Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1185–1194
https://doi.org/10.4134/BKMS.b160276
pISSN: 1015-8634 / eISSN: 2234-3016

REVISIT NONLINEAR DIFFERENTIAL EQUATIONS ASSOCIATED WITH EULERIAN POLYNOMIALS

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Abstract. In this paper, we present nonlinear differential equations arising from the generating function of the Eulerian polynomials. In addition, we give explicit formulae for the Eulerian polynomials which are derived from our nonlinear differential equations.

1. Introduction

For $N \in \mathbb{N}$, the generalized harmonic numbers are defined as

$$H_{N,1} = H_N = 1 + \frac{1}{2} + \cdots + \frac{1}{N}, \tag{1.1}$$

and

$$H_{N,j} = \frac{H_{N-1,j-1}}{N} + \frac{H_{N-2,j-1}}{N-1} + \cdots + \frac{H_{j-1,j-1}}{j}, \quad (2 \leq j \leq N) \quad \text{(see [10])}. \tag{1.2}$$

It is well known that the Eulerian polynomials $A_n(t)$ are defined by the following generating function

$$\frac{1-t}{e^{x(t-1)}-t} = e^{A(t)x} = \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!}, \quad \text{(see [11])}, \tag{1.3}$$

with the usual convention about replacing $A^n(t)$ by $A_n(t)$.

Thus, by (1.3), we get

$$A_0(t) = 1, \quad A_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(t) (t-1)^{n-1-k}, \quad (n \geq 1), \tag{1.4}$$

and

$$(A(t) + (t-1))^n - tA_n(t) = \begin{cases} 1-t & \text{if } n = 0, \\ 0 & \text{if } n > 0. \end{cases} \tag{1.5}$$

Received March 31, 2016; Revised November 22, 2016; Accepted February 20, 2017.
2010 *Mathematics Subject Classification.* 05A19, 11B83, 34A34.
Key words and phrases. Eulerian polynomials, higher-order Eulerian polynomials, nonlinear differential equation.

From (1.3), (1.4) and (1.5), we note that

$$\sum_{i=1}^{m} i^n t^i = \sum_{l=1}^{n} (-1)^{n+l} \binom{n}{l} \frac{t^{m+1} A_{n-l}(t)}{(t-1)^{n-l+1}} m^l + (-1)^n \frac{t(t^m-1)}{(t-1)^{n+1}} A_n(t), \tag{1.6}$$

where $m \geq 1$ and $n \geq 0$ (see [11]).

In [2], Kim and Bayad considered the following nonlinear differential equations related to Apostol-Bernoulli-Euler numbers:

$$(N-1)! y^N = \sum_{k=1}^{N} a_k(N) y^{(k-1)}, \quad (N \in \mathbb{N}), \tag{1.7}$$

where $y^{(k)} = \left(\frac{d}{dt}\right)^k y(t)$.

Recently, Kim and Kim studied the following nonlinear differential equations arising from the generating function of degenerate Euler and Bernoulli numbers. For example, with $N \in \mathbb{N}$,

$$F^{(N)} = \frac{(-1)^N}{(1+\lambda t)^N} \sum_{i=1}^{N+1} a_i(N, \lambda) F^i, \quad (\text{see [14]}), \tag{1.8}$$

where $F = F(t) = \frac{1}{(1+\lambda t)^{\frac{1}{\lambda}} - 1}$, $F^{(k)} = \left(\frac{d}{dt}\right)^k F(t)$, and

$$\begin{aligned}
& a_i(N, \lambda) \\
&= (i-1)! \lambda^{N-i+1} \sum_{m_{i-1}=0}^{N-i+1} \sum_{m_{i-2}=0}^{N-m_{i-1}-i+1} \cdots \sum_{m_1=0}^{N-m_{i-1}-\cdots-m_2-i+1} \left(N - m_{i-1} + \frac{i}{\lambda}\right)_{m_{i-1}} \\
&\times \left(N - m_{i-1} - m_{i-2} - 1 + \frac{i-1}{\lambda}\right)_{m_{i-2}} \cdots \left(N - m_{i-1} - \cdots - m_1 - i + 2 + \frac{2}{\lambda}\right)_{m_1} \\
&\times \left(\frac{1}{\lambda}\right)_{N-m_{i-1}-m_{i-2}-\cdots-m_1-i+1}.
\end{aligned}$$

We also refer to [1–19] for some related results in the area of the Eulerian polynomials. In this paper, we revisit nonlinear differential equations arising from the generating function of the Eulerian polynomials. In addition, we give explicit formulas for the Eulerian polynomials which are derived from our nonlinear differential equations.

2. The statement of main results

Let

$$F = F(x,t) = \frac{1}{e^{x(t-1)} - t}, \quad (t \neq 1). \tag{2.1}$$

All differentiations in this paper are with respect to x, while t is being fixed. That is,

$$F^{(k)} = \left(\frac{d}{dx}\right)^k F(x,t), \quad (k \in \mathbb{N}). \tag{2.2}$$

From (2.1), we note that

$$F^{(1)} = \frac{d}{dx} F(x,t) = (1-t)\left(F + tF^2\right). \tag{2.3}$$

Thus, by (2.3), we get

$$t(1-t) F^2 = F^{(1)} - (1-t) F. \tag{2.4}$$

From (2.4), we can derive the following equations:

$$2! t^2 (1-t)^2 F^3 = F^{(2)} - 3(1-t) F^{(1)} + 2!(1-t)^2 F, \tag{2.5}$$

and

$$3! t^3 (1-t)^3 F^4 = -3!(1-t)^3 F + 11(1-t)^2 F^{(1)} - 6(1-t) F^{(2)} + F^{(3)}. \tag{2.6}$$

Continuing this process, we are led to put

$$N! t^N (1-t)^N F^{N+1} = \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i)}, \tag{2.7}$$

where $N = 0, 1, 2, \ldots$.

Thus, by (2.7), we get

$$(N+1)! t^N (1-t)^N F^N F^{(1)} = \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i+1)}. \tag{2.8}$$

From (2.3) and (2.8), we have

$$(N+1)! t^N (1-t)^{N+1} F^N \left(F + tF^2\right) = \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i+1)}. \tag{2.9}$$

Thus, by (2.7) and (2.9), we get

(2.10)

$$\begin{aligned}
&(N+1)! t^{N+1} (1-t)^{N+1} F^{N+2} \\
= &-(N+1)! t^N (1-t)^{N+1} F^{N+1} + \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i+1)} \\
= &-(N+1)(1-t) \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i)} + \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i+1)} \\
= &-(N+1) \sum_{i=0}^{N} a_i(N)(1-t)^{N+1-i} F^{(i)} + \sum_{i=1}^{N+1} a_{i-1}(N)(1-t)^{N+1-i} F^{(i)}.
\end{aligned}$$

On the other hand, by replacing N by $N+1$ in (2.7), we get

$$(N+1)!t^{N+1}(1-t)^{N+1}F^{N+2}=\sum_{i=0}^{N+1}a_i(N+1)(1-t)^{N+1-i}F^{(i)}. \tag{2.11}$$

Comparing the coefficients on both sides of (2.10) and (2.11), we have

$$a_0(N+1)=-(N+1)a_0(N), \quad a_{N+1}(N+1)=a_N(N), \tag{2.12}$$

and

$$a_i(N+1)=-(N+1)a_i(N)+a_{i-1}(N), \quad (1\le i\le N). \tag{2.13}$$

Moreover, from (2.7) for $N=0$, we obtain

$$a_0(0)=1. \tag{2.14}$$

Also, comparing (2.4) with (2.7) for $N=1$, we get

$$a_1(1)=1, \quad a_0(1)=-1. \tag{2.15}$$

By (2.12), we easily get

$$\begin{aligned} &a_0(N+1)=(-1)^{N+1}(N+1)!,\\ &a_{N+1}(N+1)=a_N(N)=\cdots=a_1(1)=a_0(0)=1. \end{aligned} \tag{2.16}$$

Now, we observe that the matrix $(a_i(j))_{0\le i,j\le N}$ is given by

$$\begin{array}{c} \\ 0\\ 1\\ 2\\ 3\\ \\ N \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & & N \end{array}\\ \left[\begin{array}{cccccc} 1 & -1! & (-1)^2 2! & (-1)^3 3! & \cdots & (-1)^N N!\\ & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & 0 & & \ddots & \\ & & & & & 1 \end{array}\right] \end{array}.$$

For $i=1$ in (2.13), we have

$$\begin{aligned} &a_1(N+1)\\ &=a_0(N)-(N+1)a_1(N)\\ &=a_0(N)-(N+1)(a_0(N-1)-Na_1(N-1))\\ &=a_0(N)-(N+1)a_0(N-1)+(-1)^2(N+1)Na_1(N-1)\\ &=a_0(N)-(N+1)a_0(N-1)\\ &\quad+(-1)^2(N+1)N(a_0(N-2)-(N-1)a_1(N-2))\\ &\ \vdots \end{aligned} \tag{2.17}$$

$$= \sum_{k=0}^{N-1} (-1)^k (N+1)_k \, a_0 (N-k) + (-1)^N (N+1)_N \, a_1 (1)$$
$$= \sum_{k=0}^{N} (-1)^k (N+1)_k \, a_0 (N-k),$$

where $(x)_n = x(x-1)\cdots(x-n+1)$, $(n \geq 1)$, and $(x)_0 = 1$.

Let $i = 2$ in (2.13). Then, we see that

$$\begin{aligned} &a_2(N+1) \\ &= a_1(N) - (N+1)\, a_2(N) \\ &= a_1(N) - (N+1)\,(a_1(N-1) - N a_2(N-1)) \\ &= a_1(N) - (N+1)\, a_1(N-1) + (-1)^2 (N+1)\, N a_2(N-1) \\ &= a_1(N) - (N+1)\, a_1(N-1) \\ &\quad + (-1)^2 (N+1)\, N\,(a_1(N-2) - (N-1)\, a_2(N-2)) \\ &\ \vdots \\ &= \sum_{k=0}^{N-2} (-1)^k (N+1)_k \, a_1(N-k) + (-1)^{N-1} (N+1)_{N-1} \, a_2(2) \\ &= \sum_{k=0}^{N-1} (-1)^k (N+1)_k \, a_1(N-k). \end{aligned} \tag{2.18}$$

For $i = 3$ in (2.13), it is not difficult to show that

$$a_3(N+1) = \sum_{k=0}^{N-2} (-1)^k (N+1)_k \, a_2(N-k). \tag{2.19}$$

So, we can deduce that, for $1 \leq i \leq N$,

$$a_i(N+1) = \sum_{k=0}^{N+1-i} (-1)^k (N+1)_k \, a_{i-1}(N-k). \tag{2.20}$$

To find explicit expressions for $a_i(N+1)$, we recall the definition for the generalized harmonic numbers $H_{N,i}$, defined for all N with $N \geq i$ for each i with $1 \leq i \leq N$:

$$H_{N,1} = \frac{1}{N} + \frac{1}{N-1} + \cdots + \frac{1}{1}, \tag{2.21}$$

and

$$H_{N,i} = \frac{H_{N-1,i-1}}{N} + \frac{H_{N-2,i-1}}{N-1} + \cdots + \frac{H_{i-1,i-1}}{i}, \quad (2 \leq i \leq N). \tag{2.22}$$

From (2.16)-(2.19), we have

$$a_1(N+1) = \sum_{k=0}^{N} (-1)^k (N+1)_k a_0(N-k) \tag{2.23}$$
$$= \sum_{k=0}^{N} (-1)^k (N+1)_k (-1)^{N-k} (N-k)!$$
$$= (-1)^N (N+1)! \sum_{k=0}^{N} \frac{1}{N-k+1}$$
$$= (-1)^N (N+1)! \left(\frac{1}{N+1} + \frac{1}{N} + \cdots + \frac{1}{1} \right)$$
$$= (-1)^N (N+1)! H_{N+1,1},$$

$$a_2(N+1) = \sum_{k=0}^{N-1} (-1)^k (N+1)_k a_1(N-k) \tag{2.24}$$
$$= \sum_{k=0}^{N-1} (-1)^k (N+1)_k (-1)^{N-k-1} (N-k)! H_{N-k,1}$$
$$= (-1)^{N-1} (N+1)! \sum_{k=0}^{N-1} \frac{H_{N-k,1}}{N-k+1}$$
$$= (-1)^{N-1} (N+1)! \left(\frac{H_{N,1}}{N+1} + \frac{H_{N-1,1}}{N} + \cdots + \frac{H_{1,1}}{2} \right)$$
$$= (-1)^{N-1} (N+1)! H_{N+1,2},$$

and

$$a_3(N+1) = \sum_{k=0}^{N-2} (-1)^k (N+1)_k a_2(N-k) \tag{2.25}$$
$$= (-1)^{N-2} \sum_{k=0}^{N-2} (N+1)_k (N-k)! H_{N-k,2}$$
$$= (-1)^{N-2} (N+1)! \sum_{k=0}^{N-2} \frac{H_{N-k,2}}{N-k+1}$$
$$= (-1)^{N-2} (N+1)! \left(\frac{H_{N,2}}{N+1} + \frac{H_{N-1,2}}{N} + \cdots + \frac{H_{2,2}}{3} \right)$$
$$= (-1)^{N-2} (N+1)! H_{N+1,3}.$$

Thus, we can deduce that, for $1 \le i \le N$,

$$a_i(N+1) = (-1)^{N-i+1} (N+1)! H_{N+1,i}. \tag{2.26}$$

Therefore, we obtain the following theorem.

Theorem 1. *For $N \in \mathbb{N}$, the nonlinear differential equations*

$$N!t^N (1-t)^N F^{N+1} = \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i)}$$

have a solution $F = F(x,t) = \frac{1}{e^{x(t-1)}-t}$, $(t \neq 1)$, *where*

$$a_0(N) = (-1)^N N!, \ a_i(N) = (-1)^{N-i} N! H_{N,i}, \quad (1 \leq i \leq N).$$

For $r \in \mathbb{N}$, the higher-order Eulerian polynomials are defined by the generating function

$$\left(\frac{1-t}{e^{x(t-1)}-t}\right)^r = \sum_{n=0}^{\infty} A_n^{(r)}(t)\frac{x^n}{n!}. \tag{2.27}$$

From (2.1) and (2.27), we have

$$\begin{aligned} N!t^N (1-t)^N F^{N+1} &= N!t^N (1-t)^N \left(\frac{1}{e^{x(t-1)}-t}\right)^{N+1} \\ &= N!t^N (1-t)^{-1} \left(\frac{1-t}{e^{x(t-1)}-t}\right)^{N+1} \\ &= N!t^N (1-t)^{-1} \sum_{k=0}^{\infty} A_k^{(N+1)}(t)\frac{x^k}{k!}. \end{aligned} \tag{2.28}$$

We observe that

$$\begin{aligned} &\sum_{i=0}^{N} a_i(N)(1-t)^{N-i} F^{(i)} \\ &= \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} \left(\frac{1}{e^{x(t-1)}-t}\right)^{(i)} \\ &= \sum_{i=0}^{N} a_i(N)(1-t)^{N-i-1} \left(\frac{1-t}{e^{x(t-1)}-t}\right)^{(i)} \\ &= \sum_{i=0}^{N} a_i(N)(1-t)^{N-i-1} \sum_{k=0}^{\infty} A_{k+i}(t)\frac{x^k}{k!}. \end{aligned} \tag{2.29}$$

From Theorem 1, (2.28) and (2.29), we have

$$N!t^N (1-t)^{-1} A_k^{(N+1)}(t) = \sum_{i=0}^{N} a_i(N)(1-t)^{N-i-1} A_{k+i}(t). \tag{2.30}$$

Therefore, by (2.30), we obtain the following theorem.

Theorem 2. *For $N, k = 0, 1, 2, \dots, N$, we have*

$$N!t^N A_k^{(N+1)}(t) = \sum_{i=0}^{N} a_i(N)(1-t)^{N-i} A_{k+i}(t),$$

where $a_0(N) = (-1)^N N!$, $a_i(N) = (-1)^{N-i} N! H_{N,i}$, $\quad (1 \le i \le N)$.

Recall here that

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{j=0}^{\infty} t^j (j+1)^n, \quad (n \ge 0). \tag{2.31}$$

By applying (2.31) to Theorem 2, we have

(2.32)

$$\begin{aligned} N!t^N A_k^{(N+1)}(t) &= \sum_{i=0}^{N} a_i(N)(1-t)^{N-i}(1-t)^{k+i+1} \frac{A_{k+i}(t)}{(1-t)^{k+i+1}} \\ &= \sum_{i=0}^{N} a_i(N)(1-t)^{N+k+1} \sum_{j=0}^{\infty} t^j (j+1)^{k+i} \\ &= \sum_{i=0}^{N} a_i(N) \sum_{l=0}^{\infty} (-1)^l \binom{N+k+1}{l} t^l \sum_{j=0}^{\infty} t^j (j+1)^{k+i} \\ &= \sum_{i=0}^{N} a_i(N) \sum_{m=0}^{\infty} \left(\sum_{l=0}^{m} (-1)^l \binom{N+k+1}{l} (m-l+1)^{k+i} \right) t^m \\ &= \sum_{m=0}^{\infty} \left(\sum_{i=0}^{N} \sum_{l=0}^{m} (-1)^l \binom{N+k+1}{l} (m-l+1)^{k+i} a_i(N) \right) t^m. \end{aligned}$$

Comparing the degrees on both sides of (2.32) gives the following theorem.

Theorem 3.

(1) *For $k \ge 1$, we have*

$$N!t^N A_k^{(N+1)}(t) = \sum_{m=0}^{k+N-1} \left(\sum_{i=0}^{N} \sum_{l=0}^{m} (-1)^l \binom{N+k+1}{l} (m-l+1)^{k+i} a_i(N) \right) t^m,$$

and

$$\sum_{i=0}^{N} \sum_{l=0}^{m} (-1)^l \binom{N+k+1}{l} (m-l+1)^{k+i} a_i(N) = 0 \quad \text{for all } m \ge k+N.$$

(2) *For $k = 0$, we have*

$$N!t^N A_0^{(N+1)}(t) = \sum_{m=0}^{N} \sum_{i=0}^{N} \sum_{l=0}^{m} (-1)^l \binom{N+1}{l} (m-l+1)^i a_i(N) t^m,$$

and

$$\sum_{i=0}^{N}\sum_{l=0}^{m}(-1)^{l}\binom{N+1}{l}(m-l+1)^{i}a_{i}(N)=0 \quad \text{for all } m\geq N+1,$$

where $a_0(N)=(-1)^N N!$, $a_i(N)=(-1)^{N-i}N!H_{N,i}, \quad (1\leq i\leq N)$.

Acknowledgments. The authors would like to thank the referee for his valuable and detailed suggestions.

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