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TWO MEROMORPHIC FUNCTIONS SHARING FOUR PAIRS OF SMALL FUNCTIONS

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Abstract. The purpose of this paper is twofold. The first is to show that two meromorphic functions f and g must be linked by a quasi-Möbius transformation if they share a pair of small functions regardless of multiplicity and share other three pairs of small functions with multiplicities truncated to level 4. We also show a quasi-Möbius transformation between two meromorphic functions if they share four pairs of small functions with multiplicities truncated by 4, where all zeros with multiplicities at least $k > 865$ are omitted. Moreover the explicit Möbius-transformation between such f and g is given. Our results are improvement of some recent results.

1. Introduction

For a divisor ν on $\mathbb{C}$, we define its counting function by

$$N(r,\nu) = \int_1^r \frac{n(t)}{t}dt \quad (1 < r < \infty), \text{ where } n(t) = \sum_{|z|\le t} \nu(z).$$

For two positive integers k, M (maybe $M = \infty$), we set

$$\nu^{[M]}_{\le k}(z) = \begin{cases} \min\{M, \nu(z)\} & \text{if } \nu(z) \le k \\ 0 & \text{for otherwise,} \end{cases}$$

and write $N^{[M]}_{\le k}(r,\nu)$ for $N(r,\nu^{[M]}_{\le k})$. We will omit character $^{[M]}$ (resp. $\le k$) if $M = +\infty$ (resp. $k = +\infty$). Similarly, we define $N^{[M]}_{=k}(r,\nu)$ and $N^{[M]}_{>k}(r,\nu)$.

For a discrete set $S \subset \mathbb{C}$, we consider it as a reduced divisor and denote by $N(r,S)$ its counting function.

Let f be a nonzero holomorphic function on $\mathbb{C}$. For each $z_0 \in \mathbb{C}$, expanding f as $f(z) = \sum_{i=0}^{\infty} b_i(z-z_0)^i$ around z_0, then we define $\nu^0_f(z_0) := \min\{i \ : \ b_i \ne 0\}$.

Let φ be a non-constant meromorphic function on $\mathbb{C}$. Then there are two holomorphic functions φ_1, φ_2 without common zeros such that $\varphi = \frac{\varphi_1}{\varphi_2}$. We

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define $\nu^0_\varphi := \nu^0_{\varphi_1}$ and $\nu^\infty_\varphi := \nu^0_{\varphi_2}$. The proximity function of φ is defined by:

$$m(r,\varphi) := \frac{1}{2\pi}\int_0^{2\pi} \log^+ |\varphi(re^{i\theta})|d\theta \quad (r > 1),$$

here $\log^+ x = \max\{1, \log x\}$ for $x \in (0,\infty)$. The Nevanlinna characteristic function of φ is defined by

$$T(r,\varphi) := m(r,\varphi) + N(r,\nu^\infty_\varphi).$$

We will denote by $S(r,\varphi)$ a quantitive equal to $o(T(r,\varphi))$ for all $r \in (1,\infty)$ outside a finite Borel measure set.

Let f, a be two meromorphic functions on $\mathbb{C}$. The function a is said to be small (with respect to f) if and only if $T(r,a) = S(r,f)$. We denote by $\mathcal{R}_f$ the field of all small (with respect to f) functions on $\mathbb{C}$.

Let f and g be two meromorphic functions on $\mathbb{C}$. Let (a,b) be a pair of small (with respect to f and g) meromorphic functions on $\mathbb{C}$ and let n be a positive integer or $+\infty$.

Definition 1.1. We say that f and g share (a,b) weakly with multiplicities truncated to level n, or share (a,b) CM^*_n in another word, if

$$\min\{n, \nu^0_{f-a}(z)\} = \min\{n, \nu^0_{g-b}(z)\}$$

for all $z \in \mathbb{C}$ outside a discrete set of counting function equal to $S(r,f)+S(r,g)$.

We will say that f and g share a IM^* if $n = 1$ and say that f and g share a CM^* if $n = \infty$ and write CM^* for $CM^*_{+\infty}$.

The function f is said to be a quasi-Möbius transformation of g if there exist small (with respect to g) functions α_i $(1 \le i \le 4)$ with $\alpha_1\alpha_4 - \alpha_2\alpha_3 \not\equiv 0$ such that $f = \frac{\alpha_1 g+\alpha_2}{\alpha_3 g+\alpha_4}$. If all functions α_i $(1 \le i \le 4)$ are constants, then we say that the map f is a Möbius transformation of g. An interesting question arises here: "*Are there any quasi-Möbius transformation between f and g if they share some pairs of small functions IM^* or CM^*?*".

This problem has been studied by many authors, such as T. Czubiak-G. Gundersen [2], P. Li-C. C. Yang [3], P. Li-Y. Zhang [4], S. D. Quang-L. N. Quynh [7, 8], H. Z. Cao-T. B. Cao [1], L. Zhang-L. Yan [11] and others. We state here the recent result of P. Li and Y. Zhang, which is one of the best results available at present.

Theorem A (P. Li - Y. Zhang [4])**.** *Let f and g be non-constant meromorphic functions and a_i, b_i $(i = 1,2,3,4)$ $(a_i \ne a_j, b_i \ne b_j, i \ne j)$ be small functions (with respect to f and g). If f and g share three pairs (a_i,b_i), $(i = 1,2,3)$ CM^*, and share the fourth pair (a_4,b_4) IM^*, then f is a quasi-Möbius transformation of g.*

In this paper, we will improve the above result to the following.

Theorem 1.2. *Let f and g be non-constant meromorphic functions and a_i, b_i $(i = 1,2,3,4)$ $(a_i \ne a_j, b_i \ne b_j, i \ne j)$ be small functions (with respect to f and*

g). If f and g share the pair (a_1, b_1) IM^* and share three pairs (a_i, b_i), $(i = 2, 3, 4)$ CM_4^*, then f is a quasi-Möbius transformation of g. Moreover there is a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ such that

$$\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}}\ \text{or}\ \frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}}.$$

In the next theorem, we will consider the case where all zeros of functions $f - a_i$ with multiplicities at least $k > 865$ do not need to be counted. We prove the following.

Theorem 1.3. *Let f and g be non-constant meromorphic functions and a_i, b_i $(i = 1, 2, 3, 4)$ $(a_i \neq a_j, b_i \neq b_j, i \neq j)$ be small functions (with respect to f and g). Assume that*

$$\min\{\nu^0_{f-a_i,\le k}(z), 4\} = \min\{\nu^0_{g-b_i,\le k}(z), 4\}\ (1 \le i \le 4)$$

for all z outside a discrete set S of counting function equal to $S(r, f) + S(r, g)$. If $k > 865$, then there is a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ such that

$$\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}}\ \text{or}\ \frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}}.$$

2. Some lemmas and auxiliary results from Nevanlinna theory

Theorem 2.1 ([10], Corollary 1)**.** *Let f be a non-constant meromorphic function on $\mathbb{C}$. Let $a_1, \ldots, a_q$ $(q \geq 3)$ be q distinct small (with respect to f) meromorphic functions on $\mathbb{C}$. Then, for each $\epsilon > 0$, the following holds*

$$(q - 2 - \epsilon)T(r, f) \le \sum_{i=1}^{q} N^{[1]}(r, \nu^0_{f-a_i}) + S(r, f).$$

Theorem 2.2 ([9], Corollary 1)**.** *Let f be a non-constant meromorphic function on $\mathbb{C}$. Let $a_1, \ldots, a_q$ $(q \geq 3)$ be q distinct small (with respect to f) meromorphic functions on $\mathbb{C}$. Then the following holds*

$$||\ \frac{q}{3}T(r, f) \le \sum_{i=1}^{q} N^{[1]}(r, \nu^0_{f-a_i}) + o(T(r, f)).$$

Lemma 2.3 ([3], Lemma 7)**.** *Let f_1 and f_2 be two non-constant meromorphic functions satisfying*

$$N^{[1]}(r, \nu^0_{f_i}) + N^{[1]}(r, \nu^\infty_{f_i}) = S(r, f_1) + S(r, f_2)\ (i = 1, 2).$$

If $(f_1^s f_2^t - 1)$ is not identically zero for all integers s and t $(|s| + |t| > 0)$, then for any positive number ϵ, we have

$$N_0(r, 1; f_1, f_2) \le \epsilon(T(r, f_1) + T(r, f_2)),$$

where $N_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points.

Lemma 2.4. *Let f be a nonconstant meromorphic function and a be a small function (with respect to f). Then for each positive integer k (k may be $+\infty$) we have*

$$N^{[1]}(r,\nu^0_{f-a_i}) \le \frac{k}{k+1}N^{[1]}(r,\nu^0_{f-a_i,\le k}) + \frac{1}{k+1}T(r,f) + S(r,f).$$

Proof. We have

$$\begin{aligned} N^{[1]}(r,\nu^0_{f-a_i}) &= N^{[1]}(r,\nu^0_{f-a_i,\le k}) + N^{[1]}(r,\nu^0_{f-a_i,>k}) \\ &\le (1-\frac{1}{k+1})N^{[1]}(r,\nu^0_{f-a_i,\le k}) + \frac{1}{k+1}N(r,\nu^0_{f-a_i}) \\ &\le \frac{k}{k+1}N^{[1]}(r,\nu^0_{f-a_i,\le k}) + \frac{1}{k+1}T(r,f) + S(r,f). \end{aligned}$$

The lemma is proved. □

Let $\{H_i\}_{i=1}^q$ ($q \ge N+2$) be a set of q hyperplanes in $\mathbb{P}^N(\mathbb{C})$. We say that $\{H_i\}_{i=1}^q$ are in general position if for any $1 \le i_1 < \cdots < i_{N+1} \le q$ we have $\bigcap_{j=1}^{N+1} H_{i_j} = \emptyset$.

Theorem 2.5 ([5], Theorem 3.1)**.** *Let $f : \mathbb{C} \to \mathbb{P}^N(\mathbb{C})$ be a linearly holomorphic mapping. Let $\{H_i\}_{i=1}^q$ ($q \ge N+2$) be a set of q hyperplanes in $\mathbb{P}^N(\mathbb{C})$ in general position. Then*

$$(q-N-1)T(r,f) \le \sum_{i=1}^q N^{[N]}(r,f^*H_i) + S(r,f),$$

where f^*H_i denotes the pull back divisor of H_i by f.

3. Proof of the main theorems

Lemma 3.1. *Let f and g be two meromorphic functions on $\mathbb{C}$. Let $\{a_i\}_{i=1}^3$ and $\{b_i\}_{i=1}^3$ be two sets of small (with respect to f) meromorphic functions on $\mathbb{C}$ with $a_i \ne a_j$ and $b_i \ne b_j$ for all $1 \le i < j \le 3$. Assume that*

$$\min\{\nu^0_{f-a_i,\le k}(z), 1\} = \min\{\nu^0_{g-b_i,\le k}(z), 1\} \quad (1 \le i \le 3)$$

for all z outside a discrete subset S of counting function equal to $S(r,f)$. If $k \ge 3$, then $\|\ T(r,f) = O(T(r,g))$ and $\|\ T(r,g) = O(T(r,f))$. In particular, $S(r,f) = S(r,g)$.

Proof. By the second main theorem (Theorem 2.2), we have

$$\begin{aligned} T(r,f) &\le \sum_{i=1}^3 N^{[1]}(r,\nu^0_{f-a_i}) + S(r,f) \\ &\le \frac{k}{k+1}\sum_{i=1}^3 N^{[1]}_{\le k}(r,\nu^0_{f-a_i}) + \frac{3}{k+1}T(r,f) + S(r,f) \end{aligned}$$

$$= \frac{k}{k+1}\sum_{i=1}^{3}(N^{[1]}_{\leq k}(r,\nu^0_{g-b_i}) + N(r,\nu_S)) + \frac{3}{k+1}T(r,f) + S(r,f)$$
$$\leq \frac{3k}{k+1}T(r,g) + \frac{3}{k+1}T(r,f) + S(r,f) + S(r,g).$$

This implies that $\|\ T(r,f) = O(T(r,g))$. Similarly, we have $\|\ T(r,g) = O(T(r,f))$. The lemma is proved. □

Lemma 3.2. *Let f and g be non-constant meromorphic functions and a_i, b_i $(i = 1, 2, 3, 4)$ $(a_i \neq a_j, b_i \neq b_j, i \neq j)$ be small functions (with respect to f and g) such that*

$$\min\{\nu^0_{f-a_i,\leq k}(z), 1\} = \min\{\nu^0_{g-b_i,\leq k}(z), 1\}\ (1 \leq i \leq 4)$$

for all z outside a discrete subset S of counting function equal to $S(r,f) + S(r,g)$. Assume that f is a quasi-Möbius transformation of g. If $k \geq 3$, then there is a permutation (i_1, i_2, i_3, i_4) of $(1, 2, 3, 4)$ such that

$$\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}} \ \textit{or}\ \frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}} = \frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}}.$$

Proof. By Lemma 3.1 we have $S(r,f) = S(r,g)$. Suppose that there is only one index $i_0 \in \{1, 2, 3, 4\}$ such that $N^{[1]}_{\leq k}(r,\nu^0_{f-a_{i_0}}) \neq s(r,f)$. Then by Theorem 2.1, we see that

$$(2-\epsilon)T(r,f) \leq N^{[1]}(r,\nu^0_{f-a_{i_0}}) + \sum_{\substack{1\leq i\leq 4\\ i\neq i_0}} N^{[1]}(r,\nu^0_{f-a_i}) + S(r,f)$$
$$\leq T(r,f) + \frac{k}{k+1}\sum_{\substack{1\leq i\leq 4\\ i\neq i_0}} N^{[1]}_{\leq k}(r,\nu^0_{f-a_i}) + \frac{3}{k+1}T(r,f) + S(r,f) \tag{3.3}$$
$$\leq \big(1 + \frac{3}{k+1}\big)T(r,f) + S(r,f),\ \forall\epsilon > 0.$$

It implies that $2 \leq 1 + \frac{3}{k+1}$, i.e., $k \leq 2$. This is a contradiction.

Therefore, there are at least two indices i_1, i_2 in $\{1, 2, 3, 4\}$ so that

$$N^{[1]}_{\leq k}(r,\nu^0_{f-a_{i_j}}) = N^{[1]}_{\leq k}(r,\nu^0_{f-a_{i_j}}) + S(r,f) \neq S(r,f)\ (1 \leq j \leq 2). \tag{3.4}$$

Denote by i_3, i_4 the remaining indices, i.e., $\{i_1, i_2, i_3, i_4\} = \{1, 2, 3, 4\}$. We set

$$F = \frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}},\quad G = \frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}},$$
$$A = \frac{a_{i_4}-a_{i_1}}{a_{i_4}-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}},\quad B = \frac{b_{i_4}-b_{i_1}}{b_{i_4}-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}}.$$

Then we easily have the following assertions:

- $T(r,F) = T(r,f) + S(r,f),\ T(r,G) = T(r,g) + S(r,f)$,
- $T(r,A) = T(r,B) + S(r,f) = S(r,f)$,

- $\min\{\nu^0_{F-a,\le k}(z),1\} = \min\{\nu^0_{G-b,\le k}(z),1\}$, $(a,b) \in \{(0,0),(1,1),(\infty,\infty),(A,B)\}$ and for all z outside a discrete set of counting function equal to $S(r,f)$.

Since f and g are quasi-Möbius transformations of each other, then so are F and G. Hence, there exist four small functions (with respect to f) $\alpha,\beta,\gamma,\lambda$ with $\alpha\lambda-\beta\gamma\neq 0$ such that

$$G=\frac{\alpha F+\beta}{\gamma F+\lambda}.$$

By the assumption, we have $0=\frac{\beta(z)}{\lambda(z)}$ for all $z\in \operatorname{Supp}\big(\nu^0_{f-a_{i_1},\le k}\big)$ outside a discrete set of counting function equal to $S(r,f)$. This implies that $\frac{\beta}{\lambda}\equiv 0$ i.e., $\beta\equiv 0$. Similarly, we have $\frac{\gamma(z)}{\alpha(z)}=0$ for all $z\in \operatorname{Supp}\big(\nu^0_{f-a_{i_2},\le k}\big)$ outside a discrete set of counting function equal to $S(r,f)$, and hence $\frac{\gamma}{\alpha}\equiv 0$, i.e., $\gamma\equiv 0$. Therefore, we obtain $G=\frac{\alpha}{\lambda}F$.

We now suppose that $\frac{\alpha}{\lambda}\notin\{1,B,\frac{B}{A}\}$. It is easy to see that:

- $N^{[1]}_{\le k}(r,\nu^0_{G-1})\le N^{[1]}(r,\nu^0_{\frac{\alpha}{\gamma}-1})+S(r,f)=S(r,f)$,
- $N^{[1]}_{\le k}(r,\nu^0_{G-B})\le N^{[1]}(r,\nu^0_{\frac{\alpha}{\gamma}-\frac{B}{A}})+S(r,f)=S(r,f)$,
- $N^{[1]}_{\le k}(r,\nu^0_{G-\frac{\alpha}{\gamma}})=N^{[1]}_{\le k}(r,\nu^0_{F-1})+S(r,f)\le N^{[1]}(r,\nu^0_{\frac{\alpha}{\gamma}-1})+S(r,f)=S(r,f)$.

Similarly to (3.3), we have

$$(2-\epsilon)T(r,G)\le \big(1+\frac{3}{k+1}\big)T(r,G)+S(r,G),\ \forall\epsilon>0.$$

This implies that $k\le 2$. It is a contradiction.

Then $\frac{\alpha}{\lambda}\in\{1,B,\frac{B}{A}\}$. We have the following three cases:

Case 1. $\frac{\alpha}{\lambda}\equiv 1$. Then we have $G=F$, i.e., $\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_3}-b_{i_2}}{b_{i_3}-b_{i_1}}$. This implies the desired conclusion of the lemma.

Case 2. $\frac{\alpha}{\lambda}\equiv\frac{B}{A}$. Then we have $G=\frac{B}{A}F$, i.e., $\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_4}-a_{i_2}}{a_{i_4}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}}$. After changing indices i_3 and i_4, we get the desired conclusion of the lemma.

Case 3. $\frac{\alpha}{\lambda}\equiv B$. Then we have $G=BF$, i.e., $\frac{f-a_{i_1}}{f-a_{i_2}}\cdot\frac{a_{i_3}-a_{i_2}}{a_{i_3}-a_{i_1}}=\frac{g-b_{i_1}}{g-b_{i_2}}\cdot\frac{b_{i_4}-b_{i_2}}{b_{i_4}-b_{i_1}}$. This implies the desired conclusion of the lemma.

Therefore, the above three cases complete the proof of the lemma. □

Proposition 3.5. *Let F and G be non-constant meromorphic functions and A_i,B_i $(i=1,2,3)$ $(A_i\neq A_j, B_i\neq B_j, i\neq j)$ be small functions (with respect to F and G). Assume that F is not a quasi-Möbius transformation of G. Then for every positive integer n we have the following inequality*

$$N(r,\nu)\le N^{[1]}(r,|\nu^0_{F-A_1}-\nu^0_{G-B_1}|)+N^{[1]}(r,|\nu^0_{F-A_2}-\nu^0_{G-B_2}|)+S(r),$$

where $S(r)=o(T(r,F)+T(r,G))$ outside a finite Borel measure set of $[1,+\infty)$ and ν is the divisor defined by $\nu(z)=\max\{0,\min\{\nu^0_{F-A_3}(z),\nu^0_{G-B_3}(z)\}-1\}$.

Proof. By considering meromorphic functions $\frac{F-A_1}{F-A_2} \cdot \frac{A_3-A_2}{A_3-A_1}$ and $\frac{G-B_1}{G-B_2} \cdot \frac{B_3-B_2}{B_3-B_1}$ instead of f and g, we may assume that $A_1 = B_1 = 0$, $A_2 = B_2 = \infty$ and $A_3 = B_3 = 1$.

Since F is not a quasi-Möbius transformation of G, we have

$$H := \frac{F'}{F} - \frac{G'}{G} = \frac{(F/G)'}{(F/G)} \not\equiv 0.$$

By the lemma on logarithmic derivatives, it follows that

$$m(r, H) \leq m(r, \frac{F'}{F}) + m(r, \frac{G'}{G}) = S(r).$$

We also see that H has only simple poles and if z is a pole of H, then it must be either $\nu_F^0(z) \neq \nu_G^0(z)$ or $\nu_F^\infty(z) \neq \nu_G^\infty(z)$. Then it follows that

$$N(r, \nu_H^\infty) \leq N^{[1]}(r, |\nu_F^0 - \nu_G^0|) + N^{[1]}(r, |\nu_F^\infty - \nu_G^\infty|).$$

On the other hand, if z is a common zero of $(F-1)$ and $(G-1)$, then z will be a zero of $H = \big(\frac{F-G}{G}\big)'/(\frac{F}{G})$ with multiplicity at least $(\min\{\nu_{F-1}^0(z), \nu_{G-1}^0(z)\}-1)$. This yields that

$$N(r, \nu_H^0) \geq N(r, \nu).$$

Thus

$$\begin{aligned} N(r, \nu) \leq N(r, \nu_H^0) &\leq T(r, H) = m(r, H) + N(r, \nu_H^\infty) \\ &\leq N^{[1]}(r, |\nu_F^0 - \nu_G^0|) + N^{[1]}(r, |\nu_F^\infty - \nu_G^\infty|). \end{aligned}$$

The proposition is proved. □

Proof of Theorem 1.2. Suppose that f is not a quasi-Möbius transformation of g. By Lemma 3.1, we have $S(r, f) = S(r, g)$. For each $1 \leq i \leq 4$, we define a divisor ν_i by setting

$$\nu_i(z) = \max\{0, \min\{\nu_{f-a_i}^0(z), \nu_{g-a_i}^0(z)\} - 1\}.$$

By assumptions of the theorem and by Lemma 3.5, for a permutation (i, j, s) of $(2, 3, 4)$ we have following estimates:

$$\begin{aligned} &N_{=2}^{[1]}(r, \nu_{f-a_i}^0) + 2N_{=3}^{[1]}(r, \nu_{f-a_i}^0) + 3N_{>3}^{[1]}(r, \nu_{f-a_i}^0) \\ &\leq N(r, \nu_i) + S(r, f) \\ &\leq N^{[1]}(r, |\nu_{f-a_j}^0 - \nu_{g-b_j}^0|) + N^{[1]}(r, |\nu_{f-a_s}^0 - \nu_{g-b_s}^0|) + S(r, f) \\ &\leq N_{>3}^{[1]}(r, \nu_{f-a_j}^0) + N_{>3}^{[1]}(r, \nu_{f-a_s}^0) + S(r, f). \end{aligned}$$

From these inequalities, we easily obtain that

$$\begin{aligned} N_{=2}^{[1]}(r, \nu_{f,a_2}^0) &= N_{=3}^{[1]}(r, \nu_{f,a_3}^0) + S(r, f) \\ &= N_{>3}^{[1]}(r, \nu_{f,a_4}^0) + S(r, f) = S(r, f),\ i = 2, 3, 4. \end{aligned}$$

This yields that

$$N_{>1}^{[1]}(r, \nu_{f-a_i}^0) = S(r, f)\ (2 \leq i \leq 4).$$

Similarly, we also have

$$N^{[1]}_{>1}(r, \nu^0_{g-b_i}) = S(r, f)\ (2 \le i \le 4).$$

We set $f_1 = \frac{f-a_2}{f-a_3}\frac{g-a_3}{g-a_2}$ and $f_2 = \frac{f-a_2}{f-a_4}\frac{g-a_4}{g-a_2}$. Then it is easy to see that

$$N^{[1]}(r, \nu^0_{f_1}) + N^{[1]}(r, \nu^\infty_{f_1}) \le N^{[1]}_{>1}(r, \nu^0_{f-a_2}) + N^{[1]}_{>1}(r, \nu^0_{f-a_3}) = S(r, f).$$

This means the sets of multiple zeros of $f - a_i$ and $g - a_i$ are of counting functions equal to $S(r, f)$. Therefore, for $i = 2, 3, 4$, we have

$$\nu^0_{f-a_i}(z) = \nu^0_{g-a_i}(z) \in \{0, 1\}$$

for all z outside a discrete set of counting function equal to $S(r, f)$. Hence, f and g share pair (a_i, b_i) weakly with counting multiplicities for $i = 2, 3, 4$. By Theorem A, we have that f is a quasi-Möbius transformation of g. This is a contradiction.

Therefore, the supposition is untrue. Hence f is a quasi-Möbius transformation of g. With the help of Lemma 3.2, we have the conclusion of the theorem. □

Proof of Theorem 1.3. Suppose that f is not a quasi-Möbius transformation of g. By Lemma 3.1, we have $S(r, f) = S(r, g)$.

For each $1 \le i \le 4$, we define a divisor ν_i and μ_i by setting

$$\nu_i(z) = \max\{0, \min\{\nu^0_{f-a_i}(z), \nu^0_{g-a_i}(z)\} - 1\},$$
$$\mu_i(z) = \min\{1, |\nu^0_{f-a_i}(z) - \nu^0_{g-a_i}(z)|\}.$$

Take three indices i, j, t in $\{1, 2, 3, 4\}$. By Lemma 3.5 and by the assumptions of the theorem, we easily have the following

$$\begin{aligned} 3N^{[1]}(r, \mu_i) &\le 3(N^{[1]}_{>3}(r, \nu_{f-a_i,\le k}) + N^{[1]}_{>k}(r, \nu_{f-a_i}) + N^{[1]}_{>k}(r, \nu_{g-b_i})) \\ &\le N(r, \nu_i) + 3(N^{[1]}_{>k}(r, \nu_{f-a_i}) + N^{[1]}_{>k}(r, \nu_{g-b_i})) + S(r, f) \\ &\le 3(N^{[1]}_{>k}(r, \nu_{f-a_i}) + N^{[1]}_{>k}(r, \nu_{g-b_i})) + N(r, \mu_j) + N(r, \mu_t) + S(r, f). \end{aligned}$$

Summing-up both sides of the above inequality over all subsets $\{i, j, t\}$ of $\{1, 2, 3, 4\}$, we obtain

$$\sum_{i=1}^{4} N(r, \mu_i) \le \sum_{i=1}^{4} 3(N^{[1]}_{>k}(r, \nu_{f-a_i}) + N^{[1]}_{>k}(r, \nu_{g-b_i})) + S(r, f). \tag{3.6}$$

We set:

- $c_1 = \frac{a_3-a_2}{a_2-a_1},\ c_2 = \frac{a_3-a_1}{a_2-a_1},\ c'_1 = \frac{b_3-b_2}{b_2-b_1},\ c'_2 = \frac{b_3-b_1}{b_2-b_1}$,
- $F_1 = c_1(f - a_1),\ F_2 = c_2(f - a_2),\ G_1 = c'_1(g - b_1),\ G_2 = c'_2(g - b_2)$,
- $h_1 = \frac{F_1}{G_1},\ h_2 = \frac{F_2}{G_2},\ h_3 = \frac{F_1-F_2}{G_1-G_2} = \frac{b_2-b_1}{a_2-a_1} \cdot \frac{f-a_3}{g-b_3}$,
- $\alpha = \frac{c_1(a_4-a_1)}{c_2(a_4-a_2)},\ \beta = \frac{c'_1(b_4-b_1)}{c'_2(b_4-b_2)}$,
- $h_4 = \frac{F_1-\alpha F_2}{G_1-\beta G_2} = \frac{c_1(a_1-a_2)(f-a_4)/(a_4-a_2)}{c'_1(b_1-b_2)(g-b_4)/(b_4-b_2)} = \frac{(a_3-a_2)(b_4-b_2)}{(a'_3-a'_2)(b'_4-b'_2)} \cdot \frac{f-a_4}{g-b_4}$.

It is easy to see that $c_1 \neq c_2,\ c'_1 \neq c'_2, \alpha \neq 1,\ \beta \neq 1$ and all $c_i, c'_i\ (1 \leq i \leq 2)$ are small with respect to f and

$$N^{[1]}(r,\nu^0_{h_i}) + N^{[1]}(r,\nu^\infty_{h_i}) = N^{[1]}(r,\mu_i) + S(r,f)\ (1 \leq i \leq 4). \tag{3.7}$$

From the definition of functions $F_i, G_i\ (1 \leq i \leq 2)$, we have the following equations system:

$$\begin{cases} F_1 \qquad\quad - h_1G_1 \qquad\qquad = 0 \\ \qquad F_2 \qquad\quad - \ h_2G_2 \quad = 0 \\ F_1 - \ \ F_2 - h_3G_1 + \ h_3G_2 \quad = 0 \\ F_1 - \alpha F_2 - h_4G_1 + h_4\beta G_2 \quad = 0. \end{cases}$$

This implies that

$$\det\begin{pmatrix} 1 & 0 & -h_1 & 0 \\ 0 & 1 & 0 & -h_2 \\ 1 & -1 & -h_3 & h_3 \\ 1 & -\alpha & -h_4 & h_4\beta \end{pmatrix} = 0.$$

Then

$$(1-\alpha)h_1h_2 - h_1h_3 + \beta h_1h_4 + \alpha h_2h_3 - h_2h_4 + (1-\beta)h_3h_4 = 0. \tag{3.8}$$

Denote by $\mathcal{I}$ the set of all subsets $I = \{i,j\}$ of the set $\{1,2,3,4\}$. For $I \in \mathcal{I}$, we define the function h_I as follows:

$$h_{\{1,2\}} = (1-\alpha)h_1h_2,\ h_{\{1,3\}} = -h_1h_3,\ h_{\{1,4\}} = \beta h_1h_4,$$
$$h_{\{2,3\}} = \alpha h_2h_3,\ h_{\{2,4\}} = -h_2h_4,\ h_{\{3,4\}} = (1-\beta)h_3h_4.$$

Then we have

$$\sum_{I\in\mathcal{I}} h_I = 0.$$

Take a meromorphic function d on $\mathbb{C}$ such that $dh_I\ (I \in \mathcal{I})$ are all holomorphic functions on $\mathbb{C}$ without common zero. Then it is easy to see that

$$\begin{aligned}\sum_{I\in\mathcal{I}} N^{[1]}(r,dh_I) &\leq 3\sum_{i=1}^4 (N^{[1]}(r,\nu^0_{h_i}) + N^{[1]}(r,\nu^\infty_{h_i})) + S(r,f) \\ &= 3\sum_{i=1}^4 N^{[1]}(r,\mu_i) + S(r,f).\end{aligned}$$

Take $I_0 \in \mathcal{I}$. Then

$$dh_{I_0} = -\sum_{I\neq I_0} dh_I.$$

Denote by t the minimum number satisfying the following: There exist t elements $I_1,\ldots,I_t \in \mathcal{I}$ and t nonzero constants $b_v \in \mathbb{C}\ (1 \leq v \leq t)$ such that $dh_{I_0} = \sum_{v=1}^t b_v dh_{I_v}$.

By the minimality of t, then the family $\{dh_{I_1},\ldots,dh_{I_t}\}$ is linearly independent over $\mathbb{C}$.

Case 1. $t = 1$. Then $\frac{h_{I_0}}{h_{I_1}} \in \mathbb{C} \setminus \{0\}$.

Case 2. $t \geq 2$. Consider the linearly non-degenerate holomorphic mapping $h : \mathbb{C} \to \mathbb{P}^{t-1}(\mathbb{C})$ with the representation $h = (dh_{I_1} : \cdots : dh_{I_t})$. Applying Theorem 2.5, we have

$$
\begin{aligned}
T_h(r) &\leq \sum_{v=1}^{t} N^{[t-1]}_{dh_{I_v}}(r) + N^{[t-1]}_{dh_{I_0}}(r) + S(r,f) \\
&\leq (t-1)\sum_{v=1}^{t} N^{[1]}_{dh_{I_v}}(r) + (t-1)N^{[1]}_{dh_{I_0}}(r) + S(r,f) \\
&\leq 3(t-1)\sum_{i=1}^{4} N(r,\mu_i) + S(r,f) \\
&\leq 12\sum_{i=1}^{4} N(r,\mu_i) + S(r,f) \text{ (since } t \leq 5).
\end{aligned} \tag{3.9}
$$

We define the following rational functions:

$$
\begin{aligned}
&H_1(X) = \frac{c_1(X-a_1)}{c_1'(X-b_1)},\ H_2(X) = \frac{c_2(X-a_2)}{c_2'(X-b_2)}, \\
&H_3(X) = \frac{b_2-b_1}{a_2-a_1}\cdot\frac{X-a_3}{X-b_3}, \\
&H_4(X) = \frac{(a_3-a_2)(b_4-b_2)}{(a_3'-a_2')(b_4'-b_2')}\cdot\frac{X-a_4}{X-b_4}.
\end{aligned}
$$

For each $I \subset \{1,\ldots,4\}$, put $I^c = \{1,\ldots,4\} \setminus I$. For $0 \leq u, v \leq t,\ u \neq v$ and $i \in ((I_v \cup I_u) \setminus (I_u \cap I_v))^c$, we see that

$$
\begin{aligned}
T(r, \frac{h_{I_u}}{h_{I_v}}) &= T\left(r, \frac{\prod_{j\in I_u} h_j}{\prod_{j\in I_v} h_j}\right) + S(r,f) \\
&\geq N\left(r, \nu^0_{\frac{\prod_{j\in I_u\setminus I_v} h_j}{\prod_{j\in I_v\setminus I_u} h_j} - \frac{\prod_{j\in I_u\setminus I_v} H_j(a_i)}{\prod_{j\in I_v\setminus I_u} H_j(a_i)}}\right) + S(r,f) \\
&\geq N^{[1]}_{\leq k}(r, \nu^0_{f-a_i}) + S(r,f).
\end{aligned}
$$

Similarly, we have

$$T(r, \frac{h_{I_u}}{h_{I_v}}) \geq N^{[1]}_{\leq k}(r, \nu^0_{g-b_i}) + S(r,f).$$

Therefore

$$T(r, \frac{h_{I_u}}{h_{I_v}}) \geq \frac{1}{2}(N^{[1]}_{\leq k}(r, \nu^0_{f-a_i}) + N^{[1]}_{\leq k}(r, \nu^0_{g-b_i})) + S(r,f).$$

Since $(I_0\cup I_1\setminus(I_0\cap I_1))^c\cup(I_1\cup I_2\setminus(I_1\cap I_2))^c\cup(I_2\cup I_0\setminus(I_2\cap I_0))^c=\{1,\dots,4\}$, we have

$$\begin{aligned}3T(r,h)&\geq T(r,\frac{h_{I_0}}{h_{I_1}})+T(r,\frac{h_{I_1}}{h_{I_2}})+T(r,\frac{h_{I_2}}{h_{I_0}})\\&\geq\frac{1}{2}(N^{[1]}_{\leq k}(r,\nu^0_{f-a_i})+N^{[1]}_{\leq k}(r,\nu^0_{g-b_i}))+S(r,f)\ (1\leq i\leq 4).\end{aligned}$$

Thus we have

$$\begin{aligned}&\sum_{i=1}^{4}(N^{[1]}_{\leq k}(r,\nu^0_{f-a_i})+N^{[1]}_{\leq k}(r,\nu^0_{g-b_i}))\\&\leq 24T(r,h)+S(r,f)\\&\leq 288\sum_{i=1}^{4}N(r,\mu_i)+S(r,f)\\&\leq 864\sum_{i=1}^{4}(N^{[1]}_{>k}(r,\nu^0_{f-a_i})+N^{[1]}_{>k}(r,\nu^0_{g-b_i}))+S(r,f).\end{aligned}$$

By Yamanoi's second main theorem (Theorem 2.1), for every $\epsilon>0$ we have

$$\begin{aligned}(2-\epsilon)T(r)&\leq\sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}N^{[1]}(r,\nu^0_u)+S(r,f)\\&=\sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}(N^{[1]}_{\leq k}(r,\nu^0_u)+N^{[1]}_{>k}(r,\nu^0_u))+S(r,f)\\&\leq\sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}(\frac{865}{k+865}N^{[1]}_{\leq k}(r,\nu^0_u)\\&\quad+(\frac{k}{k+865}864+1)N^{[1]}_{>k}(r,\nu^0_u))+S(r,f)\ \text{(by the above inequality)}\\&\leq\sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}\frac{865}{k+865}(N^{[1]}_{\leq k}(r,\nu^0_u)+N_{>k}(r,\nu^0_u))+S(r,f)\\&\leq\sum_{i=1}^{4}\sum_{u=f-a_i,g-b_i}\frac{865}{k+865}N(r,\nu^0_u)+S(r,f)\\&\leq\frac{4\cdot 865}{k+865}T(r)+S(r,f).\end{aligned}$$

Letting $r\to+\infty$, we get

$$2-\epsilon\leq\frac{4\cdot 865}{k+865}.$$

Since the above inequality holds for every $\epsilon > 0$, letting $\epsilon \to 0$ we get

$$2 \leq \frac{4 \cdot 865}{k + 865}, \text{ i.e., } k \leq 865.$$

This is a contradiction.

Then from Case 1 and Case 2, it follows that for each $I \in \mathcal{I}$, there is $J \in \mathcal{I} \setminus \{I\}$ such that $\frac{h_I}{h_J} \in \mathbb{C} \setminus \{0\}$. We consider the following two cases:

Case a. There exist $I = \{i, j\}, J = \{i, l\}, j \neq l$, $\frac{h_I}{h_J}$ = constant. Then $h_j = ah_l$ with a is a nonzero meromorphic function in $\mathcal{R}_f$. Therefore, f is a quasi-Möbius transformation of g. This contradicts the supposition that f is not a quasi-Möbius transformation of g

Case b. There exist nonzero constants b, c such that $h_{\{1,2\}} = bh_{\{3,4\}}$ and $h_{\{1,3\}} = ch_{\{2,4\}}$, i.e.,

$$(1 - \alpha)h_1h_2 = b(1 - \beta)h_3h_4 \text{ and } h_1h_3 = ch_2h_4.$$

Then $\left(\frac{h_1}{h_4}\right)^2 = \frac{bc(1-\beta)}{1-\alpha} \in \mathcal{R}_f$. This implies that $\frac{h_1}{h_4} \in \mathcal{R}_f$. Hence f is a quasi-Möbius transformation of g. This is a contradiction.

From the above two cases, we get the contradiction to the supposition. Hence f is a quasi-Möbius transformation of g. With the help of Lemma 3.2, we have the desired conclusion of the theorem. □

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