aligned}
&(T^*)^k T^k g_j \\
&= (T^*)^k \sum_n \frac{\omega_1(n-(k-1)N)}{\omega_1(n-kN)} \frac{\omega_1(n-(k-2)N)}{\omega_1(n-(k-1)N)} \cdots \frac{\omega_1(n)}{\omega_1(n-k)} b_n z^{n-kN} w^k \\
&= (T^*)^k \sum_n \frac{\omega_1(n)}{\omega_1(n-kN)} b_n z^{n-kN} w^k \\
&= (T^*)^k \sum_n \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)}-kN)} b_n z^{n-kN} w^k \\
&= \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)}-kN)} (T^*_{\overline{z}^N w})^k \sum_n b_n z^{n-kN} w^k
\end{aligned}$$

$$= \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - kN)} \sum_n \frac{\omega_2(1)}{\omega_2(0)} \frac{\omega_2(2)}{\omega_2(1)} \cdots \frac{\omega_2(k)}{\omega_2(k-1)} b_n z^n$$
$$= \frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - kN)} \cdot \frac{\omega_2(k)}{\omega_2(0)} g_j,$$

using the definition of g_j and Lemma 1.1. Therefore

$$\left(\frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - kN)} \cdot \frac{\omega_2(k)}{\omega_2(0)} - (T^*)^k T^k \right) g_j = 0. \tag{1}$$

For each natural number $j = 2, 3, \ldots, \tilde{N}$, we choose an integer k_j such that

$$\frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_j N)} \neq \frac{\omega_1(M)}{\omega_1(M - k_j N)}$$

and put

$$C_j = \frac{\omega_2(k_j)}{\omega_2(0)} \left(\frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_j N)} - \frac{\omega_1(M)}{\omega_1(M - k_j N)} \right).$$

We will generate the sequence of functions in X inductively as follows;

$$G_2 = \left(\frac{\omega_1(M^{(2)})}{\omega_1(M^{(2)} - k_2 N)} \cdot \frac{\omega_2(k_2)}{\omega_2(0)} - (T^*)^{k_2} T^{k_2} \right) \frac{1}{C_2} G$$

and

$$G_j = \left(\frac{\omega_1(M^{(j)})}{\omega_1(M^{(j)} - k_j N)} \cdot \frac{\omega_2(k_j)}{\omega_2(0)} - (T^*)^{k_j} T^{k_j} \right) \frac{1}{C_j} G_{j-1}.$$

For example, we have

$$G_2 = g_1 + \frac{1}{C_2} \cdot \frac{\omega_2(k_2)}{\omega_2(0)} \sum_{i=3}^{\tilde{N}} \left(\frac{\omega_1(M^{(2)})}{\omega_1(M^{(2)} - k_2 N)} - \frac{\omega_1(M^{(i)})}{\omega_1(M^{(i)} - k_2 N)} \right) g_i.$$

We note that the function g_2 vanishes but the function g_1 never vanishes from the equality (1) in this calculation.

More generally, let

$$A(j,i) = \prod_{l=2}^{j-1} \frac{\frac{\omega_1(M^{(l)})}{\omega_1(M^{(l)} - k_l N)} - \frac{\omega_1(M^{(i)})}{\omega_1(M^{(i)} - k_l N)}}{\frac{\omega_1(M^{(l)})}{\omega_1(M^{(l)} - k_l N)} - \frac{\omega_1(M)}{\omega_1(M - k_l N)}},$$

and we obtain

$$G_{j-1} = g_1 + \sum_{i=j}^{\tilde{N}} A(j,i) g_i$$

for $3 \leq j \leq \tilde{N}$ and $G_{\tilde{N}} = g_1$ which is in X. It is obvious that g_1 contains the term z^M and is transparent. □

3. Main results

Now we state our main result.

Theorem 3.1. *Let $X \subset X_0^\perp$ be a reducing subspace of $T_{\overline{z}^N w}$ on $H^2(\mathbb{D}^2, d\mu)$. Then the reducing subspace X contains the minimal reducing subspace X_p where p is a transparent polynomial. Moreover X is the minimal reducing subspace of $T_{\overline{z}^N w}$ if and only if there exists a transparent polynomial p such that $X = X_p$.*

Proof. Let X be a reducing subspace of $T_{\overline{z}^N w}$. From Proposition 2.3, there exists a transparent polynomial p. By Lemma 2.1, X_p is the smallest reducing subspace containing p and therefore $X_p \subset X$. In addition, if X is minimal, then it is clear that $X = X_p$. The converse is true from Proposition 2.2. □

In the rest of this paper, we will show some examples. First we consider the case of the weighted Bergman space $A^2_\alpha(\mathbb{D}^2)$, where

$$\omega_1(n) = \omega_2(n) = \frac{n!\Gamma(2+\alpha)}{\Gamma(2+\alpha+n)}$$

for $\alpha > -1$.

Corollary 3.2. *Let $X \subset X_0^\perp$ be a reducing subspace of $T_{\overline{z}^N w}$ on $A^2_\alpha(\mathbb{D}^2)$. The reducing subspace X is minimal if and only if X is in the form of*

$$X_n = \mathrm{Span}\{z^{n-lN}w^l; 0 \le l \le \frac{n}{N}\}$$

for any natural number $n \ge N$.

Proof. It is enough to show that any pair of two distinct multi-indice in I is not equivalent. Assume

$$\frac{\omega_1(m)}{\omega_1(m-k)} = \frac{\omega_1(n)}{\omega_1(n-k)}$$

for $0 < k \le m$ and $0 < k \le n$. This equality implies

$$\frac{m!\Gamma(2+\alpha+m-k)}{(m-k)!\Gamma(2+\alpha+m)} = \frac{n!\Gamma(2+\alpha+n-k)}{(n-k)!\Gamma(2+\alpha+n)}.$$

In particular, when $k = 1$, we obtain the equality

$$\frac{m}{1+\alpha+m} = \frac{n}{1+\alpha+n},$$

which implies $m = n$. Thus we conclude that any pair of two distinct multi-indice in I is not equivalent. Therefore every transparent polynomial is a monomial. □

Next we consider the case of the abstract Hardy space. Let $C(\mathbb{D}^2)$ be the algebra of complex-valued continuous functions on $\overline{\mathbb{D}}^2$ and A a uniform algebra on $\overline{\mathbb{D}}^2$ containing $|z|$. A probability measure m on $\overline{\mathbb{D}}^2$ denotes a representing measure for some complex homomorphism. The abstract Hardy space $H^2 =$

$H^2(m)$ determined by A is defined to be closure of A in $L^2 = L^2(m)$. In this case, $\int fg dm = \int f dm \int g dm$ holds true for $f, g \in H^2$.

Lemma 3.3. *Any pair of two multi-indices in I is equivalent if $H^2(\mathbb{D}^2, d\mu)$ is a closed subspace of the abstract Hardy space $H^2(m)$.*

Proof. Put $r = \int_{\mathbb{D}^2} |z|^2 dm$. Using the equality $\int fg dm = \int f dm \int g dm$, we have

$$\omega_1(n) = \int_{\mathbb{D}^2} |z|^{2n} dm = (\int_{\mathbb{D}^2} |z|^2 dm)^n = r^n.$$

Therefore for all m, n,

$$\frac{\omega_1(m)}{\omega_1(m-kN)} = \frac{r^m}{r^{m-kN}} = r^{kN} = \frac{r^n}{r^{n-kN}} = \frac{\omega_1(n)}{\omega_1(n-kN)}. \qquad \square$$

It is obvious to see that Corollary 3.4 follows from Lemma 3.3.

Corollary 3.4. *Every reducing subspace $X \subset X_0^\perp$ of $T_{\overline{z}^N w}$ in the weighted Hardy space which is a closed subspace of $H^2(m)$ is minimal.*

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