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DISTRIBUTION OF THE VALUES OF THE DERIVATIVE OF THE DIRICHLET L-FUNCTIONS AT ITS a-POINTS

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Abstract. In this paper, we study the value distribution of the derivative of a Dirichlet L-function $L'(s,\chi)$ at the a-points $\rho_{a,\chi}=\beta_{a,\chi}+i\gamma_{a,\chi}$ of $L(s,\chi)$. We give an asymptotic formula for the sum

$$\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\leq T} L'\left(\rho_{a,\chi},\chi\right) X^{\rho_{a,\chi}} \quad \text{as} \quad T\to\infty,$$

where X is a fixed positive number and χ is a primitive character mod q. This work continues the investigations of Fujii [4–6], Garunkštis & Steuding [8] and the authors [12].

1. Introduction

Let $L(s,\chi)$ be the Dirichlet L-function associated with a primitive character χ mod q and a be a nonzero complex number. The zeros of $L(s,\chi)-a$, which will be denoted by $\rho_{a,\chi}=\beta_{a,\chi}+i\gamma_{a,\chi}$, are called the a-points of $L(s,\chi)$. First, we note that there is an a-point near any trivial zero $s=-2n$ if $\chi(-1)=1$ and $s=-2n-1$ if $\chi(-1)=1$ for sufficiently large n. Apart from these a-points there are only finitely many other a-points in the half-plane $Re(s)=\sigma\leq 0$. The a-points with $\beta_{a,\chi}\leq 0$ are said to be trivial. All other a-points lie in a strip $0<Re(s)<A$, where A is a constant depending on a; these numbers are called the nontrivial a-points. The number of these a-points satisfies a Riemann-von Mangoldt type formula (we refer to [20, §7.2] for the proof of this formula which is stated for functions in a subclass of the Selberg class including the Dirichlet L-functions $L(s,\chi)$), namely

$$N_{a,\chi}(T)=\sum_{\substack{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\leq T\\ \beta_{a,\chi}>0}} 1=\frac{T}{2\pi}\log\left(\frac{qT}{2\pi c_a e}\right)+O(\log T), \tag{1}$$

where $c_a=m$ if $a=1$ and $c_a=1$ otherwise, with $m=\min\{n\geq 2:\ \chi(n)\neq 0\}$. Here and in the sequel the error term depends on q, however, the main

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term is essentially independent of a. Moreover, $N_{a,\chi}(T) \sim N_\chi(T)$ as $T \to \infty$, where $N_\chi(T) = N_{0,\chi}(T)$ denotes the number of nontrivial zeros $\rho_\chi = \beta_\chi + i\gamma_\chi$ of $L(s,\chi)$ satisfying $0 < \gamma_\chi < T$. Gonek [10] proved that, if the Riemann Hypothesis holds for $L(s,\chi)$, then at least $\left(\frac{1}{2} + o(1)\right) \frac{T}{2\pi} \log \frac{T}{2\pi}$ of the nontrivial a-points with ordinates in $(0,T)$ of the function $L(s,\chi)$ associated with a primitive character χ are simple and lie to the left of the line $Re(s) = 1/2$.

In this paper, we continue the investigations of Fujii [4–6], Garunkštis & Steuding [8] and the authors [12]. Actually, we are interested in the sums

$$\sum_{\rho_\chi;\ 0<\gamma_\chi\le T} L'\left(\rho_\chi, \chi\right) X^{\rho_\chi}, \qquad \sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\le T} L'\left(\rho_{a,\chi}, \chi\right) X^{\rho_{a,\chi}},$$

where X is a fixed positive number and χ mod q is a primitive character. Our method is based on a formula stated by Garunkštis and Steuding in [8, §6, Remark ii)] with the function $f(s) = L'(s,\chi)X^s$. There are several reasons why the above sum with the parameter X is of interest. The first one is that the estimation of this sum can be used to study the normal distribution of the values of $\log|L'(\rho_{a,\chi},\chi)|$[1], the second one is to study the vertical distribution of a-points of $L(s,\chi)$. Recall that in the case of $a = 0$, recently Fujii studied in [4] sums over the nontrivial zeros of $L(s,\chi)$. He showed that under the Riemann hypothesis, for $X > 1$,

$$\lim_{T\to\infty} \frac{1}{T/2\pi} \sum_{0<\gamma\le T} \left[X^{1/2+i\gamma}\left(L(1/2+i\gamma,\chi) - 1\right) - \xi(X)\right]$$
$$= \begin{cases} M(X,\chi) & \text{if } \ X \text{ is rational,} \\ 0 & \text{if } \ X \text{ is irrational,} \end{cases}$$

where $L(s,\chi)$ is a Dirichlet L-function with primitive Dirichlet character χ (mod q ≥ 3), $\xi(X)$ and $M(X,\chi)$ are some constants. Furthermore, Garunkštis, Grahl and Steuding [7] obtained more subtle information on the value distribution of Dirichlet L-functions by considering certain discrete moments

$$\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}<T} L(\rho_{a,\chi}, \psi).$$

Their formula extends a previous result due to Fujii [2, 3].

Our main result is stated in the following:

Theorem 1. *Let X be a positive number. Then*

$$\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\le T} L'\left(\rho_{a,\chi}, \chi\right) X^{\rho_{a,\chi}} = -\frac{aT}{2\pi}\log^2\left(\frac{qT}{2\pi}\right) + \frac{aT}{\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{aT}{\pi}$$

[1] In the case $a = 0$, to study the normal distribution of the values of $\log|\zeta'(1/2+i\gamma_n)|$, Hiary and Odlyzko [11] have been studying the behavior of the sum $\sum_{T\le\gamma_n\le T+H} \zeta'(1/2+i\gamma_n)e^{2\pi i n x}$ as a function of x, where $\rho = \beta + i\gamma$ denotes a non-trivial zero of $\zeta(s)$ and γ_n is the nth positive imaginary part of a zero ρ. To do so, they approximated the last sum by $\sum_{T\le\gamma_n\le T+H} \zeta'(1/2+i\gamma_n)e^{2\pi i\tilde{\gamma}_n x}$, where $\tilde{\gamma}_n = \frac{1}{2\pi}\gamma_n \log\frac{T}{2\pi}$.

$$
\begin{aligned}
& -\Delta(X)\chi\left[\Delta(X)X\right]\log(X)\left\{\frac{T}{4\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{4\pi}+\frac{i\pi}{4}\frac{T}{2\pi}\right\} \\
& +\Delta(X)\chi\left[\Delta(X)X\right]\frac{T}{2\pi}\sum_{X=mn}\Lambda(n)\log(m) \\
& +\frac{X}{\sqrt{q}}\sum_{k\le\frac{qT}{2\pi X}}\log^2(k)\overline{\chi}(k)e^{2i\pi kX/q} \\
& +\frac{1}{2\sqrt{q}}X\log(X)\sum_{k\le\frac{qT}{2\pi X}}\log(k)\overline{\chi}(k)e^{2i\pi kX/q} \\
& -\left(\frac{1}{2\sqrt{q}}X\log^2(X)-\frac{i\pi}{4\sqrt{q}}X\log X\right)\sum_{k\le\frac{qT}{2\pi X}}\overline{\chi}(k)e^{2i\pi kX/q} \\
& -\frac{X}{\sqrt{q}}\sum_{k\le\frac{qT}{2\pi X}}\sum_{k=mn}\Lambda(n)\overline{\chi}(n)\overline{\chi}(m)\log(m)e^{2i\pi kX/q}+O\left(\sqrt{T}\log^3T\right),
\end{aligned}
\tag{2}
$$

where $\Delta(X)$ *is defined by*

$$
\Delta(X)=\begin{cases}1 & \textit{if } X \textit{ is an integer } \ge 1,\\ 0 & \textit{otherwise.}\end{cases}
\tag{3}
$$

Remark. For $X = 1$, we obtain Garunkštis and Steuding's results [8] in the case of the Riemann zeta function. And for $q = 1$ and $a = 0$, we obtain Fujii's results [6].

Here and in the sequel the implicit constant in the error terms may depend on a and X; the formulas of Theorem 1 are not uniform with respect to X. We note that the proofs uses standard methods: contour integration, basic properties of the functional equation for $L(s,\chi)$ and Gonek's lemma (see. Gonek [9, Lemma 5]).

2. Preliminary lemmas

To prove Theorem 1, we start with well-known results on the Dirichlet L-function $L(s,\chi)$ (see Davenport book [1]). If χ mod q is a primitive character, then

$$
\xi(s,\chi)=\left(\frac{q}{\pi}\right)^{\frac{s+\nu}{2}}\Gamma\left(\frac{s+\nu}{2}\right)L(s,\chi)
$$

satisfies the functional equation

$$
\xi(s,\chi)=\frac{\tau(\chi)}{i^{\nu}\sqrt{q}}\xi\left(1-s,\overline{\chi}\right),
\tag{4}
$$

where $\tau(\chi)=\sum_{m \mod q}\chi(m)e^{\frac{2i\pi m}{q}}$, with $\nu=\frac{1}{2}(1-\chi(-1))$. We note that

$$
\frac{\xi'}{\xi}(s,\chi)=\frac{1}{2}\log\left(\frac{q}{\pi}\right)+\frac{1}{2}\psi\left(\frac{s+\nu}{2}\right)+\frac{L'}{L}(s,\chi),
$$

where $\psi(s) = \frac{\Gamma'}{\Gamma}(s)$ and that for $|\arg(s)| < \pi - \theta$ with arbitrary fixed positive θ and for $|s| \geq \frac{1}{2}$, we have

$$\psi(s) = \log(s) + O\left(\frac{1}{|s|}\right) = \log|t| + \frac{i\pi}{2} + O\left(\frac{\sigma}{|t|}\right), \tag{5}$$

as $|t| \to \infty$. In [7, Lemma 8], Garunkštis, Grahl and Steuding proved that there exist positive constants c_1 and c_2 such that, for $\sigma \leq 0$ and $|t| > 2$,

$$|L(\sigma + it, \chi)| > \frac{c_1 |t|^{\frac{1}{2} - \sigma}}{\log^7 t} \tag{6}$$

and

$$|L(\sigma + it, \chi)| < c_2 |t|^{\frac{1}{2} - \sigma} \log t. \tag{7}$$

Furthermore, for $t > t_0$ and $1 - \frac{c}{\log(t)} \leq \sigma \leq 2$, we have (see. [7, page. 28])

$$L'(s, \chi) = O\left(\log^2 t\right). \tag{8}$$

Using partial summation and the Pólya-Vinogradov inequality, for $t \geq t_0 > 0$ and for any $\sigma > 1$, we obtain

$$L'(s, \chi) = -\sum_{n \leq t} \frac{\chi(n) \log(n)}{n^s} + O\left(t^{-\sigma} \log t\right). \tag{9}$$

From the partial fraction decomposition of $L(s, \chi)$, we get (see Davenport book [1] or [7, Equation (19)])

$$\frac{L'}{L}(\sigma + it, \chi) \ll \log^2 |t| \quad \text{for } -1 \leq \sigma \leq 2 \text{ and } |t| \geq 2. \tag{10}$$

By the functional equation and the Phragmén-Lindelöf principle, we deduce that

$$L(\sigma + it, \chi) \ll_\epsilon \begin{cases} |t|^{\frac{1}{2} - \sigma + \epsilon} & \text{if } \sigma < 0, \\ |t|^{\frac{1}{2}(1 - \sigma) + \epsilon} & \text{if } 0 \leq \sigma \leq 1, \\ |t|^\epsilon & \text{if } \sigma > 1, \end{cases}$$

as $|t| \to \infty$ and where ϵ is an arbitrarily small positive number (this is a special case of [15, Lemma 2.1] established for functions in the Selberg class in which the Dirichlet L-functions are elements). Then, by Cauchy's integral formula, we get

$$L'(s, \chi) = \frac{1}{2\pi i} \int_{\mathcal{L}} \frac{L(s, \chi)}{(\omega - s)^2} d\omega,$$

where $\mathcal{L}$ is any arbitrarily small circle with center s. Using the last bound of $L(\sigma + it, \chi)$, it follows that

$$L'(\sigma + it, \chi) \ll_\epsilon \begin{cases} |t|^{\frac{1}{2} - \sigma + \epsilon} & \text{if } \sigma < 0, \\ |t|^{\frac{1}{2}(1 - \sigma) + \epsilon} & \text{if } 0 \leq \sigma \leq 1, \\ |t|^\epsilon & \text{if } \sigma > 1. \end{cases}$$

Furthermore, for fixed complex number a, for $-1 \le \sigma \le 2$ and $|t| \ge 1$, we have [2]

$$\frac{L'(s,\chi)}{L(s,\chi)-a} = \sum_{|t-\gamma_{a,\chi}|\le 1} \frac{1}{s-\rho_{a,\chi}} + O\left(\log q(|t|+1)\right).$$

Let b be some constant which will be given below. In view of the number of nontrivial a-points (1), we obtain for $\sigma > 1-b$

$$\frac{L'^2(\sigma+it,\chi)}{L(\sigma+it,\chi)-a} \ll |t|^{(1-\sigma)/2+\epsilon} \quad \text{as} \quad |t| \ge 2. \tag{11}$$

Let

$$\Delta(s,\chi) = \frac{\tau(\chi)}{i^{\nu}\sqrt{\pi}}\left(\frac{\pi}{q}\right)^{s} \frac{\Gamma\left(\frac{1}{2}(1-s+\nu)\right)}{\Gamma\left(\frac{1}{2}(s+\nu)\right)}.$$

In the next lemma, we obtain the approximate functional equation of $L'(s,\chi)$ in the following form (which will be sufficient for our purpose).

Lemma 1. *For $t > t_0 > 0$ and $0 \le \sigma \le 1$, we have*

$$L'(s,\chi) = -\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\chi(n)\log(n)}{n^s} - \log\left(\frac{qt}{2\pi}\right)\Delta(s,\chi)\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\overline{\chi(n)}}{n^{1-s}}$$
$$+\Delta(s,\chi)\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\overline{\chi(n)}\log n}{n^{1-s}} + O\left(t^{-\frac{\sigma}{2}}\log t\right).$$

Proof. The proof uses the same argument and similar notations as Levinson [14]. According to Lavrik [13, Corollary 1 of Theorem 1] or to Rane [18], we use the following approximate functional equation of $L(s,\chi)$, if χ mod q is a primitive character, we have [3]

$$L(s,\chi) = \sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\chi(n)}{n^s} + \Delta(s,\chi)\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\overline{\chi}(n)}{n^{1-s}} + O\left(t^{-\frac{\sigma}{2}}\right),$$

where $\Delta(s,\chi)$ can be written as follows:

$$\Delta(s,\chi) := i\tau(\chi)\chi(-1)(2\pi)^{s-1}q^{-s}\Gamma(1-s)e^{-\frac{i\pi s}{2}}.$$

Writing $L(s,\chi) = f_1(s) + \Delta(s,\chi)f_2(s)$ with $f_1(s) = \sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\chi(n)}{n^s} + O\left(t^{-\frac{\sigma}{2}}\right)$ and

$$\Delta(s,\chi)f_2(s) = L(s,\chi) - f_1(s) = \Delta(s,\chi)\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\overline{\chi}(n)}{n^{1-s}}.$$

---

[2] The proof is very closely to that stated in [8, Lemma 8] with minor change and using [17, Ch. 7, Theorem 4.1] (see also [16, Lemmas 2.4 and 2.6]).

[3] An exact expression for the error term in the approximate functional equation of $L(s,\chi)$ is stated in [13, Theorem 1 and Lemma 5] and which noted R_{xy} (see also [18, page 141]). For example, with Lavrik's notation, when $x = y = \sqrt{\frac{qt}{2\pi}}$, one can see that the error term R_{xy} satisfies $R_{xy} = O_q\left(t^{-\frac{\sigma}{2}}\right)$.

Similarly to [14, page 389], we get [4]

$$\begin{aligned} L'(s,\chi) &= f_1'(s) + \Delta'(s,\chi)f_2(s) + \Delta(s,\chi)f_2'(s) \\ &= -\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\chi(n)\log(n)}{n^s} + O\left(t^{-\frac{\sigma}{2}}\log(t)\right) \\ &\quad + \Delta'(s,\chi)f_2(s) + \Delta(s,\chi)\left[\sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\overline{\chi}(n)\log(n)}{n^{1-s}}\right]. \end{aligned}$$

Since

$$\Delta'(s,\chi)f_2(s) = \frac{\Delta'}{\Delta}(s,\chi)\Delta(s,\chi)f_2(s) = \frac{\Delta'}{\Delta}(s,\chi)\left[\Delta(s,\chi)\sum_{n\le\sqrt{\frac{qt}{2\pi}}}\frac{\overline{\chi}(n)}{n^{1-s}}\right].$$

Hence, by using that for $t > t_0$

$$\frac{\Delta'}{\Delta}(\sigma+it,\chi) = -\log\left(\frac{qt}{2\pi}\right) + O\left(\frac{1}{t}\right), \tag{12}$$

we finish the proof of Lemma 1. □

Using the approximate functional equation of $L'(s,\chi)$ given in Lemma 1, we prove easily with the same argument used by Fujii in [6, Lemma 3] the following result.

Lemma 2. *Let $\delta = 1 + \frac{1}{\log T}$. Then*

$$\int_{1-\delta}^{\delta} |L'(\sigma+iT,\chi)|d\sigma \ll \sqrt{T}\log T. \tag{13}$$

Proof. From the functional equation of $L(s,\chi)$, we have

$$L'(s,\chi) = \frac{1}{\Delta(1-s,\overline{\chi})}\left(-L'(1-s,\overline{\chi}) + \frac{\Delta'}{\Delta}(1-s,\overline{\chi})L(1-s,\overline{\chi})\right).$$

Therefore

$$\int_{1-\delta}^{\delta}|L'(\sigma+iT,\chi)|d\sigma = \int_{1-\delta}^{1/2}|L'(\sigma+iT,\chi)|d\sigma + \int_{1/2}^{\delta}|L'(\sigma+iT,\chi)|d\sigma = M_1 + M_2,$$

where

$$\begin{aligned} M_1 &= \int_{1-\delta}^{1/2}\left|\frac{1}{\Delta(1-\sigma-iT,\overline{\chi})}\left(-L'(1-\sigma-iT,\overline{\chi}) + \frac{\Delta'}{\Delta}(1-\sigma-iT,\overline{\chi})L(1-\sigma-iT,\overline{\chi})\right)\right|d\sigma \\ &= \int_{1/2}^{\delta}\left|\frac{1}{\Delta(\sigma+iT,\overline{\chi})}\left(-L'(\sigma+iT,\overline{\chi}) + \frac{\Delta'}{\Delta}(\sigma+iT,\overline{\chi})L(\sigma+iT,\overline{\chi})\right)\right|d\sigma. \end{aligned}$$

[4] Furthermore, from [13, Theorem 1 and Lemma 5] the expression R_{xy} is differentiable and $\frac{dR_{xy}}{d\sigma} = O_q\left(t^{-\frac{\sigma}{2}}\log(t)\right)$.

In any strip $\sigma_1 \le \sigma \le \sigma_2$, we have uniformly as $t \to \infty$, $\left|\frac{1}{\Delta(\sigma+iT,\overline{\chi})}\right| \ll \left(\frac{qT}{2\pi}\right)^{\sigma-1/2}$. Applying the last asymptotic formula, we obtain

$$M_1 \ll \int_{1/2}^{\delta} |L'(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma + \log T \int_{1/2}^{\delta} |L(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma.$$

Then

$$M_1 + M_2 \ll \int_{1/2}^{\delta} |L'(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma + \log T \int_{1/2}^{\delta} |L(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma.$$

Let write the first integral as follows

$$\begin{aligned} &\int_{1/2}^{\delta} |L'(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma \\ &= \int_{1/2}^{1} |L'(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma + \int_{1}^{\delta} |L'(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma \\ &= M_3 + M_4. \end{aligned}$$

Using the approximate functional equation of $L'(s,\chi)$ and the fact that $|\overline{\chi}(n)| \le 1$, we get

$$\begin{aligned} &M_3 \\ &\ll \int_{1/2}^{1} \left| \sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\log(n)}{n^s} + T^{\frac12-\sigma} \sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{1}{n^{1-s}} + T^{\frac12-\sigma}\log T \sum_{n\le\sqrt{\frac{qt}{2\pi}}} \frac{\log n}{n^{1-s}} + O\left(T^{-\frac{\sigma}{2}}\log T\right) \right| T^{\sigma-\frac12} d\sigma \\ &\ll \sqrt{T}\log T. \end{aligned}$$

Now, using another approximation of $L'(s,\chi)$ as given by equation (9) above, we get

$$\begin{aligned} M_4 &\ll \int_{1}^{\delta} \left(\sum_{n\le T} \frac{|\chi(n)|\log(n)}{n^\sigma} + O\left(T^{-\sigma}\log T\right) \right) T^{\sigma-\frac12} d\sigma \\ &\ll \sqrt{T}(\delta-1) \sum_{n\le T} \frac{\log(n)}{n} \\ &\ll \sqrt{T}\log T. \end{aligned}$$

Similarly, we get

$$\int_{1/2}^{\delta} |L(\sigma+iT,\overline{\chi})| \left(\frac{qT}{2\pi}\right)^{\sigma-1/2} d\sigma \ll \sqrt{T}.$$

Hence, we obtain the assertion of Lemma 2. □

An explicit formula for the sums

$$\sum_{\rho_\chi=\beta_\chi+i\gamma_\chi;\ 0<\gamma_\chi\le T} L'\left(\rho_\chi,\chi\right)X^{\rho_\chi},$$

where ρ_χ runs over the nontrivial zeros of $L(s,\chi)$ is stated in the following:

Lemma 3. *Let X be a positive number and $\rho_\chi=\beta_\chi+i\gamma_\chi$ denotes a nontrivial zero of the Dirichlet L-function $L(s,\chi)$. Then*

$$\begin{aligned}
&\sum_{\rho_\chi;\ 0<\gamma_\chi\le T} L'\left(\rho_\chi,\chi\right)X^{\rho_\chi}\\
=\ & -\Delta(X)\chi\left[\Delta(X)X\right]\log(X)\left\{\frac{T}{4\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{4\pi}+\frac{i\pi T}{8\pi}\right\}\\
&+\Delta(X)\chi\left[\Delta(X)X\right]\frac{T}{2\pi}\sum_{X=mn}\Lambda(n)\log(m)\\
&+\frac{X}{\sqrt{q}}\sum_{k\le\frac{qT}{2\pi X}}\log^2(k)\overline{\chi}(k)e^{2i\pi kX/q}\\
&+\frac{X\log(X)}{2\sqrt{q}}\sum_{k\le\frac{qT}{2\pi X}}\log(k)\overline{\chi}(k)e^{2i\pi kX/q}\\
&-\frac{1}{\sqrt{q}}\left(\frac{1}{2}X\log^2(X)-\frac{i\pi}{4}X\log X\right)\sum_{k\le\frac{qT}{2\pi X}}\overline{\chi}(k)e^{2i\pi kX/q}\\
&-\frac{X}{\sqrt{q}}\sum_{k\le\frac{qT}{2\pi X}}\sum_{k=mn}\Lambda(n)\overline{\chi}(n)\overline{\chi}(m)\log(m)e^{2i\pi kX/q}+O\left(\sqrt{T}\log^3 T\right),
\end{aligned}\tag{14}$$

where $\Delta(X)$ is defined by equation (3).

Proof. We apply the same argument used by Fujii in [6]. Let X be a fixed positive number, $s=\sigma+it, t\in\mathbb{R}$. Suppose that $T>t_0$, where $t_0>0$ is as in equations (8) or (9) and T is not an imaginary part of any zero of the Dirichlet L-function. We consider

$$I=\frac{1}{2i\pi}\int_{\mathbf{R}}\frac{\xi'}{\xi}(s,\chi)L'(s,\chi)X^s ds,\tag{15}$$

where $\mathbf{R}$ denotes the counterclockwise oriented rectangular with vertices $\delta+iC$, $\delta+iT$, $1-\delta+iT$ and $1-\delta+iC$, with $\delta=1+\frac{1}{\log T}$. First, we have

$$\begin{aligned}
I=\ &\frac{1}{2i\pi}\int_{\delta+iC}^{\delta+iT}\frac{\xi'}{\xi}(s,\chi)L'(s,\chi)X^s ds+\frac{1}{2i\pi}\int_{\delta+iT}^{1-\delta+iT}\frac{\xi'}{\xi}(s,\chi)L'(s,\chi)X^s ds\\
&+\frac{1}{2i\pi}\int_{1-\delta+iT}^{1-\delta+iC}\frac{\xi'}{\xi}(s,\chi)L'(s,\chi)X^s ds+\frac{1}{2i\pi}\int_{1-\delta+iC}^{\delta+iC}\frac{\xi'}{\xi}(s,\chi)L'(s,\chi)X^s ds\\
=\ &I_1+I_2+I_3+I_4.
\end{aligned}\tag{16}$$

Recall that from our choice of T, we have

$$\frac{\xi'}{\xi}(\sigma+it,\chi) \ll \log^2 T \quad \text{for} \quad -1\le\sigma\le 2. \tag{17}$$

Therefore, by using Lemma 2, we deduce that

$$I_2+I_4 \ll \log^2 T\int_{1-\delta}^{\delta}\left(|L'(\sigma+iC,\chi)+|L'(\sigma+iT,\chi)|\right)X^\sigma d\sigma \ll \sqrt{T}\log^3 T. \tag{18}$$

Now, we estimate I_1. We have

$$\begin{aligned}
I_1 &= \frac{1}{2\pi}\int_C^T \frac{\xi'}{\xi}(\delta+it,\chi)L'(\delta+it,\chi)X^{\delta+it}dt\\
&= \frac{1}{2\pi}\int_C^T\left\{\frac12\log\left(\frac{q}{\pi}\right)+\frac12\psi\left(\frac{\delta+\nu+it}{2}\right)+\frac{L'}{L}(\delta+it,\chi)\right\}L'(\delta+it,\chi)X^{\delta+it}dt.
\end{aligned}$$

Formula (5) yields to

$$\begin{aligned}
I_1 &= -\frac{1}{2\pi}\int_C^T\left\{\frac12\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}-\sum_{n=1}^\infty\frac{\chi(n)\Lambda(n)}{n^{\delta+it}}+O\left(\frac1t\right)\right\}\sum_{m=1}^\infty\frac{\chi(m)\log(m)}{m^{\delta+it}}X^{\delta+it}dt\\
&= -\frac{X^\delta}{2\pi}\sum_{m=1}^\infty\frac{\chi(m)\log(m)}{m^\delta}\int_C^T\left\{\frac12\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}+O\left(\frac1t\right)\right\}\left(\frac{X}{m}\right)^{it}dt\\
&\quad +\frac{X^\delta}{2\pi}\sum_{m=1}^\infty\frac{\chi(m)\log(m)}{m^\delta}\sum_{n=1}^\infty\frac{\chi(n)\Lambda(n)}{n^\delta}\int_C^T\left(\frac{X}{mn}\right)^{it}dt = J_1+J_2,
\end{aligned}\tag{19}$$

where

$$\begin{aligned}
J_1 &= -\frac{X^\delta}{2\pi}\sum_{m=1}^\infty\frac{\chi(m)\log(m)}{m^\delta}\int_C^T\left\{\frac12\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}+O\left(\frac1t\right)\right\}\left(\frac{X}{m}\right)^{it}dt\\
&= -\Delta(X)\frac{\chi\left[\Delta(X)X\right]\log(X)}{2\pi}\int_C^T\left\{\frac12\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}\right\}dt\\
&\quad -\frac{X^\delta}{2\pi}\sum_{\substack{m=1\\ m\ne X}}^\infty\frac{\chi(m)\log(m)}{m^\delta}\int_C^T\left\{\frac12\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}\right\}\left(\frac{X}{m}\right)^{it}dt+O\left(\log^3 T\right)\\
&= J_3+J_4+O\left(\log^3 T\right)
\end{aligned}$$

with

$$\begin{aligned}
J_3 &= -\Delta(X)\frac{\chi\left[\Delta(X)X\right]\log(X)}{2\pi}\int_C^T\left\{\frac12\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}\right\}dt\\
&= -\Delta(X)\chi\left[\Delta(X)X\right]\log(X)\left\{\frac{T}{4\pi}\log\left(\frac{qT}{2\pi}\right)-\frac{T}{4\pi}+\frac{i\pi}{4}\frac{T}{2\pi}\right\}+O(1)
\end{aligned}$$

and

$$J_4 = -\frac{X^\delta}{2\pi} \sum_{\substack{m=1 \\ m \neq X}}^{\infty} \frac{\chi(m)\log(m)}{m^\delta} \int_C^T \left\{ \frac{1}{2} \log\left(\frac{qt}{2\pi}\right) + \frac{i\pi}{4} \right\} \left(\frac{X}{m}\right)^{it} dt$$

$$\ll \begin{cases} X^\delta \sum_{\substack{m=1 \\ m\neq X}}^{\infty} \frac{\log m}{m^\delta} \min\left(T\log T, \log T\left(1 + \left|\log \frac{X}{m}\right|^{-1}\right)\right) & \text{if } \ X \geq 1, \\ \sum_{m=1}^{\infty} \frac{\log m}{m^\delta} & \text{if } \ 0 < X < 1. \end{cases}$$

Hence

$$J_4 \ll \begin{cases} \log^3 T & \text{if } \ X \geq 1, \\ \log T & \text{if } \ 0 < X < 1. \end{cases}$$

Therefore

(20)

$$J_1 = -\Delta(X)\chi\left[\Delta(X)X\right]\log(X)\left\{\frac{T}{4\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{T}{4\pi} + \frac{i\pi}{4}\frac{T}{2\pi}\right\} + O\left(\log^3 T\right).$$

On the other hand, we have

$$\begin{aligned} J_2 &= \frac{X^\delta}{2\pi} \sum_{k=1}^{\infty} \frac{\chi(k)}{k^\delta} \sum_{k=mn} \Lambda(n)\log(m) \int_C^T \left(\frac{X}{k}\right)^{it} dt \\ &= \frac{X^\delta}{2\pi} T \sum_{\substack{k=1 \\ X=k}}^{\infty} \frac{\chi(k)}{k^\delta} \sum_{X=mn} \Lambda(n)\log(m) \\ &\quad + \frac{X^\delta}{2\pi} \sum_{\substack{k=1 \\ k\neq X}}^{\infty} \frac{\chi(k)}{k^\delta} \sum_{k=mn} \Lambda(n)\log(m) \int_C^T \left(\frac{X}{k}\right)^{it} dt = J_5 + J_6, \end{aligned}$$

where

$$J_5 = \Delta(X)\frac{T}{2\pi}\chi\left[\Delta(X)X\right] \sum_{X=mn} \Lambda(n)\log(m) + O(1) \tag{21}$$

and

$$J_6 \ll \begin{cases} X^\delta \sum_{\substack{k=1 \\ X\neq k}} \frac{1}{k^\delta} \sum_{k=mn} \Lambda(n)\log(m) \min\left(T, \frac{1}{\left|\log \frac{X}{k}\right|}\right) & \text{if } \ X \geq 1, \\ \sum_{k=1}^{\infty} \frac{1}{k^\delta} \sum_{nm=k} \Lambda(n)\log(m) \frac{1}{\log k} & \text{if } \ 0 < X < 1, \end{cases}$$

$$\ll \begin{cases} \log^3 T & \text{if } \ X \geq 1, \\ \log^2 T & \text{if } \ 0 < X < 1. \end{cases} \tag{22}$$

Combining the two last equations (21) and (22), we get

$$J_2 = \Delta(X)\frac{T}{2\pi}\chi\left[\Delta(X)X\right] \sum_{X=mn} \Lambda(n)\log(m) + O\left(\log^3 T\right). \tag{23}$$

Therefore, from (20) and (23), we obtain

$$I_1 = -\Delta(X)\chi\left[\Delta(X)X\right]\log(X)\left\{\frac{T}{4\pi}\log\left(\frac{qT}{2\pi}\right) - \frac{T}{4\pi} + \frac{i\pi}{4}\frac{T}{2\pi}\right\}$$

$$+\Delta(X)\frac{T}{2\pi}\chi\left[\Delta(X)X\right]\sum_{X=mn}\Lambda(n)\log(m)+O\left(\log^3 T\right). \tag{24}$$

Next, we shall evaluate I_3. First, we note that

$$\begin{aligned} L'(s,\chi) &= \left(\frac{1}{\Delta\left(1-s,\overline{\chi}\right)}L\left(1-s,\overline{\chi}\right)\right)' \\ &= \frac{1}{\Delta\left(1-s,\overline{\chi}\right)}\left(-L'\left(1-s,\overline{\chi}\right)+\frac{\Delta'}{\Delta}\left(1-s,\overline{\chi}\right)L\left(1-s,\overline{\chi}\right)\right). \end{aligned}$$

Moreover, by formula (4), we get $\frac{\xi'}{\xi}(s,\chi)=-\frac{\xi'}{\xi}\left(1-s,\overline{\chi}\right)$. Hence

$$\begin{aligned} I_3 &= -\frac{1}{2i\pi}\int_{1-\delta+iC}^{1-\delta+iT}-\frac{\xi'}{\xi}\left(1-s,\overline{\chi}\right)\left(-L'\left(1-s,\overline{\chi}\right)+\frac{\Delta'}{\Delta}\left(1-s,\overline{\chi}\right)L\left(1-s,\overline{\chi}\right)\right)\frac{X^s}{\Delta\left(1-s,\overline{\chi}\right)}ds \\ &= \frac{1}{2\pi}\int_C^T\frac{\xi'}{\xi}\left(\delta-it,\overline{\chi}\right)\left(-L'\left(\delta-it,\overline{\chi}\right)+\frac{\Delta'}{\Delta}\left(\delta-it,\overline{\chi}\right)L\left(\delta-it,\overline{\chi}\right)\right)\frac{X^{1-\delta+it}}{\Delta\left(\delta-it,\overline{\chi}\right)}dt. \end{aligned}$$

By complex conjugation, we obtain

$$\begin{aligned} \overline{I_3} &= \frac{X^{1-\delta}}{2\pi}\int_C^T\frac{\xi'}{\xi}\left(\delta+it,\chi\right)\left(-L'\left(\delta+it,\chi\right)+\frac{\Delta'}{\Delta}\left(\delta+it,\chi\right)L\left(\delta+it,\chi\right)\right)\frac{X^{-it}}{\Delta\left(\delta+it,\chi\right)}dt \\ &= \frac{X^{1-\delta}}{2\pi}\int_C^T\left\{\frac{1}{2}\log\left(\frac{qt}{2\pi}\right)+\frac{i\pi}{4}-\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^{\delta+it}}+O\left(\frac{\log t}{t}\right)\right\} \\ &\quad\times\left\{\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta+it}}-\log\left(\frac{qt}{2\pi}\right)\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta+it}}+O\left(\frac{\log t}{t}\right)\right\}\frac{X^{-it}}{\Delta(\delta+it,\chi)}dt. \end{aligned}$$

Let us write $\overline{I_3}$ as follows

$$\begin{aligned} \overline{I_3} &= \frac{X^{1-\delta}}{2\pi}\int_C^T\frac{1}{2}\log\left(\frac{qt}{2\pi}\right)\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta+it}}\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eX}\right)}dt \\ &\quad-\frac{X^{1-\delta}}{2\pi}\int_C^T\frac{1}{2}\log^2\left(\frac{qt}{2\pi}\right)\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta+it}}\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eX}\right)}dt \\ &\quad+\frac{X^{1-\delta}}{2\pi}\int_C^T\frac{i\pi}{4}\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta+it}}\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eX}\right)}dt \\ &\quad-\frac{X^{1-\delta}}{2\pi}\int_C^T\frac{i\pi}{4}\log\left(\frac{qt}{2\pi}\right)\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta+it}}\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eX}\right)}dt \\ &\quad-\frac{X^{1-\delta}}{2\pi}\int_C^T\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^{\delta+it}}\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta+it}}\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eX}\right)}dt \\ &\quad+\frac{X^{1-\delta}}{2\pi}\int_C^T\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^{\delta+it}}\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta+it}}\log\left(\frac{qt}{2\pi}\right)\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eX}\right)}dt \\ &\quad+O\left(\log^3 T\right) \end{aligned}$$

$$= H_1 + H_2 + H_3 + H_4 + H_5 + H_6 + O\left(\log^3 T\right).$$

Each of the above integrals will be evaluated by the method used by Gonek [9, Lemma 5, page 131] or Levinson [14]. First, we have [5]

$$\begin{aligned}
H_1 &= \frac{X^{1-\delta}}{2\pi}\frac{1}{2}\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta}}\int_C^T \log\left(\frac{qt}{2\pi}\right)\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}} e^{-\frac{i\pi}{4}} e^{it\log\left(\frac{qt}{2\pi eXm}\right)}dt\\
&= \frac{X}{2\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)\log(m)\log(mX)e^{-2i\pi mX/q} + O\left(\sqrt{T}\log^3 T\right)\\
&= \frac{X}{2\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)\log^2(m)e^{-2i\pi mX/q}\\
&\quad + \frac{X\log(X)}{2\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)\log(m)e^{-2i\pi mX/q} + O\left(\sqrt{T}\log^3 T\right).
\end{aligned}$$

To estimate H_2 we proceed as follows

$$\begin{aligned}
H_2 &= -\frac{1}{2}\frac{X^{1-\delta}}{2\pi}\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta}}\int_C^T \log^2\left(\frac{qt}{2\pi}\right)\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}} e^{-\frac{i\pi}{4}} e^{it\log\left(\frac{qt}{2\pi eXm}\right)}dt\\
&= -\frac{X}{2\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)\log^2(mX)e^{-2i\pi mX/q} + O\left(\sqrt{T}\log^3 T\right)\\
&= -\frac{X}{2\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)\log^2(m)e^{-2i\pi mX/q}\\
&\quad -\frac{X\log^2(X)}{2\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)e^{-2i\pi mX/q}\\
&\quad -\frac{X\log(X)}{\sqrt{q}}\sum_{m\le\frac{qT}{2\pi X}}\chi(m)\log(m)e^{-2i\pi mX/q} + O\left(\sqrt{T}\log^3 T\right).
\end{aligned}$$

---

[5]The estimation of H_1 is based on the calculation of the integral j_m

$$j_m = \int_C^T g(t)e^{i2\pi f(t)}dt,\ g(t) = \left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}},\ f(t) = \frac{t}{2\pi}\log\left(\frac{qt}{2\pi eXm}\right).$$

The saddle point $t_0 = \frac{2\pi Xm}{q}$ belongs to the segment $C \le t \le T$ for $m \le \frac{qt}{2\pi X}$. For such m, the main term of the asymptotic formula of j_m has the form

$$e^{i\pi/4}\frac{g(t_0)e^{2\pi f(t_0)}}{\sqrt{f''(t_0)}} = 2\pi e^{i\pi/4}\frac{(Xm)^{\delta}}{\sqrt{q}}\log(mX)e^{-2\pi iXm/q}.$$

For H_3, we have

$$H_3 = \frac{X^{1-\delta}}{2\pi}\frac{i\pi}{4}\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta}}\int_C^T\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eXm}\right)}dt$$
$$= \frac{i\pi}{4}\frac{X}{\sqrt{q}}\sum_{m\leq\frac{qT}{2\pi X}}\chi(m)\log(m)e^{-2i\pi mX/q} + O\left(\sqrt{T}\log^3 T\right).$$

Similarly, we obtain

$$H_4 = -\frac{X^{1-\delta}}{2\pi}\frac{i\pi}{4}\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta}}\int_C^T\log\left(\frac{qt}{2\pi}\right)\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eXm}\right)}dt$$
$$= -\frac{i\pi}{4}\frac{X\log X}{\sqrt{q}}\sum_{m\leq\frac{qT}{2\pi X}}\chi(m)e^{-2i\pi mX/q} - \frac{i\pi}{4}X\sum_{m\leq\frac{qT}{2\pi X}}\chi(m)\log(m)e^{-2i\pi mX/q}$$
$$- O\left(\sqrt{T}\log^3 T\right).$$

For H_5, one has

$$H_5 = -\frac{X^{1-\delta}}{2\pi}\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^{\delta}}\sum_{m=1}^{\infty}\frac{\chi(m)\log(m)}{m^{\delta}}\int_C^T\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eXmn}\right)}dt$$
$$= -\frac{X}{\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\sum_{k=mn}\Lambda(n)\chi(n)\chi(m)\log(m)e^{-2i\pi kX/q} + O\left(\sqrt{T}\log^3 T\right).$$

Finally, for H_6 we get

$$H_6 = \frac{X^{1-\delta}}{2\pi}\sum_{n=1}^{\infty}\frac{\chi(n)\Lambda(n)}{n^{\delta}}\sum_{m=1}^{\infty}\frac{\chi(m)}{m^{\delta}}\int_C^T\log\left(\frac{qt}{2\pi}\right)\left(\frac{qt}{2\pi}\right)^{\delta-\frac{1}{2}}e^{-\frac{i\pi}{4}}e^{it\log\left(\frac{qt}{2\pi eXmn}\right)}dt$$
$$= \frac{X\log X}{\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\sum_{k=mn}\Lambda(n)\chi(n)\chi(m)\log(k)e^{-2i\pi kX/q} + O\left(\sqrt{T}\log^3 T\right).$$
$$= \frac{X}{\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\chi(k)\log^2(k)e^{-2i\pi kX/q} + \frac{X\log X}{\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\chi(k)\log(k)e^{-2i\pi kX/q}$$
$$+ O\left(\sqrt{T}\log^3 T\right).$$

Collecting together the above results on $H_1, \ldots, H_5$ and H_6, we obtain

$$I_3 = \frac{X}{\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\log^2(k)\overline{\chi}(k)e^{2i\pi kX/q} + \frac{X\log(X)}{2\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\log(k)\overline{\chi}(k)e^{2i\pi kX/q}$$
$$-\left(\frac{X\log^2(X)}{2\sqrt{q}} - \frac{i\pi}{4}\frac{X\log X}{\sqrt{q}}\right)\sum_{k\leq\frac{qT}{2\pi X}}\overline{\chi}(k)e^{2i\pi kX/q}$$

$$-\frac{X}{\sqrt{q}}\sum_{k\leq\frac{qT}{2\pi X}}\sum_{k=mn}\Lambda(n)\overline{\chi}(n)\overline{\chi}(m)\log(m)e^{2i\pi kX/q}+O\left(\sqrt{T}\log^3 T\right). \tag{25}$$

Finally, by using equations (24) and (25), we finish the proof of Lemma 3. □

3. Proof of Theorem 1

Let X be a fixed positive real number and a be a complex number. We write $s=\sigma+it$, $\rho_{a,\chi}=\beta_{a,\chi}+i\gamma_{a,\chi}$ with real $\sigma, t, \beta_{a,\chi}$ and $\gamma_{a,\chi}$. By the theorem of residues (or Cauchy's theorem), we get

$$\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\leq T} L'\left(\rho_{a,\chi},\chi\right)X^{\rho_{a,\chi}}=\frac{1}{2i\pi}\int_{\mathbf{R}}\frac{L'^2(s,\chi)}{L(s,\chi)-a}X^s ds, \tag{26}$$

where the integration is taken over a rectangular contour in counterclockwise direction denoted by $\mathbf{R}$ according to the location of the nontrivial a-points of $L(s,\chi)$ which will be specified below. In view of formula (1), the ordinates of the a-points cannot lie too dense. For any large $T_0\geq 0$, we can find a real number $T\in[T_0,T_0+1[$ such that $\min_{\rho_{a,\chi}}|T-\gamma_{a,\chi}|\gg\frac{1}{\log T}$. We shall distinguish the case $a\neq 1$ and $a=1$. Let us suppose that $a\neq 1$. We may choose the counterclockwise oriented rectangular $\mathbf{R}$ with vertices $1-b+i$, $B+i$, $B+iT$, $1-b+iT$, where B is a large constant which will be chosen below and $b=1+\frac{1}{\log T}$, at the expense of a small error for disregarding at most finitely many nontrivial a-points below $Im(s)=1$ and for counting finitely many trivial a-points to the right of $Re(s)=1-b$. Then, we have

$$\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\leq T} L'\left(\rho_{a,\chi},\chi\right)X^{\rho_{a,\chi}}=\frac{1}{2i\pi}\int_{\mathbf{R}}\frac{L'^2(s,\chi)}{L(s,\chi)-a}X^s ds+O(1).$$

Hence

$$\begin{aligned}&\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\leq T} L'\left(\rho_{a,\chi},\chi\right)X^{\rho_{a,\chi}}\\&=\frac{1}{2i\pi}\left\{\int_{B+i}^{B+iT}+\int_{B+iT}^{1-b+iT}+\int_{1-b+iT}^{1-b+i}\right\}\frac{L'^2(s,\chi)}{L(s,\chi)-a}X^s ds\\&\quad+\frac{1}{2i\pi}\int_{1-b+i}^{B+i}\frac{L'^2(s,\chi)}{L(s,\chi)-a}X^s ds+O(1)\\&=I_1+I_2+I_3+I_4+O(1).\end{aligned}$$

It is easy to see from equation (11) that $I_2, I_4\ll T^{\frac{1}{2}+\epsilon}$. Now, let us estimate the two integrals I_1 and I_3. Recall that, for $\sigma\longrightarrow\infty$, we have $L(s,\chi)=1+o(1)$ and $L'(s,\chi)\ll 2^{-\sigma}$ uniformly in t. Hence, there are no a-points for sufficiently large σ provided that $a\neq 1$. For the case $a=1$, we define $m=\min\{n\geq 2;\chi(n)\neq 0\}$. We observe, for $\sigma\longrightarrow+\infty$, $L(s,\chi)-1=\frac{\chi(m)}{m^{\sigma+it}}(1+o(1))$. Hence, in both cases, $a\neq 0$ or $a=1$, we choose B a fixed constant sufficiently large such that there are no a-points of $L(s,\chi)$ in the half-plane $\sigma > B-1$ (see [7, Equations (20) and (21)]). Therefore, we deduce that

$$\int_B^{B+iT} \frac{L'^2(s,\chi)}{L(s,\chi)-a} X^s ds \ll X^B T^{-2\log 2} \log T \ll T^{-2\log 2} \log T.$$

Then

$$\sum_{\rho_{a,\chi};\ 0<\gamma_{a,\chi}\le T} L'(\rho_{a,\chi},\chi) X^{\rho_{a,\chi}} = -\frac{1}{2i\pi}\int_{1-b}^{1-b+iT} \frac{L'^2(s,\chi)}{L(s,\chi)-a} X^s ds + O\left(T^{\frac{1}{2}+\epsilon}\right).$$

It remains to evaluate the integral over the left vertical line segment $[1-b, 1-b+iT]$ of the rectangular $\mathbf{R}$. Using the same argument as above, for $s \in [1-b+iA, 1-b+iT]$ where $A = A(a) > 0$, we get the geometric series expansion

$$\frac{L'(s,\chi)}{L(s,\chi)-a} = \frac{L'}{L}(s,\chi)\left\{1 + \frac{a}{L(s,\chi)} + \sum_{k=2}^{\infty}\left(\frac{a}{L(s,\chi)}\right)^k\right\}.$$

Since integration over $[1-b+i, 1-b+iA]$ yields to a bounded error term, the integral I_3 becomes

$$\begin{aligned} &I_3 \\ &= \frac{1}{2i\pi}\int_{1-b+iT}^{1-b+iA}\left\{\frac{L'^2}{L}(s,\chi)X^s + a\left(\frac{L'}{L}(s,\chi)\right)^2 X^s + \frac{L'^2}{L}(s,\chi)\sum_{k=2}^{+\infty}\left(\frac{a}{L(s,\chi)}\right)^k X^s\right\} ds + O(1) \\ &= J_1 + J_2 + J_3 + O(1). \end{aligned}$$

In order to estimate the third integral J_3, we use equations (6) and (10) (similar computation was done in [7, integral $\mathcal{J}_3$ page 30]) to obtain

$$\begin{aligned} J_3 &= -\frac{aX^{1-b}}{2\pi}\int_A^T \left(\frac{L'}{L}(1-b+it,\chi)\right)^2 \sum_{l=1}^{+\infty}\left(\frac{a}{L(1-b+it,\chi)}\right)^l X^{it} dt \\ &\ll T^{\frac{1}{2}+\epsilon}, \quad \epsilon > 0. \end{aligned}$$

Next, let us consider the integral J_2. By the functional equation satisfied by $L(s,\chi)$, we write the integral J_2 as

$$\begin{aligned} J_2 &= -\frac{a}{2i\pi}\int_{1-b+iA}^{1-b+iT}\left(\frac{\Delta'}{\Delta}(s,\chi) - \frac{L'}{L}(1-s,\overline{\chi})\right)^2 X^s ds \\ &= -\frac{a}{2i\pi}\int_{1-b+iA}^{1-b+iT}\left\{\left(\frac{\Delta'}{\Delta}(s,\chi)\right)^2 - 2\frac{\Delta'}{\Delta}(s,\chi)\frac{L'}{L}(1-s,\overline{\chi}) + \left(\frac{L'}{L}(1-s,\overline{\chi})\right)^2\right\} X^s ds \\ &= N_1 + N_2 + N_3. \end{aligned}$$

We have

$$N_1 = -\frac{a}{2\pi}X^{1-b}\int_A^T\left(-\log\left(\frac{qt}{2\pi}\right) + O\left(\frac{1}{t}\right)\right)^2 X^{it} dt$$

$$= -\frac{a}{2\pi}X^{1-b}\int_A^T\left(\log^2\left(\frac{qt}{2\pi}\right)+O\left(\frac{\log\left(\frac{qt}{2\pi}\right)}{t}\right)+O\left(\frac{1}{t}\right)\right)X^{it}dt$$
$$= -\frac{aT}{2\pi}\log^2\frac{qT}{2\pi}+\frac{aT}{\pi}\log\frac{qT}{2\pi}-\frac{aT}{\pi}+O\left(\log^2 T\right).$$

Furthermore

$$N_2 = \frac{a}{i\pi}\int_{1-b+iA}^{1-b+iT}\frac{\Delta'}{\Delta}(s,\chi)\frac{L'}{L}\left(1-s,\overline{\chi}\right)X^s ds$$
$$= \frac{a}{\pi}\int_A^T\frac{\Delta'}{\Delta}(1-b+it,\chi)\frac{L'}{L}\left(b-it,\overline{\chi}\right)X^{1-b+it}dt$$
$$= \frac{a}{\pi}\int_A^T\left(\log\left(\frac{qt}{2\pi}\right)+O\left(\frac{1}{t}\right)\right)\sum_{n=2}^{\infty}\frac{\overline{\Lambda}(n)\overline{\chi}(n)}{n^{b-it}}X^{1-b+it}dt$$
$$= \frac{a}{\pi}X^{1-b}\sum_{n=2}^{\infty}\frac{\overline{\Lambda}(n)\overline{\chi}(n)}{n^b}\int_A^T\left(\log\left(\frac{qt}{2\pi}\right)+O\left(\frac{1}{t}\right)\right)e^{it\log(Xn)}dt.$$

An integration by parts yields to

$$\int_A^T\left(\log\left(\frac{qt}{2\pi}\right)+O\left(\frac{1}{t}\right)\right)e^{it\log(Xn)}dt = O\left(\log T\right).$$

So

$$N_2 \ll \log T\sum_{n=2}^{\infty}\frac{\overline{\Lambda(n)\chi(n)}}{n^b}X^{1-b} \ll \log T\left|\frac{L'}{L}\left(b,\overline{\chi}\right)\right|X^{1-b} \ll \log^3 T.$$

The same argument gives

$$N_3 = -\frac{a}{2i\pi}\int_{1-b+iA}^{1-b+iT}\left(\frac{L'}{L}\left(1-s,\overline{\chi}\right)\right)^2 X^s ds$$
$$= -\frac{a}{2\pi}\int_A^T\left(\frac{L'}{L}\left(b-it,\overline{\chi}\right)\right)^2 X^{1-b+it}dt$$
$$= -\frac{a}{2\pi}X^{1-b}\sum_{m,n=2}^{+\infty}\frac{\overline{\Lambda}(m)\overline{\Lambda}(n)}{(mn)^b}\int_A^T e^{it\log(Xmn)}dt$$
$$\ll \sum_{m,n=2}^{+\infty}\frac{\overline{\Lambda}(m)\overline{\Lambda}(n)}{(mn)^b}\left|\int_A^T e^{it\log(Xmn)}\right|dt \ll \log^2 T.$$

From the above estimations of $N_1,\ N_2$ and N_3, we obtain

$$J_2 = -\frac{aT}{2\pi}\log^2\frac{qT}{2\pi}+\frac{aT}{\pi}\log\frac{qT}{2\pi}-\frac{aT}{\pi}+O\left(\log^3 T\right).$$

Finally, to end the proof of Theorem 1, it remains to evaluate J_1 given by

$$J_1 = \frac{1}{2i\pi}\int_{1-b+iT}^{1-b+iA}\frac{L'^2}{L}(s,\chi)X^s ds$$

which can be evaluated, firstly, up to an error term as a contour integral

$$J_1 = \frac{1}{2i\pi}\int_R \frac{L'^2}{L}(s,\chi)X^s ds + O\left(T^{1/2+\epsilon}\right)$$

and, secondly, as a sum of residues

$$J_1 = \sum_{0<\gamma_{a,\chi}<T} L'(\rho_\chi,\chi)X^{\rho_\chi} + O\left(T^{1/2+\epsilon}\right),$$

where $\rho_\chi = \beta_\chi + i\gamma_\chi$ stands for nontrivial zeros of $L(s,\chi)$. To finish the proof of Theorem 1 for $a \neq 1$, we note that the last sum of residues was evaluated in Lemma 3.

For $a = 1$, we consider the function $l(s) = q^s(L(s,\chi)-1)$ in place of $L(s,\chi)-a$. Furthermore, we have $\frac{l'}{l}(s) = \log q + \frac{L'(s,\chi)}{L(s,\chi)-1}$. This implies that the constant term does not contribute by integration over a closed contour and we use the same argument as in the case $a \neq 1$.

4. Concluding remarks

We believe that we can extend our result to higher derivatives of $L(s,\chi)$. The a-points of an L-function $L(s)$ are the roots of the equation $L(s) = a$. We refer to Steuding book [20, chapter 7] for some results about a-points of L-functions from the Selberg class. Therefore, it is an interesting question to extend Theorem 1 to other classes of Dirichlet L-functions (the Selberg class [19] with some further conditions) and its higher derivatives and to refine the error term in Theorem 1 under the Riemann hypothesis. These problems will be considered in a sequel to this article.

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