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SUFFICIENT CONDITION FOR THE DIFFERENTIABILITY OF THE RIESZ-NÁGY-TAKÁCS SINGULAR FUNCTION

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Abstract. We give some sufficient conditions for the null and infinite derivatives of the Riesz-Nágy-Takács (RNT) singular function. Using these conditions, we show that the Hausdorff dimension of the set of the infinite derivative points of the RNT singular function coincides with its packing dimension which is positive and less than 1 while the Hausdorff dimension of the non-differentiability set of the RNT singular function does not coincide with its packing dimension 1.

1. Introduction

Many authors [4–7, 13] studied the characterization of the non-differentiability set of the Cantor singular function, which can be defined by the self-similar measure supported on the Cantor set which satisfies the SSC (strong separation condition) [3], and computed the Hausdorff and packing dimensions of the set of its infinite derivative points and its non-differentiability set. In particular, it was shown [4–6, 13] that the Hausdorff dimension of the non-differentiability set of the Cantor singular function is $(\frac{\log 2}{\log 3})^2$ while its packing dimension is $\frac{\log 2}{\log 3}$. Further it also can be shown that the Hausdorff dimension $\frac{\log 2}{\log 3}$ of the set of the infinite derivative points of the Cantor singular function coincides with its packing dimension since the set of the infinite derivative points is the relative complement of the non-differentiability set with respect to the Cantor set [5]. Recently, using the metric number theory, J. Paradís *et al.* [11] checked if the derivative of the Riesz-Nágy-Takács (RNT) singular function, which can be defined by the self-similar measure supported on the unit interval which satisfies the OSC (open set condition) [3], is null or infinite at a point when the singular function has a derivative at the point. In fact,

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they did not use the sufficient conditions for its null derivative and its infinite derivative since they checked it under the condition that the RNT singular function has a derivative. More recently, we [2] gave necessary conditions for the null derivative and the infinite derivative of the RNT singular function and a sufficient condition for its non-differentiability using the multifractal relation between the distribution set [2] and the local dimension set [2, 3, 9] by a self-similar measure related to the construction of the RNT singular function. As a result, we generalized their results. Further we found upper and lower bound for Hausdorff dimension $\dim(N)$ and packing dimension $\mathrm{Dim}(N)$ for the non-differentiability set N of the RNT singular function. Also we found upper and lower bound for Hausdorff dimension $\dim(D_\infty)$ and packing dimension $\mathrm{Dim}(D_\infty)$ for the set D_∞ of its infinite derivative points. More concretely, there is a solution $r(1)$ of the equation $g(r,p) = \frac{r\log p+(1-r)\log(1-p)}{r\log a+(1-r)\log(1-a)} = 1$ with respect to r where a, p generate the RNT singular function. For such equation $g(r,p)$ and the solution $r(1)$, we had:

$$0 < g(r(1), r(1)) \le \dim(N) \le \mathrm{Dim}(N) = 1,$$

and

$$\dim(D_\infty) \le \mathrm{Dim}(D_\infty) \le g(r(1), r(1)) < 1.$$

Naturally we compare the geometrical structure of the set of the infinite derivative points with that of the non-differentiability set. For this, we used to use the dimension regularity [12] that the packing dimension of a set coincides with its Hausdorff dimension. In this paper, we prove that the set of the infinite derivative points is dimension regular while the non-differentiability set is not dimension regular. Although many approaches to study the differentiability of the RNT singular function have been tried, nothing is known about the sufficient conditions for its null derivative and its infinite derivative. In this paper, we give sufficient conditions for the infinite derivative and the null derivative of the RNT singular function using digital distribution of a code which represents each point in the unit interval. Using the sufficient condition for the infinite derivative, we find that the Hausdorff dimension of the set of the infinite derivative points is the same as the aforementioned upper bound for its packing dimension, which implies that the set of the infinite derivative points is dimension regular, which is our main result. Explicitly,

$$\dim(D_\infty) = \mathrm{Dim}(D_\infty) = g(r(1), r(1)).$$

We show that the derivative of the RNT singular function which is not the identity function is zero almost everywhere using the fact that every normal point [3, 8, 10] satisfies the sufficient condition for the null derivative. Further, using the sufficient condition for the null derivative, we show that the Hausdorff dimension of the non-differentiability set of the RNT singular function is less than 1, which implies that the non-differentiability set is not dimension regular, which is our another main result. Explicitly,

$$g(r(1), r(1)) \leq \dim(N) < \mathrm{Dim}(N) = 1.$$

Conclusively, noting our two main results, we find that the RNT singular function and the Cantor singular function have similar properties: the set of infinite derivative points is dimension regular and the non-differentiability set is not dimension regular.

2. Preliminaries

Let $\mathbb{N}$ be the set of positive integers. For the probability vectors $(a_1, a_2) \in (0,1)^2$ and $\mathbf{p} = (p_1, p_2) \in (0,1)^2$,

$$[0,1] = \bigcup_{k=1}^{2} S_k([0,1]),$$

where $S_1(x) = a_1 x$ and $S_2(x) = a_2 x + a_1$ and the self-similar measure [3,9] $\gamma_{\mathbf{p}}$ is the unique probability measure on the unit interval $[0,1]$ such that

$$\gamma_{\mathbf{p}} = \sum_{i=1}^{2} p_i \gamma_{\mathbf{p}} \circ S_i^{-1}.$$

Each point $t(\in [0,1])$ has a code $\omega = (k_1, k_2, \ldots) \in \Sigma = \{1,2\}^{\mathbb{N}}$ satisfying

$$\{t\} = \bigcap_{n=1}^{\infty} K_{\omega|n}$$

for $K_{\omega|n} = K_{k_1,\ldots,k_n} = S_{k_1} \circ \cdots \circ S_{k_n}([0,1])$ [9] where $\omega|n$ denotes the truncation of ω to the nth place. In such case, we sometimes write $\pi(\omega)$ for such t using the natural projection $\pi : \Sigma \to [0,1]$. We also write the cylinder $c_n(t)$ for such $K_{\omega|n}$ and denote $\omega(l) = k_l$. We note that $\omega|l = k_1, \ldots, k_l = \omega(1), \ldots, \omega(l)$. We also note that if t is not an end point of $c_n(t)$, then it has a unique code ω where $\pi(\omega) = t$.

For the code $\omega \in \{1,2\}^{\mathbb{N}}$, we define the number of the digit k in $\omega|n$

$$n_k(\omega|n) = \sharp\{1 \leq l \leq n : \omega(l) = k\}$$

for $k = 1, 2$ and denote $A(x_n(\omega))$ the set of the accumulation points [3] of the probability vector sequence $\{x_n(\omega)\}_{n=1}^{\infty}$ where

$$x_n(\omega) = \left(\frac{n_1(\omega|n)}{n}, \frac{n_2(\omega|n)}{n} \right)$$

for the code ω. We also define $A_1(x_n(\omega))$ to be the projection of $A(x_n(\omega))$ to the first axis. Therefore $A_1(x_n(\omega))$ is the set of the accumulation points of the sequence

$$\left\{ \frac{n_1(\omega|n)}{n} \right\}_{n=1}^{\infty}$$

for the code ω. Noting $A_1(x_n(\omega)) = [x, y] \subset [0, 1]$ since $A(x_n(\omega))$ is a continuum [3, 10], we can also define a distribution set

$$F[x, y] = \{\omega : A_1(x_n(\omega)) = [x, y]\}.$$

3. Relation between distribution and differentiability

It is not difficult to show that the RNT function [1, 2, 11] can be defined as

$$f(t) = \gamma_{\mathbf{p}}([0, t])$$

for $t \in [0, 1]$, using the self-similar measure $\gamma_{\mathbf{p}}$ on the unit interval $[0, 1]$ where the probability vector $\mathbf{p} = (p_1, p_2) \in (0, 1)^2$. We denote $z_k(\omega, n)$ the position of the n-th occurrence of entry k in the code ω similarly as in [13].

Remark 1. From the same arguments in [13], we easily see that for each $k \in \{1, 2\}$

$$\overline{\lim}_{n\to\infty} \frac{n_k(\omega|n)}{n} = \overline{\lim}_{n\to\infty} \frac{n}{z_k(\omega, n)}$$

and

$$\underline{\lim}_{n\to\infty} \frac{n_k(\omega|n)}{n} = \underline{\lim}_{n\to\infty} \frac{n}{z_k(\omega, n)}.$$

Remark 2. We note that $A_1(x_n(\omega))$ for ω satisfying $\pi(\omega) = t$ where t is an end point of $c_n(t)$ is $\{0\}$ or $\{1\}$. Further the RNT function f satisfying $a_1 \neq p_1$ is not differentiable at the end point [2] and the end-points for the cylinders are countable. Therefore we can disregard them for the calculation of the Hausdorff and packing dimensions of the non-differentiability set.

From now on, if there is no particular mention, we only consider ω satisfying $\pi(\omega) = t$ where t is not an end point of $c_n(t)$.

Lemma 1. *Let $\{d_n\}$ be a real sequence and t_0 be a real number. For $t > t_0$, we assume that there is a positive integer $n(t)$ such that $H(t) \le d_{n(t)}$ for the real valued function $H(t)$. We also assume that $n(t) \uparrow \infty$ as $t \downarrow t_0$. Then we have*

$$\overline{\lim}_{t\downarrow t_0} H(t) \le \overline{\lim}_{n\to\infty} d_n.$$

Similarly for $t < t_0$, we assume that there is a positive integer $n(t)$ such that $H(t) \le d_{n(t)}$ for the real valued function $H(t)$. We also assume that $n(t) \uparrow \infty$ as $t \uparrow t_0$. Then we have

$$\overline{\lim}_{t\uparrow t_0} H(t) \le \overline{\lim}_{n\to\infty} d_n.$$

Proof. Assume that $n(t) \uparrow \infty$ as $t \downarrow t_0$ and there is a positive integer $n(t)$ such that $H(t) \le d_{n(t)}$ for the real valued function $H(t)$. For any $\delta > 0$, noting $n(t) \ge n(t_0 + \delta)$ for $t_0 < t \le t_0 + \delta$, we have

$$\sup_{t_0<t\le t_0+\delta} H(t) \le \sup_{t_0<t\le t_0+\delta} d_{n(t)} \le \sup_{n\ge n(t_0+\delta)} d_n.$$

This gives

$$\overline{\lim}_{t\downarrow t_0} H(t) = \lim_{\delta\downarrow 0} \sup_{t_0<t\le t_0+\delta} H(t) \le \lim_{\delta\downarrow 0} \sup_{n\ge n(t_0+\delta)} d_n = \overline{\lim}_{n\to\infty} d_n.$$

Similarly the dual part follows. □

From now on we assume that $\frac{p_1}{a_1} > 1$ which implies $\frac{p_2}{a_2} < 1$ for convenience since the following theorems and corollary hold for the condition $\frac{p_1}{a_1} < 1$ similarly.

Theorem 2. *Let* $A_1(x_n(\omega)) = [x, y]$ *with* $0 < x \le y < r_1$ *where*

$$(\frac{p_1}{a_1})^{r_1}(\frac{p_2}{a_2})^{1-r_1} = 1.$$

If

$$1 \le \frac{y}{x} < \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_2} + 1$$

and

$$1 \le \frac{1-x}{1-y} < \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_1} + 1,$$

then

$$f'(\pi(\omega)) = 0.$$

Proof. Consider $\pi(\omega') = t' > \pi(\omega) = t$. Then there exists n such that $\omega'|n = \omega|n$ and $\omega'|(n+1) \ne \omega|(n+1)$ with $I_n = \pi(\omega|n)$. Let $\epsilon > 0$ such that $y+\epsilon < r_1$. Since $\omega \in F[x, y]$ that is

$$x = \underline{\lim}_{n\to\infty} \frac{n}{z_1(\omega, n)} \le \overline{\lim}_{n\to\infty} \frac{n}{z_1(\omega, n)} = y,$$

$\frac{z_1(\omega,n+1)}{z_1(\omega,n)} < \frac{y}{x} + \epsilon$ for all large n. Clearly for all large n

$$\prod_{l=1}^{z_1(\omega,n)} (\frac{p_{\omega(l)}}{a_{\omega(l)}})^{1/z_1(\omega,n)} = (\frac{p_1}{a_1})^{\frac{n}{z_1(\omega,n)}}(\frac{p_2}{a_2})^{\frac{z_1(\omega,n)-n}{z_1(\omega,n)}} \le (\frac{p_1}{a_1})^{y+\epsilon}(\frac{p_2}{a_2})^{1-(y+\epsilon)}$$

since

$$\overline{\lim}_{n\to\infty} \frac{n}{z_1(\omega, n)} \le y.$$

For $I_m = K_{\omega|m}$, so satisfying $\bigcap_{m=1}^{\infty} I_m = \{t\}$, we have for all large n

$$\begin{aligned} \frac{f(\pi(\omega')) - f(\pi(\omega))}{\pi(\omega') - \pi(\omega)} &\le \frac{|f(I_{z_1(\omega,n)-1})|}{a_2^{z_1(\omega,n+1)-z_1(\omega,n)} a_1 |I_{z_1(\omega,n)-1}|} \\ &= \frac{1}{p_1 a_2^{z_1(\omega,n+1)-z_1(\omega,n)}} \prod_{l=1}^{z_1(\omega,n)} \frac{p_{\omega(l)}}{a_{\omega(l)}} \\ &= \frac{1}{p_1}\Big(\frac{1}{a_2^{\frac{z_1(\omega,n+1)}{z_1(\omega,n)}-1}}\Big((\frac{p_1}{a_1})^{\frac{n}{z_1(\omega,n)}}(\frac{p_2}{a_2})^{\frac{z_1(\omega,n)-n}{z_1(\omega,n)}}\Big)^{z_1(\omega,n)} \end{aligned}$$

$$\leq \frac{1}{p_1}\Big(\frac{1}{a_2^{\frac{y}{x}+\epsilon-1}}\Big(\frac{p_1}{a_1}\Big)^{y+\epsilon}\Big(\frac{p_2}{a_2}\Big)^{1-(y+\epsilon)}\Big)^{z_1(\omega,n)}.$$

This and the above lemma give for $t=\pi(\omega)$ and $t'=\pi(\omega')$,

$$0 \leq \overline{\lim}_{t'\downarrow t}\frac{f(t')-f(t)}{t'-t} \leq \overline{\lim}_{n\to\infty}\frac{1}{p_1}\Big(\frac{1}{a_2^{\frac{y}{x}-1}}\Big(\frac{p_1}{a_1}\Big)^{y}\Big(\frac{p_2}{a_2}\Big)^{1-y}\Big)^{z_1(\omega,n)} = 0$$

since $\frac{y}{x} < \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_2}+1 \iff \frac{1}{a_2^{\frac{y}{x}-1}}(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y} < 1$ and $z_1(\omega,n)\uparrow\infty$ as $n\uparrow\infty$.

Now, since $\omega\in F[x,y]$ that is

$$x = \underline{\lim}_{n\to\infty}\frac{n}{z_1(\omega,n)} \leq \overline{\lim}_{n\to\infty}\frac{n}{z_1(\omega,n)} = y,$$

we have

$$\begin{aligned}1-y &= \underline{\lim}_{n\to\infty}\frac{n}{z_2(\omega,n)} = 1-\overline{\lim}_{n\to\infty}\frac{n}{z_1(\omega,n)}\\ &\leq \overline{\lim}_{n\to\infty}\frac{n}{z_2(\omega,n)} = 1-\underline{\lim}_{n\to\infty}\frac{n}{z_1(\omega,n)} = 1-x.\end{aligned}$$

Therefore $\frac{z_2(\omega,n+1)}{z_2(\omega,n)} < \frac{1-x}{1-y}+\epsilon$ for all large n. Clearly for all large n

$$\prod_{l=1}^{z_2(\omega,n)}\Big(\frac{p_{\omega(l)}}{a_{\omega(l)}}\Big)^{1/z_2(\omega,n)} = \Big(\frac{p_1}{a_1}\Big)^{\frac{z_2(\omega,n)-n}{z_2(\omega,n)}}\Big(\frac{p_2}{a_2}\Big)^{\frac{n}{z_2(\omega,n)}} \geq \Big(\frac{p_1}{a_1}\Big)^{x-\epsilon}\Big(\frac{p_2}{a_2}\Big)^{1-(x-\epsilon)}$$

since

$$\overline{\lim}_{n\to\infty}\frac{n}{z_2(\omega,n)} \leq 1-x.$$

Similarly for $t'<t$, by

$$1 \leq \frac{1-x}{1-y} < \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_1}+1,$$

$$0 \leq \overline{\lim}_{t'\uparrow t}\frac{f(t')-f(t)}{t'-t} \leq \overline{\lim}_{n\to\infty}\frac{1}{p_2}\Big(\frac{1}{a_1^{\frac{1-x}{1-y}-1}}\Big(\frac{p_1}{a_1}\Big)^{y}\Big(\frac{p_2}{a_2}\Big)^{1-y}\Big)^{z_2(\omega,n)} = 0$$

follows from the similar arguments above. □

Remark 3. In the above theorem,

$$\min\left\{\frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_2}, \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_1}\right\} > 0$$

since $(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y} < 1$ from $y<r_1$.

Remark 4. For every normal point [3, 8, 10] $\pi(\omega)$, the derivative of the RNT function satisfying $a_1\neq p_1$ at $\pi(\omega)$ is 0 since $A_1(x_n(\omega)) = \{a_1\}$ and $a_1<r_1$. This also gives a direct proof that the derivative of the RNT function satisfying $a_1\neq p_1$ is zero almost everywhere.

Dually we have a sufficient condition for the infinite derivative. We need the following dual lemma without proof.

Lemma 3. *Let $\{d_n\}$ be a real sequence and t_0 be a real number. For $t > t_0$, we assume that there is a positive integer $n(t)$ such that $H(t) \geq d_{n(t)}$ for the real valued function $H(t)$. We also assume that $n(t) \uparrow \infty$ as $t \downarrow t_0$. Then we have*

$$\underline{\lim}_{t\downarrow t_0} H(t) \geq \underline{\lim}_{n\to\infty} d_n.$$

Similarly for $t < t_0$, we assume that there is a positive integer $n(t)$ such that $H(t) \geq d_{n(t)}$ for the real valued function $H(t)$. We also assume that $n(t) \uparrow \infty$ as $t \uparrow t_0$. Then we have

$$\underline{\lim}_{t\uparrow t_0} H(t) \geq \underline{\lim}_{n\to\infty} d_n.$$

Theorem 4. *Let $A_1(x_n(\omega)) = [x, y]$ with $r_1 < x \leq y < 1$ where*

$$(\frac{p_1}{a_1})^{r_1}(\frac{p_2}{a_2})^{1-r_1} = 1.$$

If

$$1 \leq \frac{y}{x} < \frac{\log(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x}}{-\log p_2} + 1$$

and

$$1 \leq \frac{1-x}{1-y} < \frac{\log(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x}}{-\log p_1} + 1,$$

then

$$f'(\pi(\omega)) = \infty.$$

Proof. Consider $\pi(\omega') = t' > \pi(\omega) = t$. Then there exists n such that $\omega'|n = \omega|n$ and $\omega'|(n+1) \neq \omega|(n+1)$ with $I_n = \pi(\omega|n)$. Let $\epsilon > 0$ such that $x - \epsilon > r_1$. Since $\omega \in F[x, y]$ that is

$$x = \underline{\lim}_{n\to\infty} \frac{n}{z_1(\omega, n)} \leq \overline{\lim}_{n\to\infty} \frac{n}{z_1(\omega, n)} = y,$$

$\frac{z_1(\omega,n+1)}{z_1(\omega,n)} < \frac{y}{x} + \epsilon$ for all large n. Clearly for all large n

$$\prod_{l=1}^{z_1(\omega,n)} (\frac{p_{\omega(l)}}{a_{\omega(l)}})^{1/z_1(\omega,n)} = (\frac{p_1}{a_1})^{\frac{n}{z_1(\omega,n)}}(\frac{p_2}{a_2})^{\frac{z_1(\omega,n)-n}{z_1(\omega,n)}} \geq (\frac{p_1}{a_1})^{x-\epsilon}(\frac{p_2}{a_2})^{1-(x-\epsilon)}$$

since

$$\underline{\lim}_{n\to\infty} \frac{n}{z_1(\omega, n)} \geq x.$$

For $I_m = K_{\omega|m}$, so satisfying $\bigcap_{m=1}^{\infty} I_m = \{t\}$, we have for all large n

$$\frac{f(\pi(\omega')) - f(\pi(\omega))}{\pi(\omega') - \pi(\omega)} \geq \frac{p_2^{z_1(\omega,n+1)-z_1(\omega,n)} p_1 |f(I_{z_1(\omega,n)-1})|}{|I_{z_1(\omega,n)-1}|}$$

$$= a_1 p_2^{z_1(\omega,n+1)-z_1(\omega,n)} \prod_{l=1}^{z_1(\omega,n)} \frac{p_{\omega(l)}}{a_{\omega(l)}}$$
$$= a_1 (p_2^{\frac{z_1(\omega,n+1)}{z_1(\omega,n)}-1} (\frac{p_1}{a_1})^{\frac{n}{z_1(\omega,n)}} (\frac{p_2}{a_2})^{\frac{z_1(\omega,n)-n}{z_1(\omega,n)}})^{z_1(\omega,n)}$$
$$\geq a_1 (p_2^{\frac{y}{x}+\epsilon-1} (\frac{p_1}{a_1})^{x-\epsilon} (\frac{p_2}{a_2})^{1-(x-\epsilon)})^{z_1(\omega,n)}.$$

This and the above lemma give for $t = \pi(\omega)$ and $t' = \pi(\omega')$,

$$\underline{\lim}_{t'\downarrow t} \frac{f(t') - f(t)}{t' - t} \geq \underline{\lim}_{n\to\infty} a_1 (p_2^{\frac{y}{x}-1} (\frac{p_1}{a_1})^x (\frac{p_2}{a_2})^{1-x})^{z_1(\omega,n)} = \infty$$

since $\frac{y}{x} < 1 + \frac{\log(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x}}{-\log p_2} \iff p_2^{\frac{y}{x}-1}(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x} > 1$ and $z_1(\omega,n) \uparrow \infty$ as $n \uparrow \infty$.

Now, since $\omega \in F[x,y]$ that is

$$x = \underline{\lim}_{n\to\infty} \frac{n}{z_1(\omega,n)} \leq \overline{\lim}_{n\to\infty} \frac{n}{z_1(\omega,n)} = y,$$

we have

$$1 - y = \underline{\lim}_{n\to\infty} \frac{n}{z_2(\omega,n)} = 1 - \overline{\lim}_{n\to\infty} \frac{n}{z_1(\omega,n)}$$
$$\leq \overline{\lim}_{n\to\infty} \frac{n}{z_2(\omega,n)} = 1 - \underline{\lim}_{n\to\infty} \frac{n}{z_1(\omega,n)} = 1 - x.$$

Therefore $\frac{z_2(\omega,n+1)}{z_2(\omega,n)} < \frac{1-x}{1-y} + \epsilon$ for all large n. Clearly for all large n

$$\prod_{l=1}^{z_2(\omega,n)} (\frac{p_{\omega(l)}}{a_{\omega(l)}})^{1/z_2(\omega,n)} = (\frac{p_1}{a_1})^{\frac{z_2(\omega,n)-n}{z_2(\omega,n)}} (\frac{p_2}{a_2})^{\frac{n}{z_2(\omega,n)}} \geq (\frac{p_1}{a_1})^{x-\epsilon} (\frac{p_2}{a_2})^{1-(x-\epsilon)}$$

since

$$\overline{\lim}_{n\to\infty} \frac{n}{z_2(\omega,n)} \leq 1 - x.$$

Similarly for $t' < t$, by

$$1 \leq \frac{1-x}{1-y} < \frac{\log(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x}}{-\log p_1} + 1,$$

$$\underline{\lim}_{t'\uparrow t} \frac{f(t') - f(t)}{t' - t} \geq \underline{\lim}_{n\to\infty} a_2 (p_1^{\frac{1-x}{1-y}-1} (\frac{p_1}{a_1})^x (\frac{p_2}{a_2})^{1-x})^{z_2(\omega,n)} = \infty$$

follows from the similar arguments above. □

Remark 5. In the above theorem,

$$\min\left\{ \frac{\log(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x}}{-\log p_2}, \frac{\log(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x}}{-\log p_1} \right\} > 0$$

since $(\frac{p_1}{a_1})^x(\frac{p_2}{a_2})^{1-x} > 1$ from $x > r_1$.

From now on, we put

$$g(r,p) = \frac{r\log p + (1-r)\log(1-p)}{r\log a_1 + (1-r)\log a_2}.$$

In fact, the above expression of $g(r,p)$ is same as that of $g(r,p)$ in our Introduction.

Corollary 5. *The Hausdorff and packing dimension of the set of the infinite derivative points is*

$$g(r_1, r_1),$$

where $(\frac{p_1}{a_1})^{r_1}(\frac{p_2}{a_2})^{1-r_1} = 1$.

Proof. From Theorem 13 of [2], the upper bound for the packing dimension of the set of the infinite derivative points is $g(r_1, r_1)$. From the above theorem, $f'(\pi(\omega)) = \infty$ for any $\omega \in F[r,r]$ where $r_1 < r < 1$. Since the lower bound for the Hausdorff dimension of $\bigcup_{r_1<r<1} \pi(F[r,r])$ is $g(r_1, r_1)$ [3], it follows. $\square$

Remark 6. The solution r_1 satisfying $(\frac{p_1}{a_1})^{r_1}(\frac{p_2}{a_2})^{1-r_1} = 1$ is the solution of the equation $g(r, p_1) = 1$. Therefore the solution r_1 is the same as the solution $r(1)$ in our Introduction. Noting Theorem 13 of [2], we remark

$$0 < g(r_1, r_1) < 1.$$

Using the sufficient condition for the null derivative of the RNT singular function, we have an upper bound less than 1 for the Hausdorff dimension of its non-differentiability set. For this, we need a lemma. From now on, we denote $\dim(E)$ the Hausdorff dimension of set E.

Lemma 6. *Let* $\underline{F}(x) = \cup_{x\le y\le 1} F[x,y]$ *and* $\overline{F}(y) = \cup_{0\le x\le y} F[x,y]$. *Then, for* $x > a_1$,

$$\dim(\cup_{x\le t\le 1}\pi(\overline{F}(t))) \le g(x,x).$$

Similarly, for $y < a_1$,

$$\dim(\cup_{0\le t\le y}\pi(\underline{F}(t))) \le g(y,y).$$

Proof. We note that $g(t,x) \le g(x,x) < 1$ for $a_1 < x \le t \le 1$ and $g(t,y) \le g(y,y) < 1$ for $0 \le t \le y < a_1$. It essentially follows from Theorem B (1.3) of [3] or Theorem 1.2 of [10]. $\square$

Theorem 7. *Let S be the non-differentiability set of the RNT function f satisfying $a_1 \ne p_1$. Then*

$$g(r_1, r_1) \le \dim(S) < 1,$$

where $(\frac{p_1}{a_1})^{r_1}(\frac{p_2}{a_2})^{1-r_1} = 1$.

Proof. $\dim(S) \ge g(r_1, r_1)$ follows from [2]. Therefore we only need to show $\dim(S) < 1$. We see that $f'(t) = 0$ for the point $t = \pi(\omega)$ where $\omega \in F[x,y]$ satisfying $1 \le \frac{y}{x} < \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_2} + 1$ and $1 \le \frac{1-x}{1-y} < \frac{\log(\frac{p_1}{a_1})^y(\frac{p_2}{a_2})^{1-y}}{\log a_1} + 1$ for

$0 < x \leq y < r_1$ from the above Theorem 2. That is, $f'(t) = 0$ for the point $t = \pi(\omega)$ where $\omega \in F[x, y]$ satisfying

$$x \leq y < c_1(x) = \frac{\log p_2}{\frac{\log a_2}{x} - \log \frac{p_1}{a_1} + \log \frac{p_2}{a_2}}$$

and

$$c_2(y) < x \leq y,$$

where

$$c_2(y) = (\alpha - \beta)y^2 + (2\beta - \alpha + 1)y - \beta$$

with

$$\alpha = \frac{\log \frac{p_1}{a_1}}{\log a_1} < 0$$

and

$$\beta = \frac{\log \frac{p_2}{a_2}}{\log a_1}.$$

We note that $c_1(r_1) = r_1$ and $\lim_{x \downarrow 0} c_1(x) = 0$ and $c_2(r_1) = r_1$ and $c_2(0) < 0$ with $c'(r_1) = 1 - \alpha > 1$. Noting this, we can easily show that there is $\delta > 0$ such that $f'(t) = 0$ for the point $t = \pi(\omega)$ where $\omega \in F[x, y]$ with the coordinate

$$(x, y) \in D = \{(x', y') : x' \leq y' \leq a_1 + \delta, a_1 - \delta \leq x' \leq a_1 + \delta\}.$$

Noting the above lemma, we have

$$\dim(S) \leq \max\{g(a_1 - \delta, a_1 - \delta), g(a_1 + \delta, a_1 + \delta)\} < 1$$

since $S \subset \pi(\{\omega \in F[x, y] : (x, y) \in D\})^c$. □

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