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A RANDOM DISPERSION SCHRÖDINGER EQUATION WITH NONLINEAR TIME-DEPENDENT LOSS/GAIN

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Abstract. In this paper, the limit behavior of solution for the Schrödinger equation with random dispersion and time-dependent nonlinear loss/gain: $idu+\frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2})\partial_{xx}udt+|u|^{2\sigma}udt+i\varepsilon a(t)|u|^{2\sigma_0}udt=0$ is studied. Combining stochastic Strichartz-type estimates with L^2 norm estimates, we first derive the global existence for L^2 and H^1 solution of the stochastic Schrödinger equation with white noise dispersion and time-dependent loss/gain: $idu+\Delta u\circ d\beta+|u|^{2\sigma}udt+ia(t)|u|^{2\sigma_0}udt=0$. Secondly, we prove rigorously the global diffusion-approximation limit of the solution for the former as $\varepsilon\to 0$ in one-dimensional L^2 subcritical and critical cases.

1. Introduction

We are interested in the well-posedness and asymptotic behavior of the solution for a random nonlinear Schrödinger equation (NLSE) including time-varying coefficient,

$$\begin{cases} idu+\frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2})\partial_{xx}udt+|u|^{2\sigma}udt \\ \qquad +i\varepsilon a(t)|u|^{2\sigma_0}udt=0,\ (t,x)\in[0,\infty)\times\mathbb{R}, \\ u(0,x)=u_0, \end{cases} \tag{1.1}$$

where $u_0\in H^1(\mathbb{R})$, m is a continuous real-valued centered stationary random process and models the random dispersion, $0<\sigma\leq 2$ and $0\leq\sigma_0\leq 2$. $a(t)$ is a real function on t and $a(t)>0$ or $a(t)<0$ denotes the strength of dissipation (loss) or gain, $\varepsilon>0$ is a small positive parameter independent of (t,x). The form of Eq. (1.1) describe the propagation of optical pulses in an optical fibre with random dispersion management in random media (see [3,4] for example).

Under some classical ergodic assumptions on m and some restrictions on time-varying coefficient $a(t)$, we will prove that when $\varepsilon\to 0$ the solution of

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Eq. (1.1) is close to the following stochastic NLSE:

$$(1.2)\qquad \begin{cases} idu + \Delta u \circ d\beta + |u|^{2\sigma} u dt = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0, \end{cases}$$

where β is a one-dimensional standard real-valued Brownian motion and $\circ$ is a Stratonovich product, $0 \le \sigma \le 2$.

One of our motivations to study Eq. (1.1) steps from a growing number of researches on deterministic models appearing physics, especially in the theory of Bose-Einstein condensates (BEC) in quantum mechanics and in the investigation of optical soliton in nonlinear optics. In the absence of random dispersion and meanwhile a quadratic potential is included, Eq. (1.1) turns into the case,

$$(1.3)\qquad \begin{cases} idu + (\Delta - V(x))u + \lambda |u|^{2\sigma} u dt \\ \qquad + i\varepsilon a(t)|u|^{2\sigma_0} u dt = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0. \end{cases}$$

In the context of nonlinear optics (where $V(x) = 0$), the equation models the propagation of a laser pulse within an optical fiber under the influence of additional multi-photon absorption processes [8]; in quantum mechanics, when considering the three-body interaction in collapsing BEC with harmonic confinement, a dissipative model involving a quintic nonlinear damping term is used to describe the emittance of particles from the condensate [24].

In recent years, Eq. (1.3) has been widely studied (see [6–8, 16, 20–22, 28, 29] for example). Particularly, the global well-posedness issues for the case $V(x) = \sum_j^d \omega_j x_j^2, \omega_j \ge 0$ and $\varepsilon a(t) \equiv a > 0$ have been considered in [6, 7] using energy method. Precisely, Antonelli and Sparber in [7] obtain global existence of the solution for $\sigma = 1$ and $1 < \sigma_0 \le 2$ in dimension $d \le 3$. The recent paper [6] discusses the case with more general nonlinear indexes σ, σ_0 and shows nonlinear damping prevents finite time blow-up by introducing modified energy functional and linear energy functional, including the cases of L^2 subcritical ($\sigma < \frac{2}{d}$ and $0 < \sigma_0 \le \frac{2}{d}$), L^2 critical ($\sigma = \sigma_0 = \frac{2}{d}$) and L^2 supcritical.

When $V(x) = 0$, Eq. (1.3) reduce to the following form,

$$(1.4)\qquad \begin{cases} idu + \Delta u + \lambda |u|^{2\sigma} u dt + i\varepsilon a(t)|u|^{2\sigma_0} u dt = 0, \\ u(0,x) = u_0, \end{cases}$$

Applying Kato's method and perturbation method, Feng et al. in [20] have discussed exhaustively the existence problem and limit behavior of solution for Eq. (1.4), whose results strongly depend on the ranges of $a(t)$ and indexes σ, σ_0. In [16], Darwich has studied Eq. (1.4) with a L^2 critical nonlinearity where $\sigma = \frac{2}{d}$ and $\varepsilon a(t) \equiv a > 0$ in dimension $d \le 4$. He confirms the global existence of solution if $\sigma_0 \ge 2/d$ and the existence of finite time blow-up dynamics in the log-log regime if $\sigma_0 < 2/d$. In [22, 29], when $\varepsilon a(t) \equiv a > 0$ in Eq. (1.4), numerical approaches are carried out to investigate the effect of small nonlinear damping on the properties of solution.

For $V(x) = 0$ and $\sigma_0 = 0$, Eq. (1.3) reduce to the case,

$$\begin{cases} idu + \Delta u + \lambda|u|^{2\sigma}udt + i\varepsilon a(t)udt = 0, \\ u(0,x) = u_0, \end{cases} \tag{1.5}$$

Equations of form (1.5) have also been well-studied, see [21,28,30] for example. Precisely, Feng et al. [21] have proved the global existence and blowup of solution for Eq. (1.5) under several different conditions on σ and the time-dependent coefficient $a(t)$ in both defocusing and focusing cases. Global existence and blowup results are also derived when $\varepsilon a(t) \equiv a > 0$ in [28,30].

Another motivation comes from the development of the theory of stochastic analysis and its extensive applications to physical problems. For example, the study on optical soliton in optical fiber or BEC in quantum mechanics in random media and the effect of noise on the behavior of solution for NLSE have attracted growing interests (see [1,2,5,9–14,17–19,23,26,27]). The results obtained in [10,12,13,17,18,23,27] have shown that the presence of noise really has great influence on the properties of solution especially in L^2 supercritical case ($\sigma > \frac{2}{d}$).

Our idea is also initiated by the works [2,13,18,19,23,26] on NLSE with random dispersion management in optical fiber. In particular, from a mathematical point of view, the following random NLSE have been considered in [13,18,26],

$$\begin{cases} idu + \frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2})\partial_{xx}udt + f(|u|^2)udt = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0, \end{cases} \tag{1.6}$$

where m is a centered stationary random process and models the dispersion coefficient. [26] is the first paper to study the asymptotic convergence of Eq. (1.6) with a sufficiently smooth function f by a splitting numerical scheme. Later, [13,18] extend the results to the case $f(|u|^2)u = |u|^{2\sigma}u$ ($\sigma = 1,2$). More precisely, the Strichartz-type estimates with stochastic noise are firstly established in de Bouard and Debussche [13] and Debussche and Tsutsumi [18], which are more involved to derive than the classical deterministic ones (see for instance [15,25]), and are of fundamental importance in solving the cauchy problem of NLSE. Then by the conservation of L^2 norm and Strichartz estimates, the global well-posedness of solution for Eq. (1.2) in mass subcritical case ($\sigma < \frac{2}{d}$) are proved and improved to one-dimensional mass critical situation ($\sigma = 2$) in [13,18]. Meanwhile, for any $T > 0$ the convergence of solution for (1.6) is justified in $C([0,T]; H^s(R))$ for $s < 1$ and $\sigma = 1$, and in $C([0,T]; H^1(R))$ for $\sigma = 2$ respectively.

Based on [13,18,26], Fang et al. [19] have recently studied the rigorous convergence of Eq. (1.6) with a periodic time-oscillating nonlinearity. In the forthcoming paper [23], where both time-oscillating nonlinearity and linear loss/gain are included, we make an improvement of the results in [19].

Motivated by the researches aforementioned, the main purpose of this article is to address the limit behavior of Eq. (1.1) as $\varepsilon \to 0$. Since Strichartz estimates are not directly available for Eq. (1.1), to study the asymptotic behavior of this model, we first turn to investigate the well-posedness problem of the stochastic Schrödinger equation with white noise dispersion and nonlinear time-dependent loss/gain,

$$(1.7)\quad \begin{cases} idu + \Delta u \circ d\beta + |u|^{2\sigma} udt + ia(t)|u|^{2\sigma_0} udt = 0, \ (t,x) \in [0,\infty) \times \mathbb{R}^d, \\ u(0,x) = u_0. \end{cases}$$

Due to the presence of nonlinear time-dependent disspation/gain $a(t)$, the L^2 norm of solution for Eq. (1.7) is no longer preserved, but under some conditions on time-varying coefficient $a(t)$, we can establish the a priori estimates of L^2 norm of solution (see Theorem 2.3 and Theorem 2.5 below), which plays the vital role to construct global solution of Eq. (1.7) when $0 < \sigma < \frac{2}{d}$, $0 \le \sigma_0 < \frac{2}{d}$ and $d = 1$, $\sigma = \sigma_0 = 2$.

Therefore, the goal of this work is two-fold. The first one is to prove the existence of the global solution for Eq. (1.7) when $0 < \sigma < \frac{2}{d}$, $0 \le \sigma_0 < \frac{2}{d}$ and $d = 1$, $\sigma = \sigma_0 = 2$. The second is to justify the diffusion-approximation limit of the solution of Eq. (1.1) to the solution of the limit equation (1.2) in one dimension for $0 < \sigma \le 2$ and $0 \le \sigma_0 \le 2$. The methods we exploit are totally different to the discussion of Eq. (1.4) in the energy subcritical and critical cases in [20]. Combining the generalized Strichartz-type estimates with a fixed point argument, we prove the existence of local L^2 and H^1 solution for Eq. (1.7), and establish the a priori estimates of mass dissipation. Then we extend the maximum lifetime of local solution to infinity which strongly rely on Strichartz-type estimates and the L^2 estimates. Lastly, we justify the asymptotic convergence of solution for Eq. (1.1) based on [13, 18, 19, 23, 26].

We remark that the Hamiltonian structures of Eq. (1.1) and Eq. (1.7) are destroyed due to the varying random dispersion. Unlike the deterministic cases in [6, 7, 16, 21], no proper energy functionals can be defined. Thus, no priori energy estimates and no priori estimates in H^1 norm are available for our consideration. Besides, the a priori estimates of L^2 norm of solution are not sufficient to make sure whether the global solution of Eq. (1.7) exists for $\sigma > \frac{2}{d}$ or $\sigma_0 > \frac{2}{d}$. This is the essential difficulty in the situation of L^2 supcritical in our work.

Notation. Throughout this paper, the notations we use are mostly standard. $\mathbb{R}^d$ is d-dimensional Euclidean space, $L^p(\mathbb{R}^d)$, $H^s(\mathbb{R}^d)$, $W^{1,p}(\mathbb{R}^d)$ $(r, p \ge 1, s \in \mathbb{R})$ denote the classical Sobolev spaces and $C_0^\infty(\mathbb{R})$ is a Banach space consisted of smooth enough functions with compact support. For convenience sake, we sometimes denote $L_x^p = L^p(\mathbb{R}^d)$, $H_x^s = H^s(\mathbb{R}^d)$, $W_x^{1,p} = W^{1,p}(\mathbb{R}^d)$ and for an given interval $I \subset \mathbb{R}$, we set $L_t^r L_x^p = L^r(I; L^p(\mathbb{R}^d))$, $L_t^r W_x^{1,p} = L^r(I; W^{1,p}(\mathbb{R}^d))$. When considering random variable with respect to a parameter $\omega \in \Omega$ which has value in Banach space M, we write $L_\omega^p(M) = L^p(\Omega; M)(p \ge 1)$. $L_{\mathcal{P}}^r(\Omega; M)$

denotes the subspace of predictable processes in $L^r(\Omega; M)$). Note that all the functions of these spaces are complex-valued throughout this work. $\|\cdot\|_M$ and $|\cdot|$ are used to denote the norm of a Banach space M and absolute value of a complex number respectively. B^c denotes the complement of the set B. C represents various different constant and $C_1 \wedge C_2 = \min\{C_1, C_2\}$. The number p' is the conjugate exponent of p ($p \geq 1$) which satisfies $\frac{1}{p} + \frac{1}{p'} = 1$.

A pair of real numbers (r, p) is called a Schrödinger-admissible pair if $r = \infty$ and $p = 2$ or if it satisfies the following conditions: $2 \leq r < \infty$, $2 \leq p \leq \infty$ and $\frac{2}{r} > d(\frac{1}{2} - \frac{1}{p})$.

The structure of this paper is as follows. In the next section, some preliminaries and main results are given. Section 3 is devoted to well-posedness of solution for Eq. (1.7) (or Eq. (2.4)) in L^2 and H^1. Section 4 concentrates on existence and convergence of solution for Eq. (1.1) (or Eq. (2.1)).

2. Preliminaries and main results

Consider the random NLSE with time-dependent nonlinear dissipation/gain,

(2.1)
$$\begin{cases} idu + \frac{1}{\varepsilon} m(\frac{t}{\varepsilon^2})\partial_{xx} u dt + |u|^{2\sigma} u dt + i\varepsilon a(t)|u|^{2\sigma_0} u dt = 0, \ (t,x) \in [0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0, \end{cases}$$

where $u_0 \in L^2_x$ or H^1_x, u is an unknown random process defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, depending on $x \in \mathbb{R}$ and $t > 0$. ε is a small positive parameter, $0 < \sigma \leq 2$ and $0 \leq \sigma_0 \leq 2$. The dispersion term m is a continuous real-valued centered stationary random process also related to the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. $a(t) > 0$ or $a(t) < 0$ characteristics the strength of dissipation (loss) or gain.

Under some assumptions on m and $a(t)$, for $\varepsilon \to 0$ Eq. (2.1) is expected to converge to the following limit equation,

$$\begin{cases} idu + \Delta u \circ d\beta + |u|^{2\sigma} u dt = 0, \quad (t,x) \in [0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0, \end{cases} \tag{2.2}$$

where β is a one-dimensional standard Brownian motion defined on the filtrated probability space $(\Omega, \mathcal{F}, \mathbf{P}, (\mathcal{F}_t)_{t\geq 0})$ and $\circ$ stands for a Stratonovich product. We denote by $S(t,s) = e^{i(\beta(t)-\beta(s))\Delta}$ the random dispersion propagator of the linear part of Eq. (2.2) (see [13, 18]).

In L^2 subcritical case ($\sigma < \frac{2}{d}$) and L^2 critical ($d = 1, \sigma = 2$) case, the global existence of solution of Eq. (2.2) have been studied in [13] and [18], which thank to the conservation of mass norm.

Proposition 2.1 ([13]). *Assume $0 < \sigma < \frac{2}{d}$; let $u_0 \in L^2_x$ a.s. be $\mathcal{F}_0$-measurable, then there exists a unique solution u to Eq. (2.2) with paths a.s. in $L^r_{loc}(0,\infty; L^p(\mathbb{R}^d))$ with $p = 2\sigma + 2 \leq r < \frac{4(\sigma+1)}{d\sigma}$; moreover, u has paths in*

$C(\mathbb{R}^+;L_x^2)$, a.s. and

$$\|u(t)\|_{L_x^2} = \|u_0\|_{L_x^2}, \quad a.s.$$

If in addition $u_0 \in H_x^1$, then u has paths a.s. in $C(\mathbb{R}^+;H_x^1)$.

Proposition 2.2 ([18]). *Assume $d = 1$, $\sigma = 2$; let $u_0 \in L_x^2$ a.s. be $\mathcal{F}_0$-measurable, then there exists a unique solution u to Eq. (2.2) with paths a.s. in $L_{loc}^5(0,\infty;L^{10}(\mathbb{R}))$; moreover, u has paths in $C(\mathbb{R}^+;L_x^2)$ a.s. and*

$$\|u(t)\|_{L^2(\mathbb{R})} = \|u_0\|_{L^2(\mathbb{R})}, \quad a.s.$$

If in addition $u_0 \in H^1(\mathbb{R})$, then u has paths a.s. in $C(\mathbb{R}^+;H^1(\mathbb{R}))$.

Our final objective is to study the convergence of Eq. (2.1) to Eq. (2.2), therefore we also consider the global existence and uniqueness of the solution for the stochastic NLSE with white noise dispersion:

$$\begin{cases} idu + \Delta u \circ d\beta + |u|^{2\sigma}udt + ia(t)|u|^{2\sigma_0}udt = 0, \ (t,x) \in (0,\infty)\times\mathbb{R}, \\ u(0,x) = u_0, \end{cases} \tag{2.3}$$

where $\beta(t)$, $a(t)$, σ,σ_0 are as above aforementioned. In Eq. (2.1) and Eq. (2.3) when $\sigma_0 = 0$, the corresponding cases are more easier to deal with. In fact, by scaling transformations

$$u_\varepsilon(t,x) = e^{-\varepsilon\int_0^t a(s)ds}v_\varepsilon(t,x) \text{ and } u(t,x) = e^{-\int_0^t a(s)ds}v(t,x),$$

they can be transformed to the formally equivalent equations as follows:

$$\begin{cases} idv+\frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2})\partial_{xx}vdt+e^{-2\sigma\varepsilon\int_0^t a(s)ds}|v|^{2\sigma}vdt=0, \ (t,x) \in [0,\infty)\times\mathbb{R}, \\ v(0,x)=u_0, \end{cases} \tag{2.4}$$

and

$$\begin{cases} idv + \Delta v \circ d\beta + e^{-2\sigma\int_0^t a(s)ds}|v|^{2\sigma}vdt = 0, \quad (t,x) \in [0,\infty)\times\mathbb{R}, \\ v(0,x) = u_0, \end{cases} \tag{2.5}$$

Thereby, in the subsequent sections we only consider Eq. (2.4) and Eq. (2.5) in the case $\sigma_0 = 0$.

One of our main results, i.e., the well-posedness of the solution for Eq. (2.3), is the following two theorems.

Theorem 2.3. *Assume $0 < \sigma < \frac{2}{d}$ and $0 \le \sigma_0 < \frac{2}{d}$, (r,p) is a Schrödinger-admissible pair with $p = 2\sigma+2 \le r < \frac{4(\sigma+1)}{d\sigma}$; let $u_0 \in L_x^2$ a.s. be $\mathcal{F}_0$-measurable, then*

Case 1: $0 < \sigma < \frac{2}{d}$, $\sigma_0 = 0$ and $a(t) \in L^1(0,\infty)$, there exists a unique solution u to Eq. (2.3) (or Eq. (2.5)) with paths a.s. in $L_{loc}^r(0,\infty;L^p(\mathbb{R}^d))$; moreover, u has paths in $C(\mathbb{R}^+;L_x^2)$, a.s. and

$$\|u(t)\|_{L_x^2}^2 = e^{-2\int_0^t a(s)ds}\|u_0\|_{L_x^2}^2, \quad a.s.$$

Case 2: $0 < \sigma = \sigma_0 < \frac{2}{d}$, $a(t) \in L^\infty(0,\infty)$ *and* $a(t) \geq 0$, *there exists a unique solution* u *of Eq.* (2.3) *with paths a.s. in* $L^r_{loc}(0,\infty; L^p(\mathbb{R}^d))$; *moreover,* u *has paths in* $C(\mathbb{R}^+; L^2_x)$, *a.s. and*

$$\|u(t)\|^2_{L^2_x} = \|u_0\|^2_{L^2_x} - 2\int_0^t a(s)\|u(s)\|^{2+2\sigma}_{L^{2+2\sigma}_x} ds, \quad a.s.$$

In particular,

$$\|u(t)\|^2_{L^2_x} \leq \|u_0\|^2_{L^2_x}, \quad t \geq 0, \quad a.s.$$

If in addition $u_0 \in H^1_x$ *in the two cases above, then* u *has paths a.s. in* $C(\mathbb{R}^+; H^1_x)$.

Remark 2.4. (i) If $a(t) \equiv 0$, the conclusion of Theorem 2.1 in [13] is a special case of our theorem. From this point of view, our result generalizes the conclusion of [13].

(ii) For the case $0 < \sigma \neq \sigma_0 < \frac{2}{d}$ or $\sigma = \frac{2}{d}$ and $\sigma_0 < \frac{2}{d}$ $(d > 1)$, it's worthy of considering the global well-posedness issue of Eq. (2.3).

Theorem 2.5. *Assume* $d = 1$, $0 < \sigma \leq 2$ *and* $0 \leq \sigma_0 \leq 2$, *let* $u_0 \in L^2_x$ *a.s. be* $\mathcal{F}_0$*-measurable, then*
Case 1: $\sigma = 2$, $\sigma_0 = 0$ *and* $a(t) \in L^1(0,\infty)$, *there exists a unique solution* u *to Eq.* (2.3) (*or Eq.* (2.5)) *with paths a.s. in* $L^5_{loc}(0,\infty; L^{10}(\mathbb{R}))$; *moreover,* u *has paths in* $C(\mathbb{R}^+; L^2(\mathbb{R}))$, *a.s. and*

$$\|u(t)\|^2_{L^2(\mathbb{R})} = e^{-2\int_0^t a(s)ds}\|u_0\|^2_{L^2(\mathbb{R})}, \quad a.s.$$

Case 2: $\sigma = \sigma_0 = 2$, $a(t) \in L^\infty(0,\infty)$ *and* $a(t) \geq 0$, *there exists a unique solution* u *of Eq.* (2.3) *with paths a.s. in* $L^5_{loc}(0,\infty; L^{10}(\mathbb{R}))$; *moreover,* u *has paths in* $C(\mathbb{R}^+; L^2(\mathbb{R}))$, *a.s. and*

$$\|u(t)\|^2_{L^2(\mathbb{R})} = \|u_0\|^2_{L^2(\mathbb{R})} - 2\int_0^t a(s)\|u(s)\|^6_{L^6(\mathbb{R})} ds, \quad a.s.$$

In particular,

$$\|u(t)\|^2_{L^2_x} \leq \|u_0\|^2_{L^2_x}, \quad t \geq 0, \quad a.s.$$

If in addition $u_0 \in H^1(\mathbb{R})$ *in the two cases above, then* u *has paths a.s. in* $C(\mathbb{R}^+; H^1(\mathbb{R}))$.

Remark 2.6. The theorem slightly generalizes the conclusion of [16] to white noise dispersion in one dimension.

For Eq. (2.1), we have the following convergence theorem.

Theorem 2.7. *Assume* $m(t)$ *is a continuous real-valued random process defined on a probability space* $(\Omega, \mathcal{F}, \mathbf{P})$ *and for any* $T > 0$, *the process* $t \mapsto \varepsilon\int_0^{\frac{t}{\varepsilon^2}} m(s)ds$ *converges to a standard real-valued Brownian motion in* $C([0,T])$ *in distribution. Let* $u_0 \in H^1(\mathbb{R})$ *a.s. be* $\mathcal{F}_0$*-measurable, then for any* $\varepsilon > 0$, *there exists a unique local solution* u_ε *of Eq.* (2.1) *with paths almost surely in*

$C([0,\tau_\varepsilon(u_0)); H^1(\mathbb{R}))$ *where* $[0,\tau_\varepsilon(u_0))$ *is a random interval, in the following two cases*:

(1) $0<\sigma\le 2$, $\sigma_0=0$ *and* $a(t)\in L^1(0,\infty)$;

(2) $0<\sigma=\sigma_0\le 2$ *and* $a(t)\in L^\infty(0,\infty)$.

Moreover, for any $T>0$

$$\lim_{\varepsilon\to 0}\mathbf{P}\Big(\tau_\varepsilon(u_0)\le T\Big)=0,$$

and the process $u_\varepsilon \mathbf{1}[\tau_\varepsilon > T]$ *converges in distribution to the solution* u *of Eq.* (2.2) *in* $C([0,T]; H^1(\mathbb{R}))$.

Remark 2.8. (i) When $a(t)\equiv 0$, the conclusions of Theorem 2.3 in [13] and Theorem 2.2 in [18] are special ones of our theorem. From this point of view, our result improves the conclusions of [13, 18].

(ii) We are also interested in the convergence issue for the case $0<\sigma\ne\sigma_0\le 2$, but we don't discuss it in this paper.

(iii) The convergence results can be generalized to the space $C([0,T]; H^s(\mathbb{R}))$ where $\frac{1}{2}<s\le 1$.

3. The well-posedness of solution for Eq. (2.3) in L^2 and H^1

This section is devoted to the well-posedness of solution for Eq. (2.3). It's well known that Strichartz type estimates are key tools to solve the classical NLSE in whole space $\mathbb{R}^d$. The stochastic Strichartz type estimates in random case are of the similar importance. Now, we recall the Strichartz type estimates of white noise dispersion, which are established in [13] and [18].

Lemma 3.1 ([13]). *Let* (r,p) *be an admissible pair, let* ρ *be a positive constant such that* $r'\le\rho\le r$; *there exists a constant* $c_{\rho,r,p}>0$ *such that for any* $s\in\mathbb{R}, T\ge 0$ *and* $f\in L^\rho_{\mathcal{P}}(\Omega; L^{r'}(s,s+T; L^{p'}_x))$,

$$\|\int_s^\cdot S(\cdot,\sigma)f(\sigma)d\sigma\|_{L^\rho(\Omega;L^r(s,s+T;L^p_x))}\le c_{\rho,r,p}T^\beta\|f\|_{L^\rho_{\mathcal{P}}(\Omega;L^{r'}(s,s+T;L^{p'}_x))}.$$

and for any $u_s\in L^\rho(\Omega; L^2_x))$, $\mathcal{F}_s$*-measurable,*

$$\|S(\cdot,s)u_s\|_{L^\rho(\Omega;L^r(s,s+T;L^p_x))}\le c_{r,p}T^{\frac{\beta}{2}}\|u_s\|_{L^\rho_\omega(L^2_x)}$$

with $\beta=\frac{2}{r}-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})$.

Lemma 3.2 ([13]). *Let*(r,p) *and* (γ,δ) *be two admissible pairs such that*

$$\frac{1}{\gamma}=\frac{1-\lambda}{r},\quad \frac{1}{\delta}=\frac{\lambda}{2}+\frac{1-\lambda}{p},$$

with $\lambda\in[0,1]$, *and* ρ *be a positive constant such that* $\max\{\rho,\rho'\}\le r$; *then there exists a constant* $c(r,p,\gamma,\delta,\rho)$ *such that for any* $s\in\mathbb{R}$, $T\ge 0$,

(3.1)
$$\|\int_s^\cdot S(\cdot,\sigma)f(\sigma)d\sigma\|_{L^\rho(\Omega;L^r(s,s+T;L^p_x))}\le c(r,p,\gamma,\delta,\rho)T^{\tilde\beta}\|f\|_{L^\rho(\Omega;L^{\gamma'}(s,s+T;L^{\delta'}_x))}$$

if

$$f \in L^{\rho}_{\mathcal{P}}(\Omega; L^{\gamma'}(s, s+T; L^{\delta'}_x))$$

and

(3.2)

$$\| \int_s^{\cdot} S(\cdot,\sigma) f(\sigma) d\sigma \|_{L^{\rho}(\Omega; L^{\gamma}(s,s+T;L^{\delta}_x))} \le c(r,p,\gamma,\delta,\rho) T^{\tilde{\beta}} \|f\|_{L^{\rho}(\Omega; L^{r'}(s,s+T;L^{p'}_x))}$$

if $f \in L^{\rho}_{\mathcal{P}}(\Omega; L^{r'}(s, s+T; L^{p'}_x))$. *In the latter case, we also have*

$$\int_s^{\cdot} S(\cdot,\sigma) f(\sigma) d\sigma \in L^{\rho}(\Omega; C([s, s+T]; L^2_x)).$$

Here, $\tilde{\beta} = (\frac{2}{r} - \frac{d}{2}(\frac{1}{2} - \frac{1}{p}))(1 - \frac{\lambda}{2})$. *Furthermore, for homogeneous propagation* $S(\cdot, s)u_s$, *similar estimates as* (3.1) *and* (3.2) *also hold, with* $\frac{\tilde{\beta}}{2}$ *instead of* $\tilde{\beta}$ *on top of* T *on the right hand side.*

Lemma 3.3 ([18]). *There exists a constant* κ *such that for any* $s \in \mathbb{R}$, $T \ge 0$ *and* $f \in L^4_{\mathcal{P}}(\Omega; L^1(s, s+T; L^2(\mathbb{R})))$, *the mapping* $t \mapsto \int_s^{\cdot} \mathrm{S}(\cdot,\sigma) f(\sigma) d\sigma$ *belongs to* $L^4_{\mathcal{P}}(\Omega; L^5(s, s+T; L^{10}(\mathbb{R})))$, *and*

$$\| \int_s^{\cdot} \mathrm{S}(\cdot,\sigma) f(\sigma) d\sigma \|_{L^4(\Omega; L^5(s,s+T;L^{10}(\mathbb{R})))} \le \kappa T^{\frac{1}{10}} \|f\|_{L^4_{\mathcal{P}}(\Omega; L^1(s,s+T;L^2(\mathbb{R})))}$$

and for any $u_s \in L^4(\Omega; L^2(\mathbb{R}))$, $\mathcal{F}_s$*-measurable,*

$$\|S(\cdot, s) u_s\|_{L^4_{\mathcal{P}}(\Omega; L^5(s,s+T;L^{10}(\mathbb{R})))} \le c T^{\frac{1}{10}} \|u_s\|_{L^4(\Omega; L^2(\mathbb{R}))}.$$

Remark 3.4. (i) By taking the first order space derivative, the above inequalities in Lemmas 3.1-3.3 also hold when L^p_x is replaced by $W^{1,p}_x$.

(ii) When a deterministic nonzero dispersion $\mu \Delta u$ is included in the linear part of Eq. (2.2), we can derive the analogous Strichartz estimates, where the random dispersion propagator is replaced by $T(t,s) = e^{i(\beta(t) - \beta(s) + \mu(t-s))\Delta}$.

Given a smooth function $\theta \in C_0^{\infty}(\mathbb{R})$ such that

$$\theta(x) = \begin{cases} 1, & |x| \le 1, \\ 0, & |x| \ge 2. \end{cases}$$

Now, we discuss the existence problem of solution for Eq. (2.3) in the following two cases.

Case 1: $0 < \sigma < \frac{2}{d}$, $\sigma_0 = 0$ and $a(t) \in L^1(0, \infty)$.

Consider the truncation of the Itô form corresponding to Eq. (2.5),

$$\begin{cases} i dv^M + \frac{i}{2}\Delta^2 v^M dt + \Delta v^M d\beta + e^{-2\sigma \int_0^t a(s) ds} \theta_M(v^M) |v^M|^{2\sigma} v^M dt = 0, \\ v^M(0) = u_0, \end{cases} \tag{3.3}$$

and its mild form,

$$v^M(t) = S(t,0)u_0 + i \int_0^t S(t,s) e^{-2\sigma \int_0^t a(s) ds} \theta_M(v^M) |v^M|^{2\sigma} v^M(s) ds, \tag{3.4}$$

where $\theta_M(v)(t) = \theta\left(\frac{\|v\|_{L^r(0,t;L_x^p)}}{M}\right)$.

Theorem 3.5. *Assume* $0 < \sigma < \frac{2}{d}$ *and* $\sigma_0 = 0$, (r,p) *is a Schrödinger-admissible pair with* $p = 2\sigma + 2 \leq r < \frac{4(\sigma+1)}{d\sigma}$. *For any* $\mathcal{F}_0$*-measurable* $u_0 \in L^r_\omega(L^2_x)$, *there exists a unique solution* v^M *of Eq.* (3.4) *in* $L^r_{\mathcal{P}}(\Omega; L^r(0,T;L^p_x)$ *for any* $T > 0$. *Moreover* v^M *is a weak solution of Eq.* (3.3) *in the sense that for any* $\varphi \in C_0^\infty(\mathbb{R})$ *and* $t \geq 0$,

$$\begin{aligned}
& i(v^M - u_0, \varphi)_{L^2_x} \\
= & -\frac{i}{2}\int_0^t (v^M, \Delta^2\varphi)_{L^2_x} ds - \int_0^t e^{-2\sigma\int_0^s a(\rho)d\rho}(\theta_M(v^M)|v^M|^{2\sigma}v^M, \varphi)_{L^2_x} ds \\
& - \int_0^t (v^M, \Delta\varphi)_{L^2_x} d\beta(s)
\end{aligned}$$

and the L^2_x *norm is preserved,*

$$\|v^M\|^2_{L^2_x} = \|u_0\|^2_{L^2_x}, \quad t \geq 0, \ a.s.$$

In addition, $v \in C([0,T]; L^2_x)$ *a.s.*

Proof. The proof of this case is similar to [13], for completeness, we list the main procedures. For convenience sake, we omit the M dependence in the proof. Define

$$J_M v(t) = S(t,0)u_0 + i\int_0^t S(t,s)e^{-2\sigma\int_0^t a(\tau)d\tau}\theta(v)(|v|^{2\sigma}v)(s)ds. \tag{3.5}$$

We can prove J_M is a strict contraction on the complete metric space,

$$X_T = L^r(\Omega; L^r(0,T;L^p(\mathbb{R}^d)))$$

equipped with the metric

$$d(v_1, v_2) = \|v_1 - v_2\|_{L^r_\omega L^r_T L^p_x},$$

where $p = 2\sigma + 2$ and $2\sigma + 2 \leq r < \frac{4(\sigma+1)}{d\sigma}$, and provided $T \leq T_0$, T_0 only depends on M. In fact, by Lemma 3.1, for $v \in X_T$, we obtain

$$\begin{aligned}
\|J_M v\|_{X_T} &\leq CT^{\frac{\beta}{2}}\|u_0\|_{L^r_\omega L^2_x} + CT^\beta \|e^{-2\sigma\int_0^s a(\tau)d\tau}\theta(v)|v|^{2\sigma}v\|_{L^r_\omega L^{r'}_T L^{p'}_x} \\
&\leq C\|u_0\|_{L^r_\omega L^2_x} + C\|e^{-2\sigma\int_0^s a(\tau)d\tau}\|_{L^\infty}T^\beta T^{1-\frac{2\sigma+2}{r}}M^{2\sigma}\|v\|_{L^r_\omega L^r_T L^p_x} \\
&< \infty.
\end{aligned}$$

Besides, for $v_1, v_2 \in X_T$, define

$$\tau_i^M = \inf\{t \in [0,T], \|v_i(t)\|_{L^r(0,t;L^p_x)} \geq 2M\}, \quad i = 1, 2,$$

and assume $\tau_1^M \leq \tau_2^M$, then we get (see [13]),

$$\begin{aligned}
& d(J_M v_1, J_M v_2) \\
= & \|J_M v_1 - J_M v_2\|_{L^r_\omega L^r_T L^p_x}
\end{aligned}$$

$$\le C\|e^{-2\sigma\int_0^s a(\tau)d\tau}\|_{L^\infty}T^\beta\Big\|\theta(v_1)|v_1|^{2\sigma}v_1-\theta(v_2)|v_2|^{2\sigma}v_2\Big\|_{L^r_\omega L^{r'}_T L^{p'}_x}$$
$$\le C\|e^{-2\sigma\int_0^s a(\tau)d\tau}\|_{L^\infty}T^\beta T^{1-\frac{2\sigma+2}{r}}M^{2\sigma}\|v_1-v_2\|_{L^r_\Omega L^{r'}_T L^p}$$

with $\beta=\frac{2}{r}-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})$. Therefore, there exists $T_0>0$ depending only on M such that for $T\le T_0$,

$$C\|e^{-2\sigma\int_0^s a(\tau)d\tau}\|_{L^\infty}T^\beta T^{1-\frac{2\sigma+2}{r}}M^{2\sigma}<1.$$

That is,

$$d(J_Mv_1,J_Mv_2)<d(v_1,v_2).$$

It means that $J_M:X_T\to X_T$ is strictly contractive for T small enough. Then by some reiterative procedures, we can obtain a unique local solution $v^M(t)$ on X_T for Eq. (3.4) (or Eq. (3.5)) for $T>0$.

It's classical to show $v^M(t)$ is a weak solution of (3.4). In fact, for any $\varphi\in C_0^\infty(\mathbb{R})$, multiplying Eq. (3.3) by φ and integrating by parts in $\mathbb{R}^d$, then

$$\begin{aligned}&i(v^M-u_0,\varphi)_{L^2_x}\\ =&-\frac{i}{2}\int_0^t(v^M,\Delta^2\varphi)_{L^2_x}ds-\int_0^t e^{-2\sigma\int_0^s a(\rho)d\rho}(\theta_M(v^M)|v^M|^{2\sigma}v^M,\varphi)_{L^2_x}ds\\ &-\int_0^t(v^M,\Delta\varphi)_{L^2_x}d\beta(s).\end{aligned}$$

Next, we prove the conservation of L^2_x norm. Let $R\ge0$, we define a regularization of solution v: $v_R=P_Rv$ by a truncation in Fourier space: $\hat v_R(t,\xi)=\theta\left(\frac{|\xi|}{M}\right)\hat v(t,\xi)$. Then by Eq. (3.3), we get

$$idv_R+\frac{i}{2}\Delta^2v_Rdt+\Delta v_Rd\beta+e^{-2\sigma\int_0^t a(s)ds}P_R(\theta(v)|v|^{2\sigma}v)dt=0.$$

By Itô formula, we obtain

$$\|v_R\|^2_{L^2_x}=\|u_0\|^2_{L^2_x}+Re\bigg(i\int_0^t e^{-2\sigma\int_0^s a(\tau)d\tau}(\theta(v)|v|^{2\sigma}v,P_Rv)ds\bigg).$$

Let $R\to\infty$, then we have in $L^{2\sigma+2}(0,T;L^{2\sigma+2}_x)$ that

$$\lim_{R\to\infty}P_Rv_R=v.$$

Therefore, $v\in L^{2\sigma+2}(0,T;L^{2\sigma+2}_x)$. Let $R\to\infty$, then

$$\lim_{R\to\infty}\|v_R\|^2_{L^2_x}=\|u_0\|^2_{L^2_x},\ t\in[0,T],\quad\text{a.s.}$$

That is, for any $t\in[0,T]$, $v\in L^2_x$ and $\|v\|^2_{L^2_x}=\|u_0\|^2_{L^2_x}$. By Eq. (3.3) and the continuity of $\|v(t)\|_{L^2_x}$ on t, we know $v\in C([0,T];L^2_x)$ and $\|v\|^2_{L^2_x}=\|u_0\|^2_{L^2_x}$ a.s. Then the conclusion of Theorem 3.5 follows. □

Case 2: $0 < \sigma = \sigma_0 < \frac{2}{d}$ and $a(t) \in L^\infty(0, \infty)$.

Consider the truncation of the Itô equation corresponding to Eq. (2.3),

$$
\begin{cases} idu^M + \frac{i}{2}\Delta^2 u^M dt + \Delta u^M d\beta + \theta_M(u^M)|u^M|^{2\sigma}u^M dt \\ \qquad + \ ia(t)\theta_M(u^M)|u^M|^{2\sigma}u^M dt = 0, \\ u^M(0) = u_0, \end{cases} \tag{3.6}
$$

and its mild form,

$$
\begin{aligned} u^M(t) = S(t,0)u_0 + i\int_0^t S(t,s)\theta_M(u^M)(|u^M|^{2\sigma}u^M)(s)ds \\ - \int_0^t S(t,s)a(s)\theta_M(u^M)(|u^M|^{2\sigma}u^M)(s)ds, \end{aligned} \tag{3.7}
$$

where $\theta_M(v)(t) = \theta\left(\frac{\|v\|_{L^r(0,t;L^p_x)}}{M}\right)$.

Then, we have the following conclusion.

Theorem 3.6. *Let $0 < \sigma = \sigma_0 < \frac{2}{d}$ and $a(t) \in L^\infty(0,\infty)$, (r,p) is a Schrödinger-admissible pair such that $p = 2\sigma+2 \le r < \frac{4(\sigma+1)}{d\sigma}$. For any $\mathcal{F}_0$-measurable $u_0 \in L^r_\omega(L^2_x)$, there exists a unique solution $u^M \in L^r_{\mathcal{P}}(\Omega; L^r(0,T;L^p_x)$ of Eq. (3.7) (or Eq. (3.6)) for any $T > 0$. Furthermore, for any $\varphi \in C_0^\infty(\mathbb{R})$ and any $t \ge 0$, u^M is a weak solution of Eq. (3.6) in the following sense,*

$$
\begin{aligned} & i(u^M - u_0, \varphi)_{L^2_x} \\ = & -\frac{i}{2}\int_0^t (u^M, \Delta^2\varphi)_{L^2_x} ds - \int_0^t (\theta_M(u^M)|u^M|^{2\sigma}u^M, \varphi)_{L^2_x} ds \\ & - i\int_0^t a(s)(\theta_M(u^M)|u^M|^{2\sigma}u^M, \varphi)_{L^2_x} ds - \int_0^t (u^M, \Delta\varphi)_{L^2_x} d\beta(s). \end{aligned}
$$

Then the L^2_x norm has the following estimate,

$$
\|u^M(t)\|^2_{L^2_x} = \|u_0\|^2_{L^2_x} - 2\int_0^t a(s)\|u^M(s)\|^{2+2\sigma}_{L^{2+2\sigma}_x} ds, \quad t \ge 0, \quad a.s.
$$

In addition, if $a(t) \ge 0$, then

$$
\|u^M(t)\|^2_{L^2_x} \le \|u_0\|^2_{L^2_x}, \quad t \ge 0, \quad a.s.
$$

and $u \in C([0,T]; L^2_x)$ a.s.

Proof. For convenience consideration, we also omit the dependence on M in the proof.

Consider also the complete metric space

$$
X_T = L^r(\Omega; L^r(0,T; L^p(\mathbb{R}^d)))
$$

equipped with the metric

$$
d(v_1, v_2) = \|v_1 - v_2\|_{L^r_\omega L^r_T L^p_x},
$$

where $p = 2\sigma + 2$ and $2\sigma + 2 \le r < \frac{4(\sigma+1)}{d\sigma}$.

Define

$$J_M u(t) = S(t,0)u_0 + i\int_0^t S(t,s)\theta(u)|u|^{2\sigma}u(s)ds$$

$$- \int_0^t S(t,s)a(s)\theta(u)|u|^{2\sigma}u(s)ds, \tag{3.8}$$

then J_M defines a strict contraction on X_T if $T \le T_0$ and T_0 only depends on M. In fact, for any $v \in X_T$ and $v_1, v_2 \in X_T$, by Lemma 3.1 the estimates below hold,

$$\begin{aligned}\|J_M v\|_{L^r_\omega L^r_T L^p_x} &\le CT^{\frac{\beta}{2}}\|u_0\|_{L^r_\omega L^2_x} + CT^\beta \|\theta(v)|v|^{2\sigma}v\|_{L^r_\omega L^{r'}_T L^{p'}_x} \\ &\quad + CT^\beta \|a(s)\theta(v)|v|^{2\sigma}v\|_{L^r_\omega L^{r'}_T L^{p'}_x} \\ &\le C\|u_0\|_{L^r_\Omega L^2_x} + C(\|a(s)\|_{L^\infty}+1)T^\beta T^{1-\frac{2\sigma+2}{r}}M^{2\sigma}\|v\|_{L^r_\omega L^r_T L^p_x} \\ &< \infty\end{aligned}$$

and

$$\begin{aligned}d(J_M v_1, J_M v_2) &= \|J_M v_1 - J_M v_2\|_{L^r_\omega L^r_T L^p_x} \\ &\le CT^\beta \left\|\left(\theta(v_1)|v_1|^{2\sigma}v_1 - \theta(v_2)|v_2|^{2\sigma}v_2\right)\right\|_{L^r_\omega L^{r'}_T L^{p'}_x} \\ &\quad + \quad CT^\beta \left\|a(s)\left(\theta(v_1)|v_1|^{2\sigma}v_1 - \theta(v_2)|v_2|^{2\sigma}v_2\right)\right\|_{L^r_\omega L^{r'}_T L^{p'}_x} \\ &\le C(\|a(s)\|_{L^\infty}+1)T^\beta T^{1-\frac{2\sigma+2}{r}}M^{2\sigma}\|v_1 - v_2\|_{L^r_\omega L^r_T L^p_x},\end{aligned}$$

where $\beta = \frac{2}{r} - \frac{d}{2}(\frac{1}{2} - \frac{1}{p})$. Therefore, there exists $T_0 > 0$ depending only on M such that for $T \le T_0$,

$$C(\|a(s)\|_{L^\infty}+1)T^\beta T^{1-\frac{2\sigma+2}{r}}M^{2\sigma} < 1.$$

That's to say, $J_M : X_T \mapsto X_T$ is strictly contractive for T small enough. Similarly, procedure, we can obtain the unique solution $u^M(t)$ of Eq. (3.7) (or Eq. (3.8)) on X_T for $T > 0$ and it is a L^2-weak solution of (3.5).

Next, we prove the estimate of L^2 norm for the solution. We define the regularization of solution u: $u_R = P_R u$ by a Fourier truncation: $\hat{u}_R(t,\xi) = \theta\left(\frac{|\xi|}{M}\right)\hat{u}(t,\xi)$. By Eq. (3.6), we obtain

$$idu_R + \frac{i}{2}\Delta^2 u_R dt + \Delta u_R d\beta + P_R(\theta(u)|u|^{2\sigma}u)dt + ia(t)P_R(\theta(u)|u|^{2\sigma}u)dt = 0,$$

Applying Itô formula to $\|u_R\|^2_{L^2_x}$, we obtain

$$\|u_R\|^2_{L^2_x} = \|u_0\|^2_{L^2_x} + Re\left(i\int_0^t (\theta(u)|u|^{2\sigma}u, P_R u)ds\right) - \int_0^t a(s)(\theta(u)|u|^{2\sigma}u, P_R u)ds.$$

Let $R \to \infty$, similarly we have in $L^{2\sigma+2}(0,T;L_x^{2\sigma+2})$

$$\lim_{R\to\infty} P_R u_R = u$$

and

$$\lim_{R\to\infty} \|u_R\|_{L_x^2}^2 = \|u_0\|_{L_x^2}^2 - \int_0^t a(s)\|u\|_{L_x^{2\sigma+2}}^{2\sigma+2} ds, \quad t\in[0,T], \ \text{a.s.}$$

If $a(t) \geq 0$, then $\|u\|_{L_x^2}^2 \leq \|u_0\|_{L_x^2}^2$. In particular, $u \in L^\infty(0,T;L_x^2)$. By Eq. (3.6), we know $u \in H_x^{-4}$ and its paths are continuous almost surely. Then u is weak continuous on t in L_x^2. Since the L_x^2 norm of $u(t)$ is continuous on t, we know $u \in C(0,T;L_x^2)$ and $\|u\|_{L_x^2}^2 \leq \|u_0\|_{L_x^2}^2$ a.s. Hence, the proof of Theorem 3.5 is completed. □

In the following, we only give the proof of Theorem 2.3. For the proof of Theorem 2.4 in the L^2 critical case ($d=1, \sigma=2$), we can utilize the method of [18] combined with Lemma 3.3 to justify it. Here we omit the details.

Proof of Theorem 2.3. We use the solutions of the truncated problems discussed above to construct the global solutions of Eq. (2.3) and Eq. (2.5).

Fix $T_0 > 0$, we construct the solution of Eq. (2.3) and Eq. (2.5) on $[0,T_0]$. Let (r,p) be a Schrödinger-admissible pair such that $2\sigma+2 = p \leq r < \frac{4(\sigma+1)}{d\sigma}$.
Case 1: $0 < \sigma \leq 2$, $\sigma_0 = 0$ and $a(t) \in L^1(0,\infty)$. Define

$$\tau^M = \inf\{t\in[0,T], \|v^M(t)\|_{L^r(0,t;L_x^p)} \geq M\},$$

then we know τ^M is nondecreasing respect to M, and $v^{M_1}(t) = v^{M_2}(t)$ on $[0, \tau^{M_1}\wedge\tau^{M_2}]$, thus the solution of Eq. (2.5) is unique.

Lemma 3.7. *There exist constants c_1, c_2 such that if*

$$T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}\|e^{-2\sigma\int_0^t a(\tau)d\tau}\|_{L^\infty(0,\infty)} \leq c_1 M^{-2r\sigma},$$

then

$$\mathbf{P}(\tau^M \leq T) \leq \frac{c_2\mathbf{E}\|u_0\|_{L_x^2}^r}{M^r}.$$

Proof. The proof is based on Lemma 5.1 in [13]. Set

$$\begin{aligned} v^M\mathbf{1}_{[0,\tau^M]}(t) =& S(t,0)u_0\mathbf{1}_{[0,\tau^M]}(t) \\ &+ i\int_0^t S(t,s)e^{-2\sigma\int_0^s a(\tau)d\tau}(|v^M|^{2\sigma}v^M)\mathbf{1}_{[0,\tau^M]}(s)ds\mathbf{1}_{[0,\tau^M]}(t), \end{aligned} \tag{3.9}$$

then for any $T \leq T_0$, by Lemma 3.1 and Hölder's inequality

$$\begin{aligned} &\mathbf{E}(\|v^M\mathbf{1}_{[0,\tau^M]}(t)\|_{L^r(0,T;L_x^p)}^r) \\ \leq& c(r,T_0)\mathbf{E}\|u_0\|_{L_x^2}^r \\ &+ cT^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}M^{2r\sigma}\|e^{-2\sigma\int_0^t a(\tau)d\tau}\|_{L^\infty(0,\infty)}^r\mathbf{E}(\|v^M\mathbf{1}_{[0,\tau^M]}(t)\|_{L^r(0,T;L_x^p)}^r), \end{aligned}$$

where thanks to $a(t) \in L^1(0,\infty)$. Thereby, if

$$cT^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}M^{2r\sigma}\|e^{-2\sigma\int_0^t a(\tau)d\tau}\|_{L^\infty(0,\infty)} \le \frac{1}{2},$$

then

$$\mathbf{E}(\|v^M \mathbf{1}_{[0,\tau^M]}(t)\|^r_{L^r(0,T;L^p_x)}) \le 2c(r,T_0)\mathbf{E}\|u_0\|^r_{L^2_x}.$$

By Markov inequality, then

$$\mathbf{P}(\tau^M \le T) \le \frac{c_2\mathbf{E}\|u_0\|^r_{L^2_x}}{M^r}.$$

□

Now, we proceed the proof of Case 1. By Theorem 3.4, $\|v^M\|^2_{L^2_x} = \|u_0\|^2_{L^2_x}$, then we can apply iteration scheme invented by [13] to construct the solution of Eq. (2.5) on $[0,T_0]$, thus we can derive the global L^2-solution of Eq. (2.5), here we omit the detailed procedures.

By the transformation $u(t,x) = e^{-\int_0^t a(s)ds}v(t,x)$ and $a(t) \in L^1(0,\infty)$, we know for any $0 < T \le \infty$, $u(t,x)$ make sense. Then $u(t,x) = e^{-\int_0^t a(s)ds}v(t,x) \in C(\mathbb{R}^+, L^2_x)$, and

$$\|u(t)\|^2_{L^2_x} = e^{-2\int_0^t a(s)ds}\|v(t)\|^2_{L^2_x} = e^{-2\int_0^t a(s)ds}\|u_0\|^2_{L^2_x}.$$

Finally, for $u_0 \in H^1_x$, by Strichartz-type estimates in Lemma 3.1 and Lemma 3.2, we can use the method of [13] to obtain the regularity conclusion $u(t,x) \in C(\mathbb{R}^+, H^1_x)$. Thus, we finish the proof of the first part of Theorem 2.3.

Case 2: $0 < \sigma = \sigma_0 < \frac{2}{d}$ and $a(t) \in L^\infty(0,\infty)$. Define

$$\bar{\tau}^M = \inf\{t \in [0,T], \|u^M(t)\|_{L^r(0,t;L^p_x)} \ge M\},$$

then we can derive the analogous unique argument of the solution for Eq. (2.3) as Case 1.

Lemma 3.8. *There exist constants $\bar{c}_1, \bar{c}_2$ such that if*

$$T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}(\|a(t)\|_{L^\infty(0,\infty)}+1) \le \bar{c}_1 M^{-2r\sigma},$$

then we have

$$\mathbf{P}(\bar{\tau}^M \le T) \le \frac{\bar{c}_2\mathbf{E}\|u_0\|^r_{L^2_x}}{M^r}.$$

Proof. Denote

$$\begin{aligned} u^M\mathbf{1}_{[0,\bar{\tau}^M]}(t) &= S(t,0)u_0\mathbf{1}_{[0,\bar{\tau}^M]}(t) \\ &\quad + i\int_0^t S(t,s)(|u^M|^{2\sigma}u^M)\mathbf{1}_{[0,\bar{\tau}^M]}(s)ds\mathbf{1}_{[0,\bar{\tau}^M]}(t) \\ &\quad - \int_0^t S(t,s)a(s)(|u^M|^{2\sigma}u^M)\mathbf{1}_{[0,\bar{\tau}^M]}(s)ds\mathbf{1}_{[0,\bar{\tau}^M]}(t). \end{aligned} \tag{3.10}$$

For any $T \le T_0$, by Strichartz estimates in Lemma 3.1 and Hölder' inequality, it follows that

$$\mathbf{E}(\|u^M\mathbf{1}_{[0,\bar{\tau}^M]}(t)\|^r_{L^r(0,T;L^p_x)})$$

$$\leq c(r,T_0)\mathbf{E}\|u_0\|^r_{L^2_x}$$
$$+ cT^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}M^{2r\sigma}(\|a(t)\|_{L^\infty}(0,\infty)+1)\mathbf{E}(\|u^M\mathbf{1}_{[0,\bar{\tau}^M]}(t)\|^r_{L^r(0,T;L^p_x)}).$$

Therefore, if

$$cT^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}M^{2r\sigma}(\|a(t)\|_{L^\infty}(0,\infty)+1)\leq \frac{1}{2},$$

then

$$\mathbf{E}(\|u^M\mathbf{1}_{[0,\bar{\tau}^M]}(t)\|^r_{L^r(0,T;L^p_x)})\leq 2c(r,T_0)\mathbf{E}\|u_0\|^r_{L^2_x}$$

by Markov inequality, we get

$$\mathbf{P}(\bar{\tau}^M\leq T)\leq \frac{c_2\mathbf{E}\|u_0\|^r_{L^2_x}}{M^r}.$$

□

We use iterative method of the time interval to construct the solution of Eq. (2.3) on $[0,T_0]$. Fix $M>0$, then $u=u^M$ defines a unique solution of Eq. (2.3) on $[0,\bar{\tau}^M]$. Set $\bar{\tau}^M_1=\bar{\tau}^M$. Consider the following equation:

$$\begin{aligned}
&u(t+\bar{\tau}^M)\\
&= S(t+\bar{\tau}^M,\bar{\tau}^M)u(\bar{\tau}^M)\\
&\quad+\ i\int_0^t S(t+\bar{\tau}^M,s+\bar{\tau}^M)\theta_M^{\bar{\tau}^M}(u)(|u(s+\bar{\tau}^M)|^{2\sigma}u(s+\bar{\tau}^M))ds\\
&\quad-\int_0^t S(t+\bar{\tau}^M,s+\bar{\tau}^M)a(s+\bar{\tau}^M)\theta_M^{\bar{\tau}^M}(u)|u(s+\bar{\tau}^M)|^{2\sigma}u(s+\bar{\tau}^M)ds.
\end{aligned}$$

Repeating the procedures of Theorem 3.6, then we can derive a solution u^M_2 of Eq. (2.3). In fact, assume

$$\bar{\tau}^M_2=\inf\{t\in[0,T],\|u^M_2\|_{L^r(\bar{\tau}^M,t+\bar{\tau}^M;L^p_x)}\geq M\}.$$

Then we obtain a solution of the non-truncated equation on $[\bar{\tau}^M,\bar{\tau}^M+\bar{\tau}^M_2]$. That is, it defines a solution u of Eq. (2.3) on $[0,\bar{\tau}^M+\bar{\tau}^M_2]$, and we have $u=u^M$ on $[0,\bar{\tau}^M]$ and $u=u^M_2$ on $[\bar{\tau}^M,\bar{\tau}^M+\bar{\tau}^M_2]$. By Lemma 3.8 and the a priori estimate of L^2_x norm in Theorem 3.6, if $T^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}(\|a(t)\|_{L^\infty(0,\infty)}+1)\leq \bar{c}_1M^{-2r\sigma}$, then

$$\mathbf{P}(\bar{\tau}^M_2\leq T|\mathcal{F}_{\bar{\tau}^M})\leq \frac{\bar{c}_2\mathbf{E}\|u_{\bar{\tau}^M}\|^r_{L^2_x}}{M^r}\leq \frac{\bar{c}_2\mathbf{E}\|u_0\|^r_{L^2_x}}{M^r}.$$

Continue the iterative schemes above repeatly, then we can construct the solution of non-truncated equation (2.3) on $[0,T^M_n]$, where $T^M_n=\bar{\tau}^M+\bar{\tau}^M_2+\cdots+\bar{\tau}^M_n$. That is, $u=u^M$ on $[0,\bar{\tau}^M]$, $u=u^M_2$ on $[\bar{\tau}^M,\bar{\tau}^M+\bar{\tau}^M_2]$, …, $u=u^M_n$ on $[T^M_{n-1},T^M_n]$. By Lemma 3.8 we get

$$\mathbf{P}(\bar{\tau}^M_n\leq T|\mathcal{F}_{T^M_{n-1}})\leq \frac{c_2\mathbf{E}\|u_0\|^r_{L^2_x}}{M^r}.$$

Notice that

$$\mathbf{P}(\lim_{n\to\infty}\bar{\tau}_n^M=0)=\lim_{\varepsilon\to 0}\lim_{N\to\infty}\mathbf{P}(\bar{\tau}_n^M\le\varepsilon,\forall n\ge N).$$

Choose M large enough and $\varepsilon^{-\frac{dr\sigma}{4(\sigma+1)}+r-2\sigma}(\|a(t)\|_{L^\infty(0,\infty)}+1)\le c_1M^{-2r\sigma}$, it follows that

$$\mathbf{P}(\bar{\tau}_n^M\le\varepsilon|\mathcal{F}_{T_{n-1}^M})\le\frac{1}{2}.$$

Thus, according to the method of [13] we conclude

$$\mathbf{P}(\bar{\tau}_n^M\le\varepsilon,\forall n\ge N)\le\lim_{M\to\infty}\frac{1}{2^{M-N}}=0.$$

Thereby

$$\mathbf{P}(\lim_{n\to\infty}\bar{\tau}_n^M=0)=0,$$

which means $\lim_{n\to\infty}T_n^M=\lim_{n\to\infty}(\bar{\tau}^M+\bar{\tau}_2^M+\cdots+\bar{\tau}_n^M)=\infty$ almost surely. Then we construct the global L^2-solution of Eq. (2.3) in the second case.

For $u_0\in H_x^1$, by (3.8) and Lemma 3.1, taking the first order space derivative, then

$$\begin{aligned}\|J_M\nabla u\|_{L_\omega^rL_T^rL_x^p}\le{}&C_1T^{\frac{\beta}{2}}\|\nabla u_0\|_{L_\omega^rL_x^2}\\&+C_2T^{1-\frac{2\sigma}{r}-\frac{d\sigma}{4(\sigma+1)}}(\|a(t)\|_{L^\infty(0,\infty)}+1)\|\nabla u\|_{L_\omega^rL_T^rL_x^p}\end{aligned}$$

therefore

$$\begin{aligned}\|J_Mu\|_{L_\omega^rL_T^rW_x^{1,p}}\le{}&C_1T^{\frac{\beta}{2}}\|u_0\|_{L_\omega^rH_x^1}\\&+C_2T^{1-\frac{2\sigma}{r}-\frac{d\sigma}{4(\sigma+1)}}(\|a(t)\|_{L^\infty(0,\infty)}+1)\|u\|_{L_\omega^rL_T^rW_x^{1,p}}\end{aligned}$$

by Lemma 3.2, the solution $u(t,x)\in C(\mathbb{R}^+,H_x^1)$. Consequently, the conclusion of Theorem 2.3 in the second case follows. □

4. Existence and convergence of solution for Eq. (2.1)

This section is devoted to the existence and global convergence of solution for Eq. (2.1), i.e., the proof Theorem 2.7.

Proof of Theorem 2.7. The proof is based on [13, 18, 19, 26]. We first study the asymptotic convergence for truncated equation of Eq. (2.1).
Case 1: $0<\sigma\le 2$, $\sigma_0=0$ and $a(t)\in L^1(0,\infty)$.

The conclusion of this case is not difficult to prove, where thanks to the fact: for any $f\in L^r(\Omega;L^1(0,T;H^1(\mathbb{R})))$, $0<T<\infty$, $a(t)\in L^1(0,\infty)$ and for any $\delta>0$

$$\lim_{\varepsilon\to 0}\mathbf{P}(\|\int_0^t(e^{-2\sigma\varepsilon\int_0^s a(\tau)d\tau}-1)S_\beta(t,s)f(s)ds\|_{H^1(\mathbb{R})}>\delta)=0,\quad\forall t\in[0,T].$$

We ignore the details and concentrate on the proof of the following situation.
Case 2: $0<\sigma=\sigma_0\le 2$ and $a(t)\in L^\infty(0,\infty)$.

Consider the truncated equations of mild forms corresponding to Eq. (2.1) and Eq. (2.2):

$$
\begin{aligned}
u_\varepsilon^M(t) = S_\varepsilon(t,0)u_0 + i\int_0^t S_\varepsilon(t,s)F_M(|u_\varepsilon^M|^2)u_\varepsilon^M(s)ds \\
- \varepsilon\int_0^t S_\varepsilon(t,s)a(s)F_M(|u_\varepsilon^M|^2)u_\varepsilon^M(s)ds,
\end{aligned} \tag{4.1}
$$

$$
u^M(t) = S(t,0)u_0 + i\int_0^t S(t,s)F_M(|u^M|^2)u^M)(s)ds, \tag{4.2}
$$

where $F_M(x) = \theta(\frac{x}{M})x^\sigma$ and $S_\varepsilon(t,s)$ denotes the dispersion propagator of the linear equation

$$
idv + \frac{1}{\varepsilon}m(\frac{t}{\varepsilon^2})\partial_{xx}vdt = 0.
$$

For the sake of the existence of solution of Eq. (4.1), we consider the NLSE below:

$$
\begin{cases}
idu + \dot{n}(t)\partial_{xx}udt + F_M(|u^M|^2)u^M(t)dt \\
\qquad +i\varepsilon a(s)F_M(|u^M|^2)u^M(s) = 0, \ (t,x) \in [0,\infty)\times\mathbb{R}, \\
u(0) = u_0,
\end{cases} \tag{4.3}
$$

and its mild form,

$$
\begin{aligned}
u_n^M(t) = S_n(t,0)u_0 + i\int_0^t S_n(t,s)F_M(|u_n^M|^2)u_n^M(s)ds \\
- \varepsilon\int_0^t S_n(t,s)a(s)F_M(|u_n^M|^2)u_n^M(s)ds,
\end{aligned} \tag{4.4}
$$

where $F_M(x)$ is as above, $n(t) \in C([0,T])$, $S_n(t,s) = e^{i(n(t)-n(s))\partial_{xx}}$ is the evolution operator of

$$
idu + \dot{n}(t)\partial_{xx}udt = 0.
$$

Since $a(t) \in L^\infty(0,\infty)$, $F_M(|u^M|^2)u^M$ is smooth enough with compact support and $S_n(t,s)$ is isometric on $H^1(\mathbb{R})$, then for $\forall\ \varepsilon > 0$ and $t < T < \infty$, there exists a unique solution $u_n^M(t)$ of Eq. (4.4) in $C([0,T];H^1(\mathbb{R}))$ by a fixed point argument. Thus the solution $u_\varepsilon^M(t)$ of Eq. (4.1) in $C([0,T];H^1(\mathbb{R}))$ exists.

According to the assumption that the process $\frac{1}{\varepsilon}\int_0^t m(\frac{t}{\varepsilon^2})ds$ converges to $\beta(t)$ in $C([0,T])$ in distribution for fixed $T > 0$, by Skorohod Theorem, we can find a new probability space and a class of random variables n_ε defined on it such that: n_ε is equal to $\frac{1}{\varepsilon}\int_0^t m(\frac{t}{\varepsilon^2})ds$ in law in $C([0,T])$ for any $\varepsilon > 0$ and n_ε converges to a standard real-valued Brownian motion $\tilde{\beta}$ in $C([0,T])$ almost surely, where $\tilde{\beta}$ has the same law as $\beta(t)$ in $C([0,T])$. Now, we consider the stochastic equations corresponding to (4.1) and (4.2):

$$
u_{n_\varepsilon}^M(t) = S_{n_\varepsilon}(t,0)u_0 + i\int_0^t S_{n_\varepsilon}(t,s)F_M(|u_{n_\varepsilon}^M|^2)u_\varepsilon^M(s)ds
$$

$$-\varepsilon\int_0^t S_{n_\varepsilon}(t,s)a(s)F_M(|u^M_{n_\varepsilon}|^2)u^M_{n_\varepsilon}(s)ds, \tag{4.5}$$

$$u^M_{\tilde\beta}(t)=S_{\tilde\beta}(t,0)u_0+i\int_0^t S_{\tilde\beta}(t,s)F_M(|u^M_{\tilde\beta}|^2)u^M_{\tilde\beta}(s)ds, \tag{4.6}$$

where $F_M(x)=\theta(\frac{x}{M})x^\sigma$. Since n_ε (equal to $\int_0^t m(\frac{t}{\varepsilon^2})ds$) is continuous in $C([0,T])$, we deduce from the aforementioned debate that for $u_0\in H^1(\mathbb{R})$, there exists a unique solution $u^M_{n_\varepsilon}(t)$ of Eq. (4.5) in $C([0,T];H^1(\mathbb{R}))$ a.s.

For convenience, we omit the dependence on M in the following. For any $0<t\le T<\infty$,

$$\begin{aligned}
&u_{n_\varepsilon}(t)-u_{\tilde\beta}(t)\\
=\;& S_{n_\varepsilon}(t,0)u_0-S_{\tilde\beta}(t,0)u_0\\
&+i\int_0^t S_{n_\varepsilon}(t,s)F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}(s)ds-i\int_0^t S_{\tilde\beta}(t,s)F(|u_{\tilde\beta}|^2)u_{\tilde\beta}(s)ds\\
&-\varepsilon\int_0^t S_{n_\varepsilon}(t,s)a(s)F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}(s)ds\\
=\;& S_{n_\varepsilon}(t,0)u_0-S_{\tilde\beta}(t,0)u_0\\
&+i\int_0^t (S_{n_\varepsilon}(t,s)-S_{\tilde\beta}(t,s))F(|u_{\tilde\beta}|^2)u_{\tilde\beta}(s)ds\\
&+\;i\int_0^t S_{n_\varepsilon}(t,s)(F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}-F(|u_{\tilde\beta}|^2)u_{\tilde\beta})(s)ds\\
&-\varepsilon\int_0^t (S_{n_\varepsilon}(t,s)-S_{\tilde\beta}(t,s))a(s)F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}(s)ds\\
&-\varepsilon\int_0^t S_{\tilde\beta}(t,s)a(s)F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}(s)ds\\
=\;& I+II+III+IV+V.
\end{aligned} \tag{4.7}$$

For any $u_0\in H^1(\mathbb{R})$, we can prove $n\to S_n(t,0)u_0$ is continuous from $C([0,T])$ to $C([0,T];H^1(\mathbb{R}))$ (see [18]). Therefore, for $\forall\ \delta>0$, there exists $\delta_1>0$ such that when $\|n_\varepsilon-\tilde\beta\|_{C([0,T])}\le\delta_1$, we have

$$\|I\|_{C([0,T];H^1(\mathbb{R}))}=\|S_{n_\varepsilon}(t,0)u_0-S_{\tilde\beta}(t,0)u_0\|_{C([0,T];H^1(\mathbb{R}))}\le\delta. \tag{4.8}$$

Since the mapping $v\to F(|v|^2)v$ is continuous on $H^1(\mathbb{R})$, by a compactness argument (see Section 5 in [18]), we conclude that for any $\delta>0$, there exists $\delta_2>0$ such that when $\|n_\varepsilon-\tilde\beta\|_{C([0,T])}\le\delta_2$,

$$\begin{aligned}
\|II\|_{H^1(\mathbb{R})}&=\left\|\int_0^t (S_{n_\varepsilon}(t,s)-S_{\tilde\beta}(t,s))(F(|v_{\tilde\beta}|^2)v_{\tilde\beta})(s)ds\right\|_{H^1(\mathbb{R})}\\
&\le 3CT\delta.
\end{aligned} \tag{4.9}$$

For the fourth term IV, since $a(t) \in L^\infty(0,\infty)$, similar to II for any $\delta > 0$ (such as $\delta = \varepsilon$), we conclude that there exists $\delta_3 > 0$ such that when $\|n_\varepsilon - \tilde{\beta}\|_{C([0,T])} \leq \delta_3$,

$$\begin{aligned} \|IV\|_{H^1(\mathbb{R})} &= \varepsilon \left\| \int_0^t (S_{n_\varepsilon}(t,s) - S_{\tilde{\beta}}(t,s)) a(s) F(|u_{\tilde{\beta}}|^2) u_{\tilde{\beta}}(s) ds \right\|_{H^1(\mathbb{R})} \\ &\leq 3C \|a(t)\|_{L^\infty(0,\infty)} \varepsilon T \delta. \end{aligned} \tag{4.10}$$

For the third term III, since $S_{n_\varepsilon}(t,s)$ is isometric on $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$, we get

$$\begin{aligned} \|III\|_{L^2(\mathbb{R})} &\leq \int_0^t \|(F(|u_{n_\varepsilon}|^2) u_\varepsilon - F(|u_{\tilde{\beta}}|^2) u_{\tilde{\beta}})(s)\|_{L^2} ds \\ &\leq C(M) \int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2} ds \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \|III\|_{H^1(\mathbb{R})} &\leq \left\| \int_0^t S_{n_\varepsilon}(t,s) (F(|u_{n_\varepsilon}|^2) u_{n_\varepsilon} - F(|u_{\tilde{\beta}}|^2) u_{\tilde{\beta}})(s) ds \right\|_{H^1(\mathbb{R})} \\ &\leq C \int_0^t \|F(|u_{n_\varepsilon}|^2) v_{n_\varepsilon} - F(|u_{\tilde{\beta}}|^2) u_{\tilde{\beta}})\|_{H^1} ds \end{aligned} \tag{4.12}$$

then for $\lambda = \min\{1, 2\sigma\} \leq 1$, we obtain (see Section 3 in [19]),

$$\begin{aligned} &\|\nabla(F(|v_{n_\varepsilon}|^2) u_{n_\varepsilon} - F(|u_{\tilde{\beta}}|^2) u_{\tilde{\beta}}))\|_{L^2} \\ &\leq C(M) \|u_{n_\varepsilon} - v_{\tilde{\beta}}\|_{H^1} + C(M) \|u_{\tilde{\beta}}\|_{H^1} (\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^\infty}^\lambda). \end{aligned}$$

Set $C_0 = C(\theta, M, \|\phi_1\|_{L^\infty}, \sup_{t\in[0,T]} \|u_{\tilde{\beta}}\|_{H^1})$, then we can estimate (4.12) as follows,

$$\|III\|_{H^1(\mathbb{R})} \leq C_0 \int_0^t (\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} + \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^\infty}^\lambda) ds.$$

For $0 < \sigma \leq \frac{1}{2}$, $\lambda = 2\sigma$, then we have

$$\begin{aligned} \|III\|_{H^1} &\leq C_0 \int_0^t (\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} + \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2}^\sigma \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1}^\sigma) ds \\ &\leq C(T) \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2}^{\frac{\sigma}{1-\sigma}} + C \int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} ds, \end{aligned} \tag{4.13}$$

where thanks to Gagliardo-Nirenberg's inequality and Young's inequality. For $\frac{1}{2} \leq \sigma \leq 2$, then $\lambda = 1$, by Sobolev embedding we get

$$\begin{aligned} \|III\|_{H^1} &\leq C_0 \int_0^t (\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} + \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^\infty}) ds \\ &\leq C(T) \int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} ds. \end{aligned} \tag{4.14}$$

For the fifth term V, by the isometry property of $S_{\tilde{\beta}}$, then for any $\delta > 0$ ($\delta = \varepsilon$), we obtain the following

$$\begin{aligned}\|V\|_{H^1(\mathbb{R})} &= \left\|\varepsilon \int_0^t S_{\tilde{\beta}}(t,s)a(s)F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}(s)ds\right\|_{H^1(\mathbb{R})} \\ &\leq \varepsilon \|a(t)\|_{L^\infty(0,\infty)} \int_0^t \|F(|u_{n_\varepsilon}|^2)u_{n_\varepsilon}\|_{H^1_x} ds \\ &\leq C\|a(t)\|_{L^\infty(0,\infty)} T\varepsilon, \qquad (4.15)\end{aligned}$$

where the constant C depends on $\|u_{n_\varepsilon}\|_{H^1(\mathbb{R})}$.

Now we can show u_{n_ε} converges to $u_{\tilde{\beta}}$ in $C([0,T];H^1(\mathbb{R}))$ in distribution. Since n_ε converges to $\tilde{\beta}$ in $C([0,T])$ almost surely, denote $A_\varepsilon(\delta) = \{\omega : \|n_\varepsilon - \tilde{\beta}\|_{C([0,T])} < \delta\}$, then for $\forall \delta > 0$

$$\lim_{\varepsilon\to 0} \mathbf{P}\Big(A^c_\varepsilon(\delta)\Big) = 0.$$

Hence, for any $\bar{\delta} > 0$ and $A_\varepsilon(\delta_1 \wedge \delta_2) = \{\omega : \|n_\varepsilon - \tilde{\beta}\|_{C([0,T])} < \delta_1 \wedge \delta_2\}$, where δ_1 and δ_2 are as in (4.8) and (4.9), we can choose $\bar{\varepsilon} > 0$ such that for $\forall\ 0 < \varepsilon < \bar{\varepsilon}$,

$$\mathbf{P}\Big(A_\varepsilon(\delta_1 \wedge \delta_2)\Big) > 1 - \bar{\delta}. \qquad (4.16)$$

By the definition of A_ε together with (4.8), (4.9), (4.10), for $\varepsilon < \bar{\varepsilon}$ and $\omega \in A_\varepsilon$, we then have

$$\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2} \leq C(T)\delta + C\int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2} ds.$$

We obtain by Gronwall's inequality

$$\sup_{t\in[0,T]} \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2} \leq C(T)e^{CT}\delta.$$

By the definition of A_ε and (4.8)-(4.10), (4.13), for $0 < \sigma \leq \frac{1}{2}$, it holds that

$$\begin{aligned}\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} &\leq C(T)\delta + C(T)\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L^2}^{\frac{\sigma}{1-\sigma}} + C\int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} ds \\ &\leq C(T)\delta^{\frac{\sigma}{1-\sigma}} + C\int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} ds.\end{aligned}$$

By Gronwall's inequality again, we derive

$$\sup_{t\in[0,T]} \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} \leq C(T)e^{CT}\delta^{\frac{\sigma}{1-\sigma}}. \qquad (4.17)$$

For $\frac{1}{2} \leq \sigma \leq 2$, by (4.8)-(4.10), (4.14), (4.15) and Gronwall's inequality we have

$$\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} \leq C(T)\delta + C\int_0^t \|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{H^1} ds \leq C(T)e^{CT}\delta. \qquad (4.18)$$

Hence, by (4.16)-(4.18) we then gain that

$$\mathbf{P}\Big(\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L_T^\infty(H^1)} > C(T)e^{CT}\delta^{\frac{\sigma}{1-\sigma}}\Big) \le \mathbf{P}\Big(A_\varepsilon^c\Big) \le \bar{\delta} \;\; \text{for} \;\; 0 < \sigma \le \frac{1}{2},$$

and

$$\mathbf{P}\Big(\|u_{n_\varepsilon} - u_{\tilde{\beta}}\|_{L_T^\infty(H^1)} > C(T)e^{CT}\delta\Big) \le \mathbf{P}\Big(A_\varepsilon^c\Big) \le \bar{\delta} \;\; \text{for} \;\; \frac{1}{2} \le \sigma \le 2,$$

which indicate that the solution $u_{n_\varepsilon}^M$ of Eq. (4.5) converges to the solution $u_{\tilde{\beta}}^M$ of Eq. (4.6) in $C([0,T];H^1(\mathbb{R}))$ in provability. We know by the above discussion that $u_{n_\varepsilon}^M$ is equal to u_ε^M, and $u_{\tilde{\beta}}^M$ is equal to u^M in distribution respectively.

From the previous arguments, we deduce that the solution u_ε^M of

$$\begin{cases} idu + \frac{1}{\varepsilon}m\Big(\frac{t}{\varepsilon^2}\Big)\partial_{xx}udt + \theta\Big(\frac{|u|^2}{M}\Big)|u|^{2\sigma}udt \\ \qquad +i\varepsilon a(s)\theta\Big(\frac{|u|^2}{M}\Big)|u|^{2\sigma}udt = 0, \;\; x \in \mathbb{R}, \;\; t > 0, \\ u(0) = u_0, \end{cases}$$

converges in distribution in $C([0,T];H^1(\mathbb{R}))$ to the solution u^M of

$$\begin{cases} idu + \partial_{xx}u \circ d\beta + \theta\Big(\frac{|u|^2}{M}\Big)|u|^{2\sigma}udt = 0, \;\; x \in \mathbb{R}, \;\; t > 0, \\ u(0) = u_0, \end{cases}$$

By Skorohod Theorem, for each fixed M, u_ε^M converges to u^M almost surely in $C([0,T];H^1(\mathbb{R}))$. That is, the convergence of truncated equation for Eq. (4.1) to Eq. (4.2) is proved.

Now we can extend the convergence result to the original equations (2.1) and (2.2). Define

$$\tilde{\tau}_\varepsilon^M = \inf\{t \ge 0; \|u_\varepsilon^M\|_{L_x^\infty}^2 \ge M\}.$$

Then, $\tilde{\tau}_\varepsilon^M$ is increasing with respect to M and $u_\varepsilon = u_\varepsilon^M$ on $[0, \tilde{\tau}_\varepsilon^M]$, which defines a local unique solution u_ε of (2.1) with $\tau_\varepsilon = \lim_{M\to\infty} \tilde{\tau}_\varepsilon^M$. Define

$$\tilde{\tau}^M = \inf\{t \ge 0; \|u^M\|_{L_x^\infty}^2 \ge M\}.$$

Similarly, we have $u = u^M$ on $[0, \tilde{\tau}^M]$ and $\lim_{M\to\infty} \tilde{\tau}^M = \infty$ (see Theorem 2.1 ($d = 1, 0 < \sigma < 2$) and Theorem 2.2 ($d = 1, \sigma = 2$)). For some $0 < \delta \le 1$, assume

$$\tilde{\tau}^{M-1} \ge T \quad \text{and} \quad \|u_\varepsilon^M - u^M\|_{C([0,T];H_x^1)} \le \delta, \tag{4.19}$$

then $u = u^M$ on $[0,T]$. By (4.19) and Sobolev embedding, choosing δ small enough, then

$$\begin{aligned} \|u_\varepsilon^M\|_{L^\infty([0,T];L_x^\infty(\mathbb{R}))} &\le \|u^M\|_{L_T^\infty(L_x^\infty)} + C\|u_\varepsilon^M - u^M\|_{C([0,T];H_x^1)} \\ &\le (M-1)^{\frac{1}{2}} + C\delta \;\le\; M^{\frac{1}{2}}. \end{aligned}$$

Thus $T < \tilde{\tau}_\varepsilon^M \leq \tau_\varepsilon$, and $u_\varepsilon^M = u_\varepsilon, u^M = u$ on $[0,T]$, therefore

$$\|u_\varepsilon - u\|_{C([0,T];H_x^1)} \leq \delta. \tag{4.20}$$

Then, for δ small enough, by (4.19), it follows that

$$\begin{aligned} &\mathbf{P}(\tau_\varepsilon \leq T \text{ or } \|u_\varepsilon - u\|_{C([0,T];H_x^1)} > \delta) \\ &= \mathbf{P}(\tau_\varepsilon \leq T) + \mathbf{P}(\tau_\varepsilon > T \text{ and } \|u_\varepsilon - u\|_{C([0,T];H^1)} > \delta) \\ &\leq \mathbf{P}(\tilde{\tau}^{M-1} < T \text{ or } \|u_\varepsilon - u\|_{C([0,T];H_x^1)} > \delta) \\ &\leq \mathbf{P}(\tilde{\tau}^{M-1} < T) + \mathbf{P}(\|u_\varepsilon - u\|_{C([0,T];H_x^1)} > \delta). \end{aligned} \tag{4.21}$$

By Theorem 2.1 and Theorem 2.2, $u \in C(\mathbb{R}^+; H^1(\mathbb{R}))$ almost surely, then

$$\lim_{M\to\infty} \mathbf{P}(\tilde{\tau}^{M-1} < T) = 0. \tag{4.22}$$

By (4.20), it is obvious that

$$\lim_{\varepsilon\to 0} \mathbf{P}(\|u_\varepsilon - u\|_{C([0,T];H_x^1)} > \delta) = 0. \tag{4.23}$$

Hence by (4.21)-(4.23)

$$\lim_{\varepsilon\to 0} \mathbf{P}(\tau_\varepsilon \leq T) \leq \lim_{\varepsilon\to 0} \mathbf{P}(\tau_\varepsilon \leq T \text{ or } \|u_\varepsilon - u\|_{C([0,T];H_x^1)} > \delta) = 0,$$

and

$$\lim_{\varepsilon\to 0} \mathbf{P}(\tau_\varepsilon > T \text{ and } \|u_\varepsilon - u\|_{C([0,T];H^1(\mathbb{R}))} > \delta) = 0.$$

Then we complete the proof. □

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