Bull. Korean Math. Soc. **54** (2017), No. 4, pp. 1123–1139
https://doi.org/10.4134/BKMS.b150862
pISSN: 1015-8634 / eISSN: 2234-3016

RING STRUCTURES CONCERNING FACTORIZATION MODULO RADICALS

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Abstract. The aim in this note is to describe some classes of rings in relation to factorization by prime radical, upper nilradical, and Jacobson radical. We introduce the concepts of *tpr* ring, *tunr* ring, and *tjr* ring in the process, respectively. Their ring theoretical structures are investigated in relation to various sorts of factor rings and extensions. We also study the structure of noncommutative tpr (tunr, tjr) rings of minimal order, which can be a base of constructing examples of various ring structures. Various sorts of structures of known examples are studied in relation with the topics of this note.

1. Structure of tpr rings

Throughout this note every ring is an associative ring with identity unless otherwise stated. Let R be a ring. The polynomial (resp., power series) ring with an indeterminate x over R is denoted by $R[x]$ (resp., $R[[x]]$) and for any polynomial (resp., power series) $f(x)$ in $R[x]$ (resp., $R[[x]]$), let $C_{f(x)}$ denote the set of all coefficients of $f(x)$. Use the notation that $\bar{R} = R/I$ and $\bar{r} = r + I$, where I is an ideal of R. $\mathbb{Z}$ ($\mathbb{Z}_n$) denotes the ring of integers (modulo n). Denote the n by n full (resp., upper triangular) matrix ring over R by $\text{Mat}_n(R)$ (resp., $U_n(R)$). Use E_{ij} for the matrix with (i, j)-entry 1 and zeros elsewhere. Following the literature, $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and $N_n(R) = \{(b_{ij}) \in D_n(R) \mid b_{11} = \cdots = b_{nn} = 0\}$.

Use $I(R)$ for the set of all nontrivial (i.e., nonzero nonunit) idempotents of R, and set $I_e(R) = I(R) \cup \{0, 1\}$. A nilpotent element is also said to be a *nilpotent* for simplicity. Let $J(R)$, $N_0(R)$, $N_*(R)$, $N^*(R)$, and $N(R)$ to denote the Jacobson radical, the Wedderburn radical (i.e., sum of all nilpotent ideals), the prime radical, the upper nilradical (i.e., sum of all nil ideals), and the set of all nilpotents in R (possibly without identity), respectively. It is well-known that $N^*(R) \subseteq J(R)$ and $N_0(R) \subseteq N_*(R) \subseteq N^*(R) \subseteq N(R)$. A ring R is usually called *semiprimitive* if $J(R) = 0$.

Received October 22, 2015; Revised June 18, 2016; Accepted July 20, 2016.
2010 *Mathematics Subject Classification.* 16N40, 16S36.
Key words and phrases. tpr ring, tunr ring, tjr ring, polynomial ring, factor ring, noncommutative ring of minimal order, nilradical, Jacobson radical.

Let R be a ring (not necessarily with identity) and consider the condition

$(*)$ $\quad a_i \in N_*(R)$ or $b_i \in N_*(R)$ for each i whenever $f(x)g(x) \in N_*(R)[x]$

for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, where we let $m = n$ by using zero coefficients if necessary. Amitsur in [2, Theorem 3] proved that $N_*(R[x]) = N_*(R)[x]$ for any ring R.

It is easy to find examples which do not satisfy the condition $(*)$ (e.g., $f(x) = (1,0)$ and $g(x) = (0,1)$ in $(\mathbb{Z} \oplus \mathbb{Z})[x]$, noting $N_*(\mathbb{Z} \oplus \mathbb{Z}) = 0$). But every ring does not satisfy the condition $(*)$, so the following definition makes sense.

Definition 1.1. A ring R (not necessarily with identity) is called *trivializable over prime radical* (simply, *tpr*) if it satisfies the condition $(*)$.

The name of tpr is originated from Cohn [8]. Every domain R is clearly tpr by the fact that $fg = 0$ for $f, g \in R[x]$, implies $f = 0$ or $g = 0$. We provide tpr rings which are not domains as follows. A ring is usually said to be *reduced* if it has no nonzero nilpotents. Following the literature, a ring is called *Abelian* if every idempotent is central. Reduced rings are easily shown to be Abelian.

Lemma 1.2. (1) *A ring R is tpr if and only if $R/N_*(R)$ is a domain.*

(2) *Let R be a domain and $E_0 = D_m(R)$ for $m \geq 1$. Then $D_n(E_0)$ is a tpr ring for all $n \geq 1$.*

(3) *Let R be a tpr ring. Then $I_e(R) = \{0, 1\}$, i.e., $I(R)$ is empty.*

(4) *If R is a tpr ring, then $N(R) = N_*(R) = N^*(R)$.*

(5) *Let R be a tpr ring. Then R is semiprime if and only if R is reduced if and only if R is a domain.*

(6) *The class of tpr rings is closed under subrings.*

Proof. (1) Let R be a ring and $ab \in N_*(R)$ for $a, b \in R$. Then $a \in N_*(R)$ or $b \in N_*(R)$ because R is tpr. The sufficiency comes from the fact that $\frac{R}{N_*(R)}[x] \cong \frac{R[x]}{N_*(R)[x]}$. Indeed, suppose that $R/N_*(R)$ is a domain and let $f(x)g(x) \in N_*(R)[x]$ for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{m} b_j x^j \in R[x]$. Then $f(x) \in N_*(R)[x]$ or $g(x) \in N_*(R)[x]$ because $\frac{R[x]}{N_*(R)[x]}$ is a domain.

(2) Let $E_0 = D_m(R)$ and $E = D_n(E_0)$ for $m, n \geq 1$. Then

$$N_*(E) = \{(a_{ij}) \in E \mid a_{ii} \in N_m(R) \text{ for all } i\},$$

so we have $E/N_*(E) \cong R$ is a domain. Thus E is a tpr ring by (1).

(3) Let $e \in I_e(R)$ and consider the equality $e(1-e) = 0$. Since R is tpr, $e \in N_*(R)$ or $1 - e \in N_*(R)$. Thus $e = 0$ or $e = 1$.

(4) is shown by (1).

(5) is an immediate consequence of (1) and (4), noting that reduced rings are semiprime.

(6) Let R be a tpr ring and S be a subring of R. Then $N(R) = N_*(R)$ by (4), so we get $N_*(S) = N_*(R) \cap S$ from $N_*(R) \cap S \subseteq N_*(S) \subseteq N(S) = N(R) \cap S = N_*(R) \cap S$.

Let $f(x)g(x) \in N_*(S)[x]$ for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in S[x]$. Then $f(x)g(x) \in N_*(R)[x]$. But, since R is tpr, $a_i \in N_*(R)$ or $b_i \in N_*(R)$. This implies that $a_i \in N_*(S)$ or $b_i \in N_*(S)$ because $N_*(S) = N_*(R) \cap S$. Thus S is tpr. □

Lemma 1.2(1) implies that $R/N_*(R)$ is a reduced ring when a given ring R is tpr. Every tpr ring is Abelian by Lemma 1.2(3). It is possible to give examples of Abelian rings but not tpr, by help of [17, Lemma 2]. Recall that for each $e \in I_e(D_n(R))$ is of the form $\begin{pmatrix} f & 0 & 0 & \cdots & 0 \\ 0 & f & 0 & \cdots & 0 \\ 0 & 0 & f & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f \end{pmatrix}$ with $f \in I_e(R)$ by [17, Lemma 2], where R is an Abelian ring. So if R is a domain, $I_e(D_n(R)) = \{0, 1\}$, noting that R is tpr by Lemma 1.2(1).

Let next $R = \mathbb{Z}_6$. Then $3^2 = 3, 4^2 = 4 \in I(\mathbb{Z}_6)$ and $\begin{pmatrix} 3 & 0 & 0 & \cdots & 0 \\ 0 & 3 & 0 & \cdots & 0 \\ 0 & 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 3 \end{pmatrix}$, $\begin{pmatrix} 4 & 0 & 0 & \cdots & 0 \\ 0 & 4 & 0 & \cdots & 0 \\ 0 & 0 & 4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 4 \end{pmatrix} \in I(D_n(R))$ when $n \geq 2$. This implies that $D_n(\mathbb{Z}_6)$ is not tpr by Lemma 1.2(3), but Abelian by [17, Lemma 2].

The converse of Lemma 1.2(4) need not hold too. Let $R = U_2(A)$ over a domain A. Then $N(R) = N_*(R)$ clearly, but $I(R)$ contains E_{11}; hence R is not tpr by Lemma 1.2(3).

Considering Lemma 1.2(1), one may ask whether a ring R is tpr when $R/N^*(R)$ is a domain. But the answer is negative by the following.

Example 1.3. (1) We refer the ring constructions and computations in [19, Example 1.2] and [20, Theorem 2.2]. Let A be a domain and $R_n = D_{2^n}(A)$, where $n \geq 1$. Define a map

$$\sigma : R_n \to R_{n+1} \text{ by } M \mapsto \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}.$$

Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $M = \sigma(M)$ for $M \in R_n$). Set $R = \varinjlim R_n$ be the direct limit of $\{R_n, \sigma_{nm}\}$, noting that $\sigma_{nm} = \sigma^{m-n}$ (when $n \leq m$) is a direct system over $I = \{1, 2, \ldots\}$. It is easily checked that $N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$, so $R/N^*(R) \cong A$.

But $N_*(R) = 0$ by [20, Theorem 2.2], entailing $N_*(R) \subsetneq N^*(R)$. Thus R is not tpr by Lemma 1.2(4). Indeed, $E_{12}^2 = 0$ but $E_{12} \notin N_*(R)$.

(2) There exists a tpr ring whose structure is similar to the ring R in (1). Let B be a ring with a nonzero nilpotent ideal K such that B/K is a domain (e.g., $B = \mathbb{Z}_4$ with $K = 2\mathbb{Z}_4$), and

$$R_n = \{(a_{ij}) \in D_{2^n}(B) \mid a_{st} \in K \text{ for all } s, t \text{ with } s < t\},$$

where $n \geq 1$. Define a map $\sigma : R_n \to R_{n+1}$ by $M \mapsto \left(\begin{smallmatrix} M & 0 \\ 0 & M \end{smallmatrix}\right)$ and let $R = \varinjlim R_n$ be the direct limit of $\{R_n, \sigma_{nm}\}$ as in (1). Then we have

$$N_0(R) = N_*(R) = N^*(R) = N(R) = \{(a_{ij}) \in R \mid a_{ii} \in K \text{ for all } i\},$$

entailing $R/N^*(R) \cong B/K$. In fact, $N_0(R)^{m^m} = 0$ when $K^m = 0$. Thus R is tpr by Lemma 1.2(1).

Note that the fact of $N_*(R) = N(R)$, in the proof of Lemma 1.2(6), need not hold for any ring R. Indeed, consider the ring R in Example 1.3(1), $0 = N_*(R) \subsetneq N^*(R)$ and $N(R) = N^*(R_n) = N^*(D_{2^n}(A)) = \{(a_{ij}) \in D_{2^n}(A) \mid a_{ii} = 0 \text{ for all } i\} \neq 0$, where R_n is a subring of R.

Armendariz [5, Lemma 1] proved that, for a reduced ring R,

$$ab = 0 \text{ for all } a \in C_{f(x)}, b \in C_{g(x)} \text{ whenever } f(x)g(x) = 0$$

where $f(x), g(x) \in R[x]$. Rege et al. [30] called a ring (possibly without identity) *Armendariz* if it satisfies this property, based on Armendariz's result. So reduced rings are clearly Armendariz. Armendariz rings are Abelian by the proof of [3, Theorem 6] or [18, Corollary 8]. But Armendariz rings need not be tpr as can be seen the non-tpr ring $\mathbb{Z} \oplus \mathbb{Z}$. Tpr rings also need not be Armendariz. Indeed, $D_n(A)$, over a domain A, is a tpr ring by Lemma 1.2(2); but $D_n(A)$ is not Armendariz by [23, Example 3] when $n \geq 4$.

Let A be an algebra, with or without identity, over a commutative ring S. Following Dorroh [9], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. This algebraic system is denoted by $A \oplus_D S$.

If a ring R is an Armendariz ring, then $N_0(R) = N_*(R) = N^*(R)$ by [22, Lemma 2.3(5)]. So it is natural to compare this with the fact that if R is a tpr ring, then $N_*(R) = N^*(R) = N(R)$ by Lemma 1.2(4). The following makes a distinction between these two facts.

Example 1.4. (1) There exists an Armendariz ring R such that R is not tpr and $N^*(R) \subsetneq N(R)$. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by b^2, and set $R = A/I$. We identify a, b with their images in R for simplicity. Then R is Armendariz by [4, Example 4.8]. Let $0 \neq s \in N(R)$. Then s is of the form either kb or btb for some $k \in K$ and $t \in R$ by the argument in [21, pages 4–5]. However ba is not a nilpotent, forcing $ba \notin N^*(R)$. This implies $N^*(R) \subsetneq N(R)$, showing that R is not tpr by Lemma 1.2(4).

(2) We refer the construction and computation in [24, Example 2.4(3)] which extends the argument in Klein [27, Example (ii)]. Let S be the factor ring of the polynomial ring $\mathbb{Z}_2[t_1, t_2, \ldots]$ with a set of commuting indeterminates $\{t_i \mid i = 1, 2, \ldots\}$ over $\mathbb{Z}_2$, modulo the ideal generated by $\{t_i^2 \mid i = 1, 2, \ldots\}$. We identify t_i's with their images in R for simplicity. It is easily checked that $N_0(S)$ is the set of all polynomials of zero constants in S. Following [24,

Example 2.4(3)], set

$$T = \begin{pmatrix} N_0(S) & S \\ N_0(S) & N_0(S) \end{pmatrix} \text{ and } R = T \oplus_D \mathbb{Z}_2.$$

Then, by the argument in [24, Example 2.4(3)], $T \oplus_D 0 = N(R) = N_*(R) = N^*(R)$ and $N_0(R) = \left(\begin{smallmatrix} N_0(S) & N_0(S) \\ N_0(S) & N_0(S) \end{smallmatrix}\right) \oplus_D 0 \subsetneq N_*(R)$. So R is not Armendariz by [22, Lemma 2.3(5)]. Indeed, for $f(x) = (\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_1 \end{smallmatrix}\right), 0) + (\left(\begin{smallmatrix} t_2 & 0 \\ 0 & t_2 \end{smallmatrix}\right), 0)x \in R[x]$, we have $f(x)^2 = 0$ and $(\left(\begin{smallmatrix} t_1 & 0 \\ 0 & t_1 \end{smallmatrix}\right), 0)(\left(\begin{smallmatrix} t_2 & 0 \\ 0 & t_2 \end{smallmatrix}\right), 0) = (\left(\begin{smallmatrix} t_1t_2 & 0 \\ 0 & t_1t_2 \end{smallmatrix}\right), 0) \neq 0$. But since $\frac{R}{N_*(R)} \cong \mathbb{Z}_2$, R is a tpr ring by Lemma 1.2(1).

Following Han et al. [12], a ring R is called *APR* if

$$f(x)g(x) \in N_*(R)[x] \text{ implies } ab \in N_*(R) \text{ for all } a \in C_{f(x)} \text{ and } b \in C_{g(x)}$$

where $f(x), g(x) \in R[x]$. So R is APR if and only if $R/N_*(R)$ is Armendariz, entailing that a semiprime ring is APR if and only if it is Armendariz. Obviously tpr rings are APR by Lemma 1.2(1). Armendariz rings are APR by [12, Theorem 1.4(2)]. Therefore the concept of APR ring is a generalization of both tpr and Armendariz. $U_n(A)$, over a reduced ring A, is neither Armendariz nor tpr when $n \geq 2$, because it is non-Abelian; but it is APR since $R/N_*(R)$ is reduced (hence Armendariz).

We argue next the case of $N_0(R) = N_*(R) = N^*(R) = N(R)$, considering the relation $N_0(R) \subsetneq N_*(R) = N^*(R) = N(R)$ in Example 1.4(2). We use $\deg f(x)$ to denote the degree of a given polynomial $f(x)$. Birkenmeier et al. in [6, Proposition 2.6] proved that if $N(R) = N_*(R)$ for a ring R, then $N(R[x]) = N_*(R)[x]$, entailing $N(R[x]) = N_*(R[x]) = N_*(R)[x] = N(R)[x]$.

Following [1, p. 130], a subset of a ring R is said to be *locally nilpotent* if its finitely generated subrings are nilpotent. Due to [1, p. 130], the Levitzki radical of R, written by $s\sigma(R)$, means the sum of all locally nilpotent ideals of R. It is well-known that $N_*(R) \subseteq s\sigma(R) \subseteq N^*(R)$. Amitsur in [2, Theorem 1] show that $J(R[x]) = N[x]$ for any ring R, where $N = J(R[x]) \cap R$ is a nil ideal of R which contains $s\sigma(R)$.

Let R be a ring. The index of nilpotency of a nilpotent $a \in R$ is the least positive integer k such that $a^k = 0$. The index of nilpotency of a subset S of R is the supremum of the indices of nilpotency of all nilpotents in S. If such a supremum is finite, t say, then S is said to be *of bounded index t of nilpotency.* Camillo et al. in [7, Corollary 4] proved that $N_0(R[x]) = N_0(R)[x]$ for any ring R.

Proposition 1.5. (1) *A ring R is tpr if and only if so is $R[x]$.*

(2) *If R is a tpr ring, then*

$$\begin{aligned} J(R[x]) &= N(R[x]) = N^*(R[x]) = s\sigma(R[x]) \\ &= N_*(R[x]) = N_*(R)[x] = s\sigma(R)[x] = N^*(R)[x] = N(R)[x]. \end{aligned}$$

(3) *Let R be a tpr ring of bounded index 2 of nilpotency. Then* $N_0(R) = N_*(R) = N^*(R) = N(R)$.

(4) *Let R be a tpr ring of bounded index 2 of nilpotency. Then*

$$J(R[x]) = N(R[x]) = N_*(R[x]) = N_0(R[x]) = N_*(R)[x] = N_0(R)[x] = N(R)[x].$$

Proof. (1) Let R be a tpr ring. Then $R/N_*(R)$ is a domain by Lemma 1.2(1), entailing that $\frac{R[x]}{N_*(R)[x]} \cong \frac{R}{N_*(R)}[x]$ is also a domain. But $N_*(R[x]) = N_*(R)[x]$ by [2, Theorem 3], so $\frac{R[x]}{N_*(R[x])}$ is a domain. Thus $R[x]$ is tpr by Lemma 1.2(1). The converse holds by Lemma 1.2(6).

(2) Let R be a tpr ring. Then $N(R) = N_*(R)$ by Lemma 1.2(4). So we get moreover $N(R[x]) = N_*(R[x]) = N_*(R)[x] = N(R)[x]$ by [2, Theorem 3] and [6, Proposition 2.6]. This yields $N(R[x]) = N^*(R[x]) = N_*(R[x]) = N_*(R)[x] = N^*(R)[x] = N(R)[x]$.

While, $J(R[x]) = N[x]$ for some nil ideal N of R by [2, Theorem 1], so $N[x] \subseteq N^*(R)[x]$ implies $J(R[x]) = N^*(R)[x]$. This completes the proof.

(3) Let R be a tpr ring of bounded index 2 of nilpotency and assume on the contrary that $N_0(R) \subsetneq N_*(R)$, say $a \in N_*(R)\backslash N_0(R)$. Then RaR is a nilpotent ideal (indeed, $(RaR)^2 = 0$) by Hong et al. [15, Lemma 11] in which they applied the method of Klein in [26]. For, $xRx = 0$ by [15, Lemma 11] because $x^2 = 0$ for all $x \in RaR$. This forces $a \in N_0(R)$, a contradiction. Thus we have $N_0(R) = N_*(R) = N^*(R) = N(R)$ by Lemma 1.2(4) since R is tpr.

(4) is shown by (2), (3), and [7, Corollary 4]. □

The converse of Proposition 1.5(2) does not hold in general. Indeed, $R = U_n(A)$, over a simple domain A, satisfies the property $J(R[x]) = N(R[x]) = N^*(R[x]) = N_*(R[x]) = N_*(R)[x] = N^*(R)[x] = N(R)[x]$ by help of [6, Proposition 2.6], but it is not tpr when $n \geq 2$ by Lemma 1.2(3), where $N(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$.

One may also ask whether a semiprime tpr ring (hence a domain by Lemma 1.2(1)) is semiprimitive, based on Proposition 1.5(2). However the answer is negative by the following.

Example 1.6. We refer the ring in [14, Example 3]. Let L be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, where p is an odd prime. We consider next the quaternions over L, R say. Then R is clearly a domain (hence tpr), so $R[x]$ is semiprimitive because $J(R[x]) = N[x]$ for some nil ideal N of R by [2, Theorem 1]. But $J(R) = pR$, so R is not semiprimitive.

A ring R (possibly without identity) is usually called *von Neumann regular* (simply, *regular*) if for every $a \in R$ there exists $b \in R$ such that $aba = a$ in [10]. It is easily shown that every regular ring R is semiprimitive (i.e., $J(R) = 0$). So we obtain the following.

Proposition 1.7. *Let R be a regular ring. Then R is APR if and only if R is reduced if and only if R is Abelian if and only if R is Armendariz.*

Proof. Use [10, Theorem 3.2] and the fact that R is APR if and only if $R/N_*(R)$ is Armendariz. □

However the tpr property need not be equivalent to the conditions in Proposition 1.7, as can be seen by the regular reduced ring $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ which is not tpr.

Following the literature, a ring R is called *π-regular* if for each $a \in R$ there exist a positive integer $n = n(a)$ and $b \in R$ such that $a^n = a^n b a^n$. Regular rings are clearly π-regular, and $J(R)$ is easily shown to be nil for every π-regular ring R. One may also ask whether π-regular APR rings are Armendariz. But the answer is negative by the ring $R = D_n(S)$ $(n \geq 4)$ over a division ring S. Indeed, R is a π-regular ring because R is local and $J(R) = N_*(R)$ is nil. Moreover, since $R/N_*(R) \cong S$ is a division ring, R is APR. But R is not Armendariz by [23, Example 3].

Proposition 1.8. (1) *Let R be a π-regular ring. Then R is tpr if and only if R is a local ring with $J(R) = N_*(R)$.*

(2) *Let R be a regular ring. Then R is tpr if and only if R is a division ring.*

Proof. (1) Suppose that R is tpr. Then $N_*(R) = N^*(R)$ by Lemma 1.2(4). But $J(R)$ is nil since R is π-regular, entailing $J(R) = N_*(R)$. Let $a \in R \backslash J(R)$. Then, since R is π-regular, $a^n = a^n b a^n$ for some $b \in R$ and $n \geq 1$. But $a^n \neq 0$, so $a^n b$ and $b a^n$ are nonzero idempotents. This implies $a^n b = 1 = b a^n$ by Lemma 1.2(3), entailing that a is a unit.

The converse comes from Lemma 1.2(1) because $R/N_*(R)$ is a division ring.

(2) is an immediate consequence of (1) since regular rings are semiprimitive. □

We investigate next the structure of noncommutative tpr rings of minimal order, which may do roles in constructing various kinds of rings. $GF(p^n)$ denotes the Galois field of order p^n. Every ring in the following is of order 16.

Example 1.9. (1) $R_1 = D_3(\mathbb{Z}_2)$ is a noncommutative tpr ring by Lemma 1.2(1) because $N_*(R_1) = J(R_1) = \left(\begin{smallmatrix} 0 & \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & 0 & \mathbb{Z}_2 \\ 0 & 0 & 0 \end{smallmatrix}\right)$ and $R_1/N_*(R_1) \cong \mathbb{Z}_2$.

(2) Following Xue [33, Example 2], let $R_2 = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^2 \end{smallmatrix}\right) \mid a, b \in GF(2^2) \right\}$. Then $N_*(R_2) = J(R_2) = \left(\begin{smallmatrix} 0 & GF(2^2) \\ 0 & 0 \end{smallmatrix}\right)$ and $R_2/N_*(R_2) \cong GF(2^2)$. So R_2 is a noncommutative tpr ring by Lemma 1.2(1).

(3) Following [33, Example 2], let $A_1 = \mathbb{Z}_4\langle x, y \rangle$ be the free algebra with noncommuting indeterminates x, y over $\mathbb{Z}_4$, and $R_3 = A_1/I$, where I is the ideal of A_1 generated by $x^3, y^3, yx, x^2 - xy, x^2 - 2, y^2 - 2, 2x, 2y$. Identify x and y with their images in R_3 for simplicity. Then $N_*(R_3) = J(R_3) = 2\mathbb{Z}_4 + (x, y)$, where (x, y) is the ideal of R_3 generated by x and y. So $R_3/N_*(R_3) \cong \mathbb{Z}_4$, not reduced, and hence R_3 is not a tpr ring.

(4) Following [33, Example 2], let $A_2 = \mathbb{Z}_2\langle x, y \rangle$ be the free algebra with noncommuting indeterminates x, y over $\mathbb{Z}_2$, and $R_4 = A_2/I$, where I is the ideal of A_2 generated by $x^3, y^3, yx, x^2 - xy, y^2 - xy$. Identify x and y with their

images in R_4 for simplicity. Then $N_*(R_4) = J(R_4) = (x, y)$, where (x, y) is the ideal of R_4 generated by x and y. So $R_4/N_*(R_4) \cong \mathbb{Z}_2$, and hence R_4 is a noncommutative tpr ring by Lemma 1.2(1).

(5) Following Xu and Xue [32, Example 7], let $A_3 = \mathbb{Z}_4\langle x, y\rangle$ be the free algebra with noncommuting indeterminates x, y over $\mathbb{Z}_4$, and $R_5 = A_3/I$, where I is the ideal of A_3 generated by $x^3, y^2, yx, x^2 - xy, x^2 - 2, 2x, 2y$. Identify x and y with their images in R_5 for simplicity. Then $N_*(R_5) = J(R_5) = 2\mathbb{Z}_4 + (x, y)$, where (x, y) is the ideal of R_5 generated by x and y. So $R_5/N_*(R_5) \cong \mathbb{Z}_4$, not reduced, and hence R_5 is not a tpr ring.

The rings R_1, R_2, and R_4 in Example 1.9 are minimal noncommutative tpr rings as we see in the following.

Theorem 1.10. *If R is a noncommutative tpr ring of minimal order, then R is of order* 16 *and is isomorphic to one of the rings R_1, R_2, and R_4 in Example 1.9.*

Proof. Let R be a tpr ring of minimal order. Then R is Abelian by Lemma 1.2(3), and any noncommutative Abelian ring of minimal order is isomorphic to one of the rings in Example 1.9 by [25, Theorem 3.3]. But R_1, R_2, and R_4 in Example 1.9 are tpr, so this completes the proof. □

The preceding example and theorem imply that a noncommutative Abelian ring of minimal order need not be tpr.

Let R be a ring and $u \in R$. Recall that u is said to be *right regular* in R if $ur = 0$ implies $r = 0$ for $r \in R$. The case of *left regular* is defined similarly; and u is said to be *regular* if it is both left and right regular (i.e., not a zero divisor). Following Lambek [28], an element a of R is called *strongly nilpotent* provided that every sequence $a_0, a_1, a_2, \ldots$, such that $a_0 = a$ and $a_{n+1} \in a_n R a_n$ for $n = 0, 1, 2, \ldots$, is ultimately zero. $N_*(R)$ is the set of all strongly nilpotent elements by [28, Proposition 3.2.1].

Proposition 1.11. *Let R be a ring and M be a multiplicatively closed subset of R consisting of central regular elements. Then R is tpr if and only if so is RM^{-1}.*

Proof. It suffices to show the necessity by Lemma 1.2(6). Let R be a tpr ring. Every element of RM^{-1} can be expressed by au^{-1} for some $a \in R$ and $u \in M$. This fact is necessary to show that

$$N(RM^{-1}) = N(R)M^{-1} \text{ and } N_*(RM^{-1}) = N_*(R)M^{-1}.$$

Clearly $N(R)M^{-1} \subseteq N(RM^{-1})$. Let $au^{-1} \in N(RM^{-1})$, $(au^{-1})^n = 0$ say. Then $0 = (au^{-1})^n = a^n u^{-n}$ implies $a^n = 0$ and so $a \in N(R)$, entailing $N(M^{-1}R) = N(R)M^{-1}$.

Now we show that $N_*(RM^{-1}) = N_*(R)M^{-1}$. In fact,

$$N_*(RM^{-1}) \subseteq N(RM^{-1}) = N(R)M^{-1} = N_*(R)M^{-1}$$

by Lemma 1.2(4) because R is tpr, and so $N_*(RM^{-1}) \subseteq N_*(R)M^{-1}$.

Conversely let $cv^{-1} \in N_*(R)M^{-1}$ with $c \in N_*(R)$ and $v \in M$. Write $c_0 = c$, $v_0 = v$, and $d_0 = c_0v_0^{-1}$. Consider a sequence d_0, $d_1 = d_0e_0f_0^{-1}d_0 = c_0e_0c_0v_0^{-2}f_0^{-1}$ ($c_1 = c_0e_0c_0$ say),

$$\begin{aligned} d_2 &= d_1e_1f_1^{-1}d_1 = (c_1v_0^{-2}f_0^{-1})e_1f_1^{-1}(c_1v_0^{-2}f_0)^{-1} \\ &= c_1e_1c_1v_0^{-2^2}f_0^{-2}f_1^{-1} \text{ (say } c_2 = c_1e_1c_1), \\ &\quad \ldots, \\ d_{n+1} &= d_ne_nf_n^{-1}d_n \\ &= (c_nv_0^{-2^n}f_0^{-2^{n-1}}f_1^{-2^{n-2}}\cdots f_n^{-1})e_nf_n^{-1}(c_nv_0^{-2^n}f_0^{-2^{n-1}}f_1^{-2^{n-2}}\cdots f_n^{-1}) \\ &= (c_ne_nc_n)(v_0^{-2^{n+1}}f_0^{-2^n}f_1^{-2^{n-1}}\cdots f_n^{-2}f_n^{-1}), \end{aligned}$$

where $e_if_i^{-1}$'s are taken arbitrarily in RM^{-1}. But $c \in N_*(R)$, so $c_he_hc_h = 0$ for some $h \geq 1$. This implies that $d_h = 0$, so $c_0v_0^{-1} \in N_*(RM^{-1})$. Therefore $N_*(RM^{-1}) = N_*(R)M^{-1}$.

Consider now $(au^{-1})(bv^{-1}) \in N_*(RM^{-1})$ for $au^{-1}, bv^{-1} \in RM^{-1}$. Then $(ab)(uv)^{-1} \in N_*(RM^{-1})$. But $ab \in R$ and $N_*(RM^{-1}) = N_*(R)M^{-1}$, forcing $ab \in N_*(R)$. Since R is tpr, $R/N_*(R)$ is a domain by Lemma 1.2(1); hence we have $a \in N_*(R)$ or $b \in N_*(R)$. This yields that $au^{-1} \in N_*(RM^{-1})$ or $bv^{-1} \in N_*(RM^{-1})$, i.e., the factor ring $RM^{-1}/N_*(RM^{-1})$ is a domain. Therefore RM^{-1} is tpr by Lemma 1.2(1). □

In the proof of Proposition 1.11, we see

$$N_*(RM^{-1}) = N_*(R)M^{-1} = N(R)M^{-1} = N(RM^{-1}).$$

Let R be a ring. Following the literature, the ring of *Laurent* polynomials, with an indeterminate x over R, consists of all formal sums $\sum_{i=k}^{n} a_ix^i$ with canonical addition and multiplication, where $a_i \in R$ and k, n are (possibly negative) integers with $k \leq n$. This ring is written by $R[x; x^{-1}]$, and called the *Laurent* polynomial ring over R.

Corollary 1.12. *A ring R is tpr if and only if $R[x]$ is tpr if and only if $R[x; x^{-1}]$ is tpr.*

Proof. Let R be a ring and $M = \{1, x, x^2, \ldots\}$. Then, clearly, M is a multiplicatively closed subset of $R[x]$ consisting of central regular elements, and note $R[x; x^{-1}] = R[x]M^{-1}$. The result follows Proposition 1.5(1) and Proposition 1.11. □

Given any tpr ring we can construct tpr rings, via factor rings of the polynomial ring over it.

Proposition 1.13. *Let R be a tpr ring. Then $R[x]/(x^n)$ is a tpr ring for all $n \geq 1$, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. Write $E = R[x]/(x^n)$. Then $N(E) = N_*(E) = N_*(R) + \bar{x}E$ because $(\bar{x}E)^n = 0$, where we express E by $R + \bar{x}E$. So $E/N_*(E) \cong R/N_*(R)$ is a domain by Lemma 1.2(1) because R is tpr. Thus E is a tpr ring. □

2. Structure of tunr rings

In this section we study the structure of rings in which the factor rings modulo upper nilradicals are domains. A ring R (not necessarily with identity) will be called *tunr* if it satisfies the condition:

$$a_i \in N^*(R) \text{ or } b_i \in N^*(R) \text{ for each } i \text{ whenever } f(x)g(x) \in N^*(R)[x]$$

for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, where we let $m = n$ by using zero coefficients if necessary.

Lemma 2.1. (1) *A ring R is tunr if and only if $R/N^*(R)$ is a domain.*

(2) *Every tpr ring is tunr.*

(3) *Let R be a domain and $E_0 = D_m(R)$ for $m \geq 1$. Then $D_n(E_0)$ is a tunr ring for all $n \geq 1$.*

(4) *Let R be a tunr ring. Then $I_e(R) = \{0, 1\}$, i.e., $I(R)$ is empty.*

(5) *If R is a tunr ring, then $N(R) = N^*(R)$.*

(6) *The class of tunr rings is closed under subrings.*

Proof. (1) The proof is almost same as one of Lemma 1.2(1).

(2) Let R be a tpr ring. Then $N_*(R) = N^*(R)$ by help of Lemma 1.2(4) because $N_*(R) \subseteq N^(R)$, implying that $R/N^*(R)$ is a domain. Thus R is tunr by (1).

The proof of (3) (resp., (4)) is almost same as one of Lemma 1.2(2) (resp., Lemma 1.2(3)).

(5) is shown by (1).

(6) Let R be a tunr ring and S be a subring of R. Then $N(R) = N^*(R)$ by (5), so we get $N^*(S) = N^*(R) \cap S$ from $N^*(R) \cap S \subseteq N^*(S) \subseteq N(S) = N(R) \cap S = N^*(R) \cap S$. The remainder of the proof is almost same as one of Lemma 1.2(6). □

One may ask whether tunr rings are tpr, considering Lemma 2.1(2). However the following answers negatively.

Example 2.2. We use the ring R in Example 1.3(1). Recall $N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$, so $R/N^*(R) \cong A$ is a domain. Thus R is tunr by Lemma 2.1(1). But R is not tpr by the argument in Example 1.3.

We see two sorts of conditions under which tunr rings are tpr as follows.

Proposition 2.3. *Let R be a ring of bounded index of nilpotency. Then R is tunr if and only if R is tpr.*

Proof. It suffices to show the necessity. Let R be tunr. Then $N^*(R) = N(R)$ by Lemma 2.1(5). By hypothesis, R is of bounded index of nilpotency, and so $N_*(R) = N^*(R)$ by the proof of [19, Proposition 1.4]. Then $R/N^*(R)$ is a domain by help of Lemma 2.1(1), and so R is tpr by Lemma 1.2(1). $\square$

The ring R in Example 1.3 is not of bounded index of nilpotency.

The following proposition shows that the APR property is a bridge between tpr and tunr.

Proposition 2.4. *Let R be a tunr ring.*

(1) *If R is an Armendariz ring, then RaR is nilpotent for all $a \in N(R)$ (i.e., $N_0(R) = N_*(R) = N^*(R) = N(R)$) and $R/N_0(R)$ is a domain.*

(2) *If R is a right Goldie ring, then $N(R)$ is nilpotent and $R/N_0(R)$ is a domain (hence, R is tpr and $N_0(R) = N_*(R) = N^*(R) = N(R)$).*

(3) *If R is an APR ring, then R is tpr.*

Proof. (1) Let R be an Armendariz ring. Then $N_0(R) = N_*(R) = N^*(R) = N(R)$ by Lemma 2.1(5) and [22, Lemma 2.3(5)]; hence $R/N_0(R)$ is a domain by Lemma 2.1(1), and RaR is nilpotent for all $a \in N(R)$.

(2) Let R be a right Goldie ring. Then $N^*(R)$ is nilpotent by [29], so $N_0(R) = N_*(R) = N^*(R) = N(R)$ by Lemma 2.1(5). Moreover $R/N_0(R)$ is a domain by Lemma 2.1(1), so R is tpr by Lemma 1.2(1).

(3) Let R be an APR ring. Then $N_*(R) = N^*(R)$ by [12, Lemma 1.1(7)], and $N(R) = N^*(R)$ by Lemma 2.1(5) because R is tunr, entailing $N(R) = N_*(R) = N^*(R)$. Hence $R/N_*(R) = R/N^*(R)$ is a domain by Lemma 2.1(5), so R is tpr by Lemma 1.2(1). $\square$

The condition of R being APR in Proposition 2.4(3) is not superfluous. Indeed, letting R be the semiprime ring in Example 1.3(1), R is not APR by help of [23, Example 3]. Moreover, $E_{12} \in N(R)$, but $RE_{12}R$ is not nilpotent because $N_0(R) = N_*(R) = 0$. But R is tunr by Example 2.2.

Considering Proposition 1.5(1), it is natural to ask whether the tunr property passes to polynomial rings. We answer this question negatively by help of Smoktunowicz [31].

Example 2.5. Smoktunowicz showed by [31, Corollary 13] that there exists a nil ring R_0 such that polynomial ring over R_0 is not nil. Smoktunowicz constructed the ring R_0 so that R_0 is an algebra over a countable field K, in [31, Theorem 12].

Let R be the Dorroh extension of R_0 by K. Then clearly $N(R) = N^*(R) = R_0$. So $R/N^*(R) \cong K$, showing that R is tunr by Lemma 2.1(1). Note that $N^*(R[x]) \subseteq N(R[x]) \subseteq N^*(R)[x] = R_0[x]$ because $\frac{R[x]}{N^*(R)[x]} \cong \frac{R}{N^*(R)}[x] \cong K[x]$.

But $R_0[x]$ is not nil by [31, Corollary 13], entailing $N^*(R[x]) \subsetneq N(R[x])$. Thus $R[x]$ is not tunr by Lemma 2.1(5).

If a tunr ring R is of bounded index of nilpotency, then R is tpr by Proposition 2.3, so $R[x]$ is tpr (hence tunr) by Proposition 1.5(1). The ring R in Example 2.5 is not of bounded index of nilpotency by the construction in the proof of [31, Theorem 12].

As an extension of tunr rings, we obtain a similar result to Proposition 1.11.

Proposition 2.6. *Let R be a ring and M be a multiplicatively closed subset of R consisting of central regular elements. Then R is tunr if and only if so is RM^{-1}.*

Proof. It suffices to show the necessity by Lemma 2.1(6). Let R be a tunr ring. $N(RM^{-1}) = N(R)M^{-1}$ by the proof of Proposition 2.6. We use this fact freely.

$N^*(R)M^{-1}$ is easily shown to be a nil ideal of RM^{-1}, so $N^*(R)M^{-1} \subseteq N^*(RM^{-1})$. We next show the converse inclusion. Let $au^{-1} \in N^*(RM^{-1})$. Then $a \in N(R)$ and moreover $a = au^{-1}u \in N^*(RM^{-1})$, entailing $RaR \subseteq N^*(RM^{-1})$. This implies that RaR is a nil ideal of R, and so $a \in N^*(R)$. Thus we get $N^*(R)M^{-1} \supseteq N^*(RM^{-1})$, obtaining $N^*(R)M^{-1} = N^*(RM^{-1})$.

Consider next $(au^{-1})(bv^{-1}) \in N^*(RM^{-1})$ for $au^{-1}, bv^{-1} \in RM^{-1}$. Then $(ab)(uv)^{-1}$ in contained in $N^*(R)M^{-1}$, so $ab \in N^*(R)$. Since R is tunr, $R/N^*(R)$ is a domain by Lemma 2.1(1); hence we have $a \in N^*(R)$ or $b \in N^*(R)$. This yields that $au^{-1} \in N^*(RM^{-1})$ or $bv^{-1} \in N^*(RM^{-1})$, i.e., the factor ring $RM^{-1}/N^*(RM^{-1})$ is a domain. Therefore RM^{-1} is tunr by Lemma 2.1(1). □

The following is similar to Corollary 1.12.

Corollary 2.7. (1) *For a ring R, $R[x]$ is tunr if and only if so is $R[x;x^{-1}]$.*

(2) *Let R be a ring of bounded index of nilpotency. Then R is tunr if and only if $R[x]$ is tunr if and only if $R[x;x^{-1}]$ is tunr if and only if R is tpr if and only if $R[x]$ is tpr if and only if $R[x;x^{-1}]$ is tpr.*

Proof. (1) follows Proposition 2.6.

(2) is shown by Proposition 2.3, Corollary 1.12, and Lemma 2.1(2, 6). □

Given any tunr ring we can construct tunr rings, via factor rings of the polynomial ring over it.

Proposition 2.8. *Let R be a tunr ring. Then $R[x]/(x^n)$ is a tunr ring for all $n \geq 1$, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. The proof is almost similar to one of Proposition 1.13. □

3. Structure of tjr rings

In this section we study the structure of rings in which the factor rings modulo Jacobson radicals are domains. A ring R (not necessarily with identity) will be called *tjr* if it satisfies the condition:

$$a_i \in J(R) \text{ or } b_i \in J(R) \text{ for each } i \text{ whenever } f(x)g(x) \in J(R)[x]$$

for $f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$, where we let $m = n$ by using zero coefficients if necessary.

Lemma 3.1. (1) *A ring R is tjr if and only if $R/J(R)$ is a domain.*

(2) *Let R be a tjr ring. Then $I_e(R) = \{0, 1\}$, i.e., $I(R)$ is empty.*

(3) *Let R be a ring such that $J(R)$ is nil. Then R is tunr if and only if R is tjr.*

(4) *If a ring R is tjr, then $N(R) \subseteq J(R)$.*

Proof. The proof of (1) (resp., (2)) is almost same as one of Lemma 1.2(1) (resp., Lemma 1.2(3)).

(3) If $J(R)$ is nil, then $N^*(R) = J(R)$, so the result follows.

(4) is an immediate consequence of (1). □

The concepts of tpr (tunr) and tjr are independent of each other by the following.

Example 3.2. (1) There exists a tpr ring but not tjr. We use the ring in [14, Example 3]. Let R_0 be the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$, where p is an odd prime; and let R be the quaternions over R_0. Then R is clearly a domain (hence tpr). But $J(R) = pR$, and $R/J(R)$ is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_p)$ by the argument in [11, Exercise 2A]. $\mathrm{Mat}_2(\mathbb{Z}_p)$ is not Abelian, so R is not tjr by Lemma 3.1(2).

(2) There exists a tjr ring but not tpr. Let A be a simple domain and consider the ring R in Example 1.3 over A. Then $N^*(R) = \{(a_{ij}) \in R \mid a_{ii} = 0 \text{ for all } i\}$, and $R/N^*(R) \cong A$ is a simple domain. This yields $N^*(R) = J(R)$, so R is tjr. But R is not tpr by the argument in Example 1.3.

(3) There exists a tjr ring but not tunr. We apply the ring construction and the argument in [16, Example 1.1]. Let F be a field and let V be a infinite dimensional left F-module with $\{v_1, v_2, \dots\}$ a basis. For the endomorphism ring $A = \mathrm{End}_F(V)$, define

$$A_1 = \{f \in A \mid \mathrm{rank}(f) < \infty \text{ and } f(v_i) = a_1v_1 + \cdots + a_iv_i \text{ for } i = 1, 2, \dots \text{ with } a_j \in F\}$$

and let R be the F-subalgebra of A generated by A_1 and 1_A. Then

$$\begin{aligned} N_*(R) &= N^*(R) = N(R) \\ &= \{g \in A_1 \mid g(v_i) = a_1v_1 + \cdots + a_{i-1}v_{i-1} \text{ for } i = 1, 2, \dots \text{ with } a_i \in F\} \end{aligned}$$

by the computation in [16, Example 1.1].

Consider next the subring

$$S = \{f(x) = \sum_{i=0}^{\infty} a_i x^i \in R[[x]] \mid a_0 \in F\}$$

of $R[[x]]$. Then $J(S) = xR[[x]]$ and $S/J(S) \cong F$, so S is tjr by Lemma 3.1(1). But S is not tunr as we see in the following.

Let

$$f(x) = E_{12}x + (E_{34} + E_{56})x^2 + \cdots + (\Sigma_{i=0}^{2^{n-1}-1} E_{(2^n+2i-1)(2^n+2i)})x^n + \cdots$$

and

$$g(x) = E_{23}x + (E_{45} + E_{67})x^2 + \cdots + (\Sigma_{i=0}^{2^{n-1}-1} E_{(2^n+2i)(2^n+2i+1)})x^n + \cdots$$

as in [16, Example 1.1]. Then $f(x), g(x) \in N(S)$ (indeed, $f(x)^2 = 0 = g(x)^2$), but $f(x)+g(x) \notin N(S)$ by the computation in [16, Example 1.1]. This implies $f(x) \notin N^*(S)$ or $g(x) \notin N^*(S)$, so $N^*(S) \subsetneq N(S)$. Therefore S is not tunr by Lemma 2.1(5).

Considering Lemma 3.1(3), one may ask whether a tjr ring R is tpr if $J(R)$ is nil. However the answer is negative by the existence of the ring R in Example 3.2(2). Note that $J(R) = N^*(R)$ is nil and R is tjr (if and only if tunr), but R is not tpr.

In the following we see another role of the ring R in Example 3.2(3). Consider Proposition 1.5(1). Then it is natural to conjecture that the tpr property passes to power series rings. However we see a counterexample in the following.

Example 3.3. Let R be the ring in Example 3.2(3), and T be the subring of R generated by $N_*(R)$ and 1_A. Then $T/N_*(T) \cong F$, so T is tpr by Lemma 1.2(1).

Consider next $T[[x]]$. Then the power series $f(x), g(x) \in R[[x]]$ are also contained in $N(T[[x]])$. Recall $f(x)+g(x) \notin N(T[[x]])$, and so $f(x) \notin N_*(T[[x]])$ or $g(x) \notin N_*(T[[x]])$. Thus $T[[x]]/N_*(T[[x]])$ is not reduced (hence not a domain), forcing $T[[x]]$ to be not tpr.

In the following arguments, we see conditions under which the tpr, tunr, and tjr properties are equivalent.

Proposition 3.4. (1) *Let R be a π-regular ring of bounded index of nilpotency. Then R is tpr if and only if R is tunr if and only if R is tjr.*

(2) *Let R be a right or left Artinian ring. Then R is tpr if and only if R is tunr if and only if R is tjr.*

(3) *If R is a noncommutative tunr ring of minimal order, then R is of order* 16 *and is isomorphic to one of the rings R_1, R_2, and R_4 in Example 1.9.*

(4) *If R is a noncommutative tjr ring of minimal order, then R is of order* 16 *and is isomorphic to one of the rings R_1, R_2, and R_4 in Example 1.9.*

Proof. (1) Recall that π-regular rings are semiprimitive. So we obtain the result by Proposition 2.3 and Lemma 3.1(3).

(2) It is well-known that if a ring R is right or left Artinian, then $J(R)$ is nilpotent. So if R is tpr, tunr, or tjr, then $J(R) = N^*(R) = N_*(R) = N(R)$ by Lemma 1.2(4), Lemma 2.1(5), and Lemma 3.1(4). Then R is clearly of bounded index of nilpotency. Therefore we obtain the equivalences among the tpr, tunr, and tjr properties by help of Lemma 1.2(1), Lemma 2.1(1), and Lemma 3.1(1).

(3) and (4) are proved by (2) and Theorem 1.10. □

$D_n(R)$, over a division ring R, satisfies the conditions in Proposition 3.4(1, 2) (refer Lemmas 1.2(2) and 2.1(3)). We see next tjr Laurent polynomial rings.

Proposition 3.5. (1) *Let R be a π-regular ring of bounded index of nilpotency. Then R is tjr if and only if $R[x]$ is tjr if and only if $R[x;x^{-1}]$ is tjr if and only if R is tunr if and only if $R[x]$ is tunr if and only if $R[x;x^{-1}]$ is tunr if and only if R is tpr if and only if $R[x]$ is tpr if and only if $R[x;x^{-1}]$ is tpr.*

(2) *Let R be a right or left Artinian ring. Then R is tjr if and only if $R[x]$ is tjr if and only if $R[x;x^{-1}]$ is tjr if and only if R is tunr if and only if $R[x]$ is tunr if and only if $R[x;x^{-1}]$ is tunr if and only if R is tpr if and only if $R[x]$ is tpr if and only if $R[x;x^{-1}]$ is tpr.*

Proof. The proof is done by Proposition 3.4(1, 2) and Corollary 2.7. □

Considering the proofs of Propositions 1.11 and 2.6, one may ask whether $J(RM^{-1}) = J(R)M^{-1}$ for a given ring R such that M is a multiplicatively closed subset of R consisting of central regular elements. However the answer is negative by the following.

Example 3.6. Let R be the ring in Example 3.2(1), i.e., R is the quaternions over R_0, where R_0 is the localization of $\mathbb{Z}$ at the prime ideal $p\mathbb{Z}$ (p is an odd prime). Recall that R is a commutative domain and $J(R) = pR$ such that $R/J(R)$ is isomorphic to $\mathrm{Mat}_2(\mathbb{Z}_p)$. Let $M = R\backslash\{0\}$. Then RM^{-1} is the field of rational numbers, $\mathbb{Q}$ say; and $J(RM^{-1}) = 0 \subsetneq \mathbb{Q} = pRM^{-1} = J(R)M^{-1}$.

Note that $J(R)$ in Example 3.6 is not nil.

Given any tjr ring we can construct tjr rings, via factor rings of the polynomial ring over it.

Proposition 3.7. *Let R be a tjr ring. Then $R[x]/(x^n)$ is a tjr ring for all $n \geq 1$, where (x^n) is the ideal of $R[x]$ generated by x^n.*

Proof. The proof is almost similar to one of Proposition 1.13. □

The Dorroh extension does a role in constructing examples of tpr, tunr, and tjr rings.

Proposition 3.8. (1) *Let A be a prime radical ring. Then the Dorroh extension $A \oplus_D \mathbb{Z}$ of A by $\mathbb{Z}$ is tpr.*

(2) *Let A be a upper nilradical ring. Then the Dorroh extension $A \oplus_D \mathbb{Z}$ of A by $\mathbb{Z}$ is tunr.*

(3) *Let A be a Jacobson radical ring. Then the Dorroh extension $A \oplus_D \mathbb{Z}$ of A by $\mathbb{Z}$ is tjr.*

Proof. Let A be a prime radical ring and consider the Dorroh extension $A \oplus_D \mathbb{Z}$, R say. Then clearly $N(R) = N_*(R)$ and $R/N_*(R) \cong \mathbb{Z}$. So R is a tpr ring by Lemma 1.2(1). The proofs of the remainder are similar. □

The nilradicals in Example 1.3(1, 2) can be used to construct tpr rings and tunr rings, via Dorroh extensions as in Proposition 3.8.

Consider next $A = \mathrm{Mat}_n(K)$ over a domain K such that $K/J(K)$ is a domain (e.g., $\mathbb{Z}$), and let $R_1 = \{\sum_{i=0}^{\infty} a_i x^i \in A[[x]] \mid a_0 \in K\}$, i.e., $R_1 = K + xA[[x]]$.

Then $J(R_1) = J(K) + xA[[x]]$ and $R_1/J(R_1) \cong K/J(K)$, so R_1 is a tjr ring by Lemma 3.1(1). Let $R_2 = \mathbb{Z} + xA[[x]]$. Then $J(R_2) = A[[x]]$ and $R_2/J(R_2) \cong \mathbb{Z}$, so R_2 is also a tjr ring. Note that R_2 is isomorphic to the Dorroh extension $xA[[x]] \oplus_D \mathbb{Z}$ of $xA[[x]]$ by $\mathbb{Z}$.

Acknowledgments. The authors must thank the referee very much for very careful reading of the manuscript and many valuable suggestions that improved the paper by much. The first named author was supported by the National Natural Science Foundation of China(11361063). And the third named author was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education(NRF-2013R1A1A2A10004687).

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