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THE t-WISE INTERSECTION OF RELATIVE THREE-WEIGHT CODES

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Abstract. The t-wise intersection is a useful property of a linear code due to its many applications. Recently, the second author determined the t-wise intersection of a relative two-weight code. By using this result and generalizing the finite projective geometry method, we will present the t-wise intersection of a relative three-weight code and its applications in this paper.

1. Introduction

The t-wise intersection of a linear code was introduced by Cohen et al. [3] [6] in order to study separating codes and independent families. A 2-wise intersecting code is usually called intersecting, which satisfies the property that any two nonzero codewords have intersecting support [7]. The support $\chi(c)$ of a codeword c is defined as the set of the nonzero coordinate positions of the codeword. The t-wise intersecting codes have many applications such as multiple access [4] and cryptography [10].

The t-wise intersecting code is also closely related to the separating code which has been used in the areas such as automatic synthesis, technical diagnosis and digital fingerprinting [5]. A binary intersecting code is equivalent to a (2,1)-separating code, whereas the 3-wise binary intersecting code is equivalent to the (2,2)-separating code, and in nonbinary case, the 3-wise intersecting property is the necessary condition of (2,2)-separation [3]. In addition, it is well known that all the nonzero codewords of a binary intersecting code are minimal [1], and the set of minimal codewords of a linear code is the key to the secret sharing scheme based on the dual of the linear code.

The t-wise intersections of a constant-weight code and a relative two-weight code have been addressed in [8] by using the finite projective geometry method [2]. A relative three-weight code was recently introduced by the second author

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and Wu [9], and it can be applied to the secret sharing scheme based on a linear code and the wiretap channel II with the coset coding scheme. A relative three-weight code is a large family of codes, including a relative two-weight code and a constant-weight code as special cases. The aim of this paper is to compute the t-wise intersection of a relative three-weight code.

Definition 1.1. Let $\mathcal{C}$ be an $[n, k]$ linear code, that is, a code with length n and dimension k. The t-wise intersection of $\mathcal{C}$ is defined as the number

$$\min\{|\bigcap_{i=1}^{i=t} \chi(c_i)| : c_1, c_2, \ldots, c_t \text{ are any } t \text{ linearly independent codewords}\}.$$

$\mathcal{C}$ is called t-wise intersecting if its t-wise intersection is nonzero.

Definition 1.2 ([9])**.** Let $\mathcal{C}_1$ be a k_1-dimensional subcode of $\mathcal{C}$, and $\mathcal{C}_2$ be a k_2-dimensional subcode, satisfying $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathcal{C}$. Then $\mathcal{C}$ is called a relative three-weight code with respect to $\mathcal{C}_1$ and $\mathcal{C}_2$, provided that $\mathcal{C}_1 \setminus \{0\}$, $\mathcal{C}_2 \setminus \mathcal{C}_1$ and $\mathcal{C} \setminus \mathcal{C}_2$ are all constant-weight codes. If these three constant-weight codes have weights ω_1, ω_2 and ω_3, respectively, then the relative three-weight code $\mathcal{C}$ is denoted by $\mathcal{C}(\omega_1, \omega_2, \omega_3)$.

The notations $\mathcal{C}$, $\mathcal{C}_1$ and $\mathcal{C}_2$ preserve the same meaning as in Definition 1.2 throughout the paper unless otherwise stated.

Definition 1.3. $\mathcal{D} \subset \mathcal{C}$ is called a *relative* (r, r_1, r_2) subcode with $r_1 \le r_2 \le r$ if $\mathcal{D}$ satisfies $\dim \mathcal{D} = r$, $\dim \mathcal{D} \cap \mathcal{C}_1 = r_1$, and $\dim \mathcal{D} \cap \mathcal{C}_2 = r_2$.

Additionally, let $\langle c_1, \ldots, c_t \rangle$ be the subcode of $\mathcal{C}$ generated by $c_1, \ldots, c_t$. Define $t_1^{\max}$ and $t_2^{\max}$ as follows.

$$t_1^{\max} := \max\{\dim(\langle c_1, \ldots, c_t\rangle \cap \mathcal{C}_1) : c_1, \ldots, c_t \text{ are any } t \text{ linearly independent codewords in } \mathcal{C}\}.$$

$$t_2^{\max} := \max\{\dim(\langle c_1, \ldots, c_t\rangle \cap \mathcal{C}_2) : c_1, \ldots, c_t \text{ are any } t \text{ linearly independent codewords in } \mathcal{C}\}.$$

Assume G to be a generator matrix of a k-dimensional q-ary linear code. The *finite geometry method* is to view the columns of G as points of the $(k-1)$-dimensional projective space $PG(k-1, q)$. Such a viewpoint induces a map $m(\cdot)$ from $PG(k-1, q)$ to the set of nonnegative integers:

$$m : PG(k-1, q) \to \boldsymbol{N}$$

where $\boldsymbol{N} = \{0, 1, 2, \ldots\}$, and for any $p \in PG(k-1, q)$, $m(p)$ is the number of occurrences of p as a column of G. $m(p)$ is called *the value of* p and the map $m(\cdot)$ is called a *value assignment* (or *value function*) [2]. This map can be extended to any subset $S \subset PG(k-1, q)$ by defining

$$m(S) = \sum_{p \in S} m(p).$$

Obviously, a value assignment and a code can be determined each other (up to code equivalence).

For an $[n,k]$ relative three-weight code $\mathcal{C}(\omega_1,\omega_2,\omega_3)$ in this paper, we fix a generator matrix G with the first k_1 and k_2 rows generating the subcodes $\mathcal{C}_1$ and $\mathcal{C}_2 \supset \mathcal{C}_1$, respectively. Assume G determines the value assignment $m(\cdot)$. In addition, let $L \subset \{1,2,\ldots,k\}$ and $p=(t_1,t_2,\ldots,t_k) \in PG(k-1,q)$, then define $P_L(p) := (v_1,v_2,\ldots,v_k)$ where $v_i = t_i$ if $i \in L$, otherwise, $v_i = 0$. Using the above notations, the geometric structure of relative three-weight codes may be given as follows.

Lemma 1.1 ([9]). *Let $\mathcal{C}(\omega_1,\omega_2,\omega_3)$ be a relative three-weight code with respect to a k_1-dimensional subcode $\mathcal{C}_1$ and a k_2-dimensional subcode $\mathcal{C}_2$, and let G and $m(\cdot)$ be defined as the above. Then $m(\cdot)$ satisfies*

$$m(p) = \begin{cases} \frac{\omega_1}{q^{k-1}}, & p \in S_1, \\ \frac{q^{k_1}\omega_2-(q^{k_1}-1)\omega_1}{q^{k-1}}, & p \in S_2, \\ \frac{q^{k_2}\omega_3-(q^{k_1}-1)\omega_1-(q^{k_2}-q^{k_1})\omega_2}{q^{k-1}}, & p \in S_3, \end{cases} \tag{1}$$

where $S_i \subset PG(k-1,q)$ for $1 \le i \le 3$ and $S_1 = \{p : P_{L_1}(p) \neq 0\}$, $S_2 = \{p : P_{L_1}(p) = 0$ and $P_{L_2}(p) \neq 0\}$, $S_3 = \{p : P_{L_1}(p) = 0$ and $P_{L_2}(p) = 0\}$, and $L_1 = \{1,2,\ldots,k_1\}$, and $L_2 = \{k_1+1,\ldots,k_2\}$.

2. Some preliminary lemmas

Note that if $\omega_1 = \omega_2$ in Definition 1.2, then $\mathcal{C}(\omega_1,\omega_2,\omega_3)$ reduces to a relative two-weight code with respect to $\mathcal{C}_2$ [8], and if $\omega_2 = \omega_3$, then $\mathcal{C}$ is a relative two-weight code with respect to $\mathcal{C}_1$, and if $\omega_1 = \omega_2 = \omega_3$, then $\mathcal{C}$ becomes a constant-weight code. Thus, a relative three-weight code is a generalization of both a relative two-weight one and a constant-weight one. The t-wise intersection of relative three-weight codes can be determined by borrowing the idea of that of a relative two-weight one and a constant-weight one.

Lemma 2.1 ([8]). *The t-wise intersection of a linear constant-weight code with weight ω is equal to $(\frac{q-1}{q})^{t-1}\omega$.*

Lemma 2.2 ([8]). *The t-wise $(1 \le t \le k)$ intersection of a relative two-weight code $\mathcal{C}(\omega_1,\omega_2)$ with respect to a subcode $\mathcal{C}_1$ (with weight ω_1) is equal to*

$$\begin{cases} (\frac{q-1}{q})^{t-1}\omega_1, & \omega_1 < \omega_2, \\ (\frac{q-1}{q})^{t-1}\omega_1 - (\frac{q-1}{q})^{t-t^{\max}-1}(\omega_1-\omega_2), & t^{\max} < t \text{ and } \omega_1 > \omega_2, \\ (\frac{q-1}{q})^{t-1}\omega_1 - (\omega_1-\omega_2), & t^{\max} = t \text{ and } \omega_1 > \omega_2, \end{cases}$$

where

$$t^{\max} = \max\{\dim(\langle c_1,\ldots,c_t\rangle \cap \mathcal{C}_1) : c_1,\ldots,c_t \text{ are any } t \text{ linearly independent codewords in } \mathcal{C}\}.$$

For a relative three-weight code $\mathcal{C}(\omega_1,\omega_2,\omega_3)$, any t linearly independent codewords $c_1,\ldots,c_t$ can be written as the following matrix operation by using the generating matrix G of $\mathcal{C}(\omega_1,\omega_2,\omega_3)$.

$$\begin{pmatrix} c_1\\ \vdots \\ c_t\end{pmatrix} = X_{t\times k}G = (X_{t\times k_1}, X_{t\times (k-k_1)})\begin{pmatrix} G_{k_1\times n}\\ G_{(k-k_1)\times n}\end{pmatrix}$$
$$= (X_{t\times k_2}, X_{t\times (k-k_2)})\begin{pmatrix} G_{k_2\times n}\\ G_{(k-k_2)\times n}\end{pmatrix}.$$

Note that $\text{rank}(X_{t\times k}) = t$, and that the block matrices $G_{k_1\times n}$ and $G_{k_2\times n}(k_1 < k_2)$ are generator matrices of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively.

Lemma 2.3. *If $\mathcal{D} = \langle c_1,\ldots,c_t\rangle$ is a relative (t,t_1,t_2) subcode of a relative three-weight code $\mathcal{C}$, then*

$$\text{rank}(X_{t\times(k-k_1)}) = t-t_1; \quad \text{rank}(X_{t\times(k-k_2)}) = t-t_2.$$

Proof. Since $\langle c_1,\ldots,c_t\rangle$ is a relative (t,t_1,t_2) subcode, there is an invertible matrix $Y_{t\times t}$ such that

$$\begin{aligned} Y_{t\times t}X_{t\times k} &= \big(Y_{t\times t}X_{t\times k_1}, Y_{t\times t}X_{t\times(k-k_1)}\big)\\ &= \big(Y_{t\times t}X_{t\times k_2}, Y_{t\times t}X_{t\times(k-k_2)}\big)\\ &= \begin{pmatrix} X'_{t_1\times k_1} & 0_{t_1\times(k-k_1)}\\ X'_{(t-t_1)\times k_1} & X'_{(t-t_1)\times(k-k_1)}\end{pmatrix}\\ &= \begin{pmatrix} X'_{t_1\times k_1} & 0_{t_1\times(k_2-k_1)} & 0_{t_1\times(k-k_2)}\\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k-k_2)}\\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)}\end{pmatrix}, \end{aligned}$$

with

$$\text{rank}(X'_{t_1\times k_1}) = t_1, \text{rank}(X'_{(t-t_1)\times(k-k_1)}) = t-t_1 \text{ and}$$
$$\text{rank}(X''_{(t-t_2)\times(k-k_2)}) = t-t_2.$$

Therefore, $\text{rank}(X_{t\times(k-k_1)}) = \text{rank}(Y_{t\times t}X_{t\times(k-k_1)}) = \text{rank}(X'_{(t-t_1)\times(k-k_1)}) = t-t_1$ and $\text{rank}(X_{t\times(k-k_2)}) = \text{rank}(Y_{t\times t}X_{t\times(k-k_2)}) = \text{rank}(X''_{(t-t_2)\times(k-k_2)}) = t-t_2$. □

3. The main result

The t-wise intersection of a relative three-weight $\mathcal{C}(\omega_1,\omega_2,\omega_3)$ is closely related to the size of ω_1, ω_2 and ω_3. By comparing the size of ω_1, ω_2 and ω_3, we may divide the analysis into six cases. In this section, we will determine the t-wise intersection of a binary relative three-weight code for all these six cases, and then we will state in a later remark how to determine the t-wise intersection of any q-ary ($q > 2$) relative three-weight code for the first four cases, and state the difficulties to determine the t-wise intersection of any q-ary relative three-weight code for the last two cases.

Let the notations be the same as in Lemma 1.1 and let $m_1 := m(p_1)$, $m_2 := m(p_2)$ and $m_3 := m(p_3)$ for $p_1 \in S_1$, $p_2 \in S_2$ and $p_3 \in S_3$. Then, it follows from (1) that

$$\begin{aligned} \omega_1 &= m_1 q^{k-1}; \\ \omega_1 - \omega_2 &= q^{k-k_1-1}(m_1 - m_2); \\ \omega_2 - \omega_3 &= q^{k-k_2-1}(m_2 - m_3). \end{aligned}$$

Theorem 3.1. *The t-wise intersection of binary relative three-weight codes $\mathcal{C}(\omega_1, \omega_2, \omega_3)$ is equal to*

(i) $(\frac{1}{2})^{t-1}\omega_1, \quad \omega_3 > \omega_2 > \omega_1.$

(ii) $$\begin{cases} (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_2^{\max}-1}(\omega_2 - \omega_3) - (\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1 - \omega_2), & \begin{cases} t_2^{\max} < t \\ \omega_1 > \omega_2 > \omega_3 \end{cases} \\ (\frac{1}{2})^{t-1}\omega_1 - (\omega_2 - \omega_3) - (\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1 - \omega_2), & \begin{cases} t_2^{\max} = t \\ t_1^{\max} < t \\ \omega_1 > \omega_2 > \omega_3 \end{cases} \\ (\frac{1}{2})^{t-1}\omega_1 - (\omega_1 - \omega_3), & \begin{cases} t_2^{\max} = t_1^{\max} = t \\ \omega_1 > \omega_2 > \omega_3. \end{cases} \end{cases}$$

(iii) $$\begin{cases} (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1 - \omega_2), \ t_1^{\max} < t \text{ and } \omega_1 > \omega_3 > \omega_2 \\ (\frac{1}{2})^{t-1}\omega_1 - (\omega_1 - \omega_2), \ t_1^{\max} = t \text{ and } \omega_1 > \omega_3 > \omega_2. \end{cases}$$

(iv) $$\begin{cases} (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1 - \omega_2), \ t_1^{\max} < t \text{ and } \omega_3 > \omega_1 > \omega_2 \\ (\frac{1}{2})^{t-1}\omega_1 - (\omega_1 - \omega_2), \ t_1^{\max} = t \text{ and } \omega_3 > \omega_1 > \omega_2. \end{cases}$$

(v) $$\begin{cases} \min\{(\frac{1}{2})^{t-1}\omega_2 - (\frac{1}{2})^{t-t_2^{\max}-1}(\omega_2 - \omega_3); (\frac{1}{2})^{t-1}\omega_1\}, & \begin{cases} t_2^{\max} < t \\ \omega_2 > \omega_3 > \omega_1 \end{cases} \\ \min\{(\frac{1}{2})^{t-1}\omega_2 - (\omega_2 - \omega_3); (\frac{1}{2})^{t-1}\omega_1\}, \ t_2^{\max} = t \text{ and } \omega_2 > \omega_3 > \omega_1. \end{cases}$$

(vi) $$\begin{cases} (\frac{1}{2})^{t-1}\omega_2 - (\frac{1}{2})^{t-t_2^{\max}-1}(\omega_2 - \omega_3), \ t_2^{\max} < t \text{ and } \omega_2 > \omega_1 > \omega_3 \\ (\frac{1}{2})^{t-1}\omega_2 - (\omega_2 - \omega_3), \ t_2^{\max} = t \text{ and } \omega_2 > \omega_1 > \omega_3. \end{cases}$$

Before giving out the detailed proof of Theorem 3.1, we introduce a key lemma which will be used in cases (v) and (vi). In these two cases, the fact that ω_2 is greater than both ω_1 and ω_3 yields $m_2 > m_1$ and $m_2 > m_3$. Expand the generator matrix G of $\mathcal{C}$ to the following form:

$$(G, G_3) = (G_1, G_2),$$

where G_1 consists of all points in $PG(k-1, 2)$ with each point repeating m_1 times, and all the points in $S_2 \cup S_3$ constitute the columns of G_2 with each point repeating $m_2 - m_1$ times. The columns of G_3 consist of all points of S_3 and each point repeats $m_2 - m_3$ times. Then, G_1 generates a k-dimensional constant-weight code $\mathcal{C}'$ with weight $m_1 2^{k-1}$ and length $l_1 = m_1(2^k - 1)$, and

G_2 generates a $(k-k_1)$-dimensional constant-weight code $\mathcal{C}''$ with weight $(m_2-m_1)2^{k-k_1-1}$ and length $l_2=(m_2-m_1)(2^{k-k_1}-1)$, and G_3 generates a $(k-k_2)$-dimensional constant-weight code $\mathcal{C}'''$ with weight $(m_2-m_3)2^{k-k_2-1}$ and length $l_3=(m_2-m_3)(2^{k-k_2}-1)$. Let $c_1,\ldots,c_t$ be any t linearly independent codewords in $\mathcal{C}$ and $\langle c_1,\ldots,c_t\rangle$ is a relative $(t,t_1,t_2)(t_1<t_2<t)$ subcode. Denote $\begin{pmatrix} c_1\\ \vdots\\ c_t\end{pmatrix}=X_{t\times k}G$, and introduce $\begin{pmatrix} c'_1\\ \vdots\\ c'_t\end{pmatrix}=X_{t\times k}G_1$, $\begin{pmatrix} c''_1\\ \vdots\\ c''_t\end{pmatrix}=X_{t\times k}G_2$, $\begin{pmatrix} c'''_1\\ \vdots\\ c'''_t\end{pmatrix}=X_{t\times k}G_3$. According to the above statement, for any $i\in\{1,2,\ldots,t\}$, we have $(c_i,c'''_i)=(c'_i,c''_i)$ with $c'_i\in\mathcal{C}',c''_i\in\mathcal{C}''$ and $c'''_i\in\mathcal{C}'''$. Besides, $c'_1,\ldots,c'_t$ are linearly independent codewords, whereas $\mathrm{rank}(c''_1,\ldots,c''_t)=t-t_1$ and $\mathrm{rank}(c'''_1,\ldots,c'''_t)=t-t_2$ by Lemma 2.3. For convenience, we will always denote *inter*, *inter*$_1$, *inter*$_2$ and *inter*$_3$ as follows.

$$\mathit{inter}:=|\bigcap_{i=1}^{t}\chi(c_i)|;\quad \mathit{inter}_1:=|\bigcap_{i=1}^{t}\chi(c'_i)|;$$
$$\mathit{inter}_2:=|\bigcap_{i=1}^{t}\chi(c''_i)|;\quad \mathit{inter}_3:=|\bigcap_{i=1}^{t}\chi(c'''_i)|.$$

Based on Lemma 2.1, we have $\mathit{inter}=\mathit{inter}_1+\mathit{inter}_2-\mathit{inter}_3$ with $\mathit{inter}_1=(\frac{1}{2})^{t-1}m_12^{k-1}$, $0\le\mathit{inter}_2\le(\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}$ and $0\le\mathit{inter}_3\le(\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1}$. Preserve the same notations aforementioned. Then, we have:

Lemma 3.1. *Assume $q=2$ and $\omega_2>\max\{\omega_1,\omega_3\}$, and let $\mathcal{D}=\langle c_1,\ldots,c_t\rangle$ be a relative (t,t_1,t_2) $(t_1<t_2<t)$ subcode of $\mathcal{C}$ with $\mathit{inter}_3\neq 0$. Then*

$$\mathit{inter}_2=(\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}.$$

Proof. Write $\begin{pmatrix} c_1\\ \vdots\\ c_t\end{pmatrix}=X_{t\times k}G$. Then, as in the proof of Lemma 2.3, there exists an invertible matrix $Y_{t\times t}$ such that

$$Y_{t\times t}X_{t\times k}=\begin{pmatrix} X'_{t_1\times k_1} & 0_{t_1\times(k_2-k_1)} & 0_{t_1\times(k-k_2)}\\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k-k_2)}\\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)}\end{pmatrix},$$

where

$$\mathrm{rank}(X'_{t_1\times k_1})=t_1,\mathrm{rank}(X''_{(t_2-t_1)\times(k_2-k_1)})=t_2-t_1 \text{ and}$$
$$\mathrm{rank}(X''_{(t-t_2)\times(k-k_2)})=t-t_2.$$

Thus,

$$Y_{t\times t}\begin{pmatrix} c''_1 & \vline & c'''_1 \\ \vdots & \vline & \vdots \\ c''_t & \vline & c'''_t \end{pmatrix} = \begin{pmatrix} 0 & \vline & 0 \\ \vdots & \vline & \vdots \\ 0 & \vline & 0 \\ \bar{c}''_{t_1+1} & \vline & 0 \\ \vdots & \vline & \vdots \\ \bar{c}''_{t_2} & \vline & 0 \\ \bar{c}''_{t_2+1} & \vline & \bar{c}'''_{t_2+1} \\ \vdots & \vline & \vdots \\ \bar{c}''_t & \vline & \bar{c}'''_t \end{pmatrix},$$

where $\mathrm{rank}(\bar{c}''_{t_1+1},\ldots,\bar{c}''_t) = t-t_1$ and $\mathrm{rank}(\bar{c}'''_{t_2+1},\ldots,\bar{c}'''_t) = t-t_2$. Obviously, we have $\mathrm{rank}\begin{pmatrix} c''_1 & \vline & c'''_1 \\ \vdots & \vline & \vdots \\ c''_t & \vline & c'''_t \end{pmatrix} = t-t_1$. Without loss of generality, let $(c''_{t_1+1},\ldots,c''_t)$ be a maximal linearly independent set of $(c''_1,\ldots,c''_t)$. Then, the last $(t-t_1)$ rows of the matrix $\begin{pmatrix} c''_1 & \vline & c'''_1 \\ \vdots & \vline & \vdots \\ c''_t & \vline & c'''_t \end{pmatrix}$, that is, $\begin{pmatrix} c''_{t_1+1} & \vline & c'''_{t_1+1} \\ \vdots & \vline & \vdots \\ c''_t & \vline & c'''_t \end{pmatrix}$, is the maximal linearly independent set of its all rows. So, there exists a matrix $\begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix}$ such that

$$\begin{pmatrix} c''_1 & \vline & c'''_1 \\ \vdots & \vline & \vdots \\ c''_{t_1} & \vline & c'''_{t_1} \end{pmatrix} = \begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix}\begin{pmatrix} c''_{t_1+1} & \vline & c'''_{t_1+1} \\ \vdots & \vline & \vdots \\ c''_t & \vline & c'''_t \end{pmatrix},$$

that is,

$$\begin{pmatrix} c''_1 \\ \vdots \\ c''_{t_1} \end{pmatrix} = \begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix}\begin{pmatrix} c''_{t_1+1} \\ \vdots \\ c''_t \end{pmatrix}, \tag{2}$$

$$\begin{pmatrix} c'''_1 \\ \vdots \\ c'''_{t_1} \end{pmatrix} = \begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix}\begin{pmatrix} c'''_{t_1+1} \\ \vdots \\ c'''_t \end{pmatrix}. \tag{3}$$

Based on (3) and $\mathrm{rank}(c'''_1,\ldots,c'''_t) = t-t_2$, without loss of generality, we assume $(c'''_{t_2+1},\ldots,c'''_t)$ to be the maximal linearly independent set of $(c'''_1,\ldots,c'''_t)$.

Then, there is a matrix $\begin{pmatrix} b_{(t_1+1)\times(t_2+1)} & \cdots & b_{(t_1+1)\times t} \\ \vdots & \ddots & \vdots \\ b_{t_2\times(t_2+1)} & \cdots & b_{t_2\times t} \end{pmatrix}$ such that

$$\begin{pmatrix} c'''_{t_1+1} \\ \vdots \\ c'''_{t_2} \end{pmatrix} = \begin{pmatrix} b_{(t_1+1)\times(t_2+1)} & \cdots & b_{(t_1+1)\times t} \\ \vdots & \ddots & \vdots \\ b_{t_2\times(t_2+1)} & \cdots & b_{t_2\times t} \end{pmatrix} \begin{pmatrix} c'''_{t_2+1} \\ \vdots \\ c'''_{t} \end{pmatrix}. \tag{4}$$

Combing (3) and (4), one gets

$$\begin{pmatrix} c'''_{1} \\ \vdots \\ c'''_{t_1} \end{pmatrix} = \begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix} \begin{pmatrix} b_{(t_1+1)\times(t_2+1)} & \cdots & b_{(t_1+1)\times t} \\ \vdots & \ddots & \vdots \\ b_{t_2\times(t_2+1)} & \cdots & b_{t_2\times t} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} c'''_{t_2+1} \\ \vdots \\ c'''_{t} \end{pmatrix}. \tag{5}$$

Since $inter_3 \neq 0$, there must be a coordinate position $j_0 \in \{1, 2, \ldots, l_3\}$ such that $j_0 \in \chi(c'''_i) \quad \forall 1 \leq i \leq t$. According to (4) and (5), we have

$$\begin{cases} \begin{pmatrix} b_{(t_1+1)\times(t_2+1)} & \cdots & b_{(t_1+1)\times t} \\ \vdots & \ddots & \vdots \\ b_{t_2\times(t_2+1)} & \cdots & b_{t_2\times t} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}, \\ \begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix} \begin{pmatrix} b_{(t_1+1)\times(t_2+1)} & \cdots & b_{(t_1+1)\times t} \\ \vdots & \ddots & \vdots \\ b_{t_2\times(t_2+1)} & \cdots & b_{t_2\times t} \\ 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \end{cases}$$

Thus,

$$\begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}. \tag{6}$$

Furthermore, denote $\bigcap_{i=t_1+1}^{t} \chi(c''_i) = \{j_1, j_2, \ldots, j_r\}$ and let $\begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$ be the matrix which consists of the j_1th, j_2th,…,and j_rth columns of matrix

$\begin{pmatrix} c''_{t_1+1} \\ \vdots \\ c''_t \end{pmatrix}$. Then, based on (2) and (6), one gets

$$\begin{pmatrix} a_{1\times(t_1+1)} & \cdots & a_{1\times t} \\ \vdots & \ddots & \vdots \\ a_{t_1\times(t_1+1)} & \cdots & a_{t_1\times t} \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}.$$

It is obviously that

$$\bigcap_{i=1}^{t} \chi(c''_i) = \bigcap_{i=t_1+1}^{t} \chi(c''_i),$$

then,

$$inter_2 = |\bigcap_{i=t_1+1}^{t} \chi(c''_i)|.$$

Since $(c''_{t_1+1}, \ldots, c''_t)$ are $t - t_1$ linearly independent codewords of constant-weight code $\mathcal{C}''$ with weight $(m_2 - m_1)2^{k-k_1-1}$, it follows $|\bigcap_{i=t_1+1}^{t} \chi(c''_i)| = (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1}$ by Lemma 2.1. Thus $inter_2 = (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1}$. □

Now we are ready to show Theorem 3.1.

Proof. (Case i) Since $\omega_3 > \omega_2 > \omega_1$, it can be checked that there holds $m_3 > m_2 > m_1$. Then, the generator matrix G of code $\mathcal{C}$ can be rewritten

$$G = \left(G_1, G_2, G_3\right),$$

where G_1 consists of all points from $PG(k-1, 2)$ and each point occurs m_1 times. All points in $S_2 \cup S_3$ constitute columns of G_2 and each point repeats $m_2 - m_1$ times. The block matrix G_3 is made up of all points in S_3 and the number of occurrence of each point as columns of G_3 is $m_3 - m_2$. So G_1 generates a k-dimensional constant-weight code $\mathcal{C}'$ with weight $m_1 2^{k-1}$ and length $l_1 = m_1(2^k - 1)$. G_2 generates a $(k - k_1)$-dimensional constant-weight code $\mathcal{C}''$ with weight $(m_2 - m_1)2^{k-k_1-1}$ and length $l_2 = (m_2 - m_1)(2^{k-k_1} - 1)$. G_3 generates a $(k - k_2)$-dimensional constant-weight code $\mathcal{C}'''$ with weight $(m_3 - m_2)2^{k-k_2-1}$ and length $l_3 = (m_3 - m_2)(2^{k-k_2} - 1)$. Let $c_1, \ldots, c_t$ be any t linearly independent codewords in $\mathcal{C}$ and $\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k}G$, then introduce $\begin{pmatrix} c'_1 \\ \vdots \\ c'_t \end{pmatrix} = X_{t\times k}G_1$, $\begin{pmatrix} c''_1 \\ \vdots \\ c''_t \end{pmatrix} = X_{t\times k}G_2$ and $\begin{pmatrix} c'''_1 \\ \vdots \\ c'''_t \end{pmatrix} = X_{t\times k}G_3$. It can be concluded that each above codeword c_i $(i = 1, 2, \ldots, t)$ can be divided into three sectors, that is, $c_i = (c'_i, c''_i, c'''_i)$ with $c'_i \in \mathcal{C}', c''_i \in \mathcal{C}''$ and $c'''_i \in \mathcal{C}'''$. Obviously, the codewords $c'_1, \ldots, c'_t$ are linearly independent. In addition, based on Lemma 2.3, the rank of codewords $c''_1, \ldots, c''_t$ is $(t - t_1)$, and the codewords

$c_1''', \ldots, c_t'''$ have rank $(t-t_2)$. Based on Lemma 2.1, we conclude that $inter_1 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$ and $0 \leq inter_2 \leq (\frac{1}{2})^{t-t_1-1}(m_2-m_1)2^{k-k_1-1}, 0 \leq inter_3 \leq (\frac{1}{2})^{t-t_2-1}(m_3-m_2)2^{k-k_2-1}$. Furthermore, $inter = inter_1 + inter_2 + inter_3$. Thus, $inter = (\frac{1}{2})^{t-1} m_1 2^{k-1}$ is reachable whenever $c_1'' = 0$ and $c_1''' = 0$. Additionally, it follows that $c_1'' = 0$ and $c_1''' = 0$ are equivalent to $c_1 \in \mathcal{C}_1$ and $c_1 \in \mathcal{C}_2$ respectively. Since $\dim(\mathcal{C}_1) \geq 1$, we can select an arbitrary nonzero codeword c_1 from $\mathcal{C}_1$ and expand it to t linearly independent codewords $c_1, \ldots, c_t$ in $\mathcal{C}$. So, in case $\omega_3 > \omega_2 > \omega_1$, the t-wise intersection of binary relative three-weight codes is $(\frac{1}{2})^{t-1} m_1 2^{k-1} = (\frac{1}{2})^{t-1}\omega_1$.

(Case ii) Based on $\omega_1 > \omega_2 > \omega_3$, we may obtain $m_1 > m_2 > m_3$. Similar to the analysis in (Case i), these matrices, G_1, G_2, G_3, are introduced.

$$G_3 = (G, G_1, G_2),$$

where the columns of G_1 are all points in $S_2 \cup S_3$ and each point appears $m_1 - m_2$ times. G_2 consists of all points in S_3 and each point appears $m_2 - m_3$ times. All the points in $PG(k-1,2)$ constitute the columns of matrix G_3 and the number of occurrence of each point as columns in G_3 is m_1. Hence, G_1 generates a $(k-k_1)$-dimensional constant-weight code $\mathcal{C}'$ with weight $(m_1 - m_2)2^{k-k_1-1}$ and length $l_1 = (m_1-m_2)(2^{k-k_1}-1)$. G_2 generates a $(k-k_2)$-dimensional constant-weight code $\mathcal{C}''$ with weight $(m_2-m_3)2^{k-k_2-1}$ and length $l_2 = (m_2-m_3)(2^{k-k_2}-1)$. G_3 generates a k-dimensional constant-weight code $\mathcal{C}'''$ with weight $m_1 2^{k-1}$ and length $l_3 = m_1(2^k-1)$. Let c_i be an arbitrary codeword of the t linearly independent codewords $c_1, \ldots, c_t$ in $\mathcal{C}$ with the matrix form $\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k}G$. Then, we have $c_i''' = (c_i, c_i', c_i'')$ for any $i \in \{1, 2, \ldots, t\}$ with $c_i' \in \mathcal{C}', c_i'' \in \mathcal{C}''$ and $c_i''' \in \mathcal{C}'''$. Besides, $\text{rank}(c_1''', \ldots, c_t''') = t$, whereas $\text{rank}(c_1', \ldots, c_t') = t - t_1$ and $\text{rank}(c_1'', \ldots, c_t'') = t - t_2$ by Lemma 2.3. Furthermore, $inter = inter_3 - inter_2 - inter_1$ with $inter_3 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \leq inter_2 \leq (\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1}$ and $0 \leq inter_1 \leq (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$ by Lemma 2.1.

Next, we state that $inter = (\frac{1}{2})^{t-1} m_1 2^{k-1} - (\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1} - (\frac{1}{2})^{t-t_1-1}(m_1-m_2)2^{k-k_1-1}$ can be reachable when $\langle c_1, \ldots, c_t \rangle$ is a relative (t, t_1, t_2) subcode and $t_1 \leq t_2 < t$. Let $c_1, \ldots, c_t$ be arbitrary t linearly independent codewords and $\langle c_1, \ldots, c_t \rangle$ is a relative (t, t_1, t_2) subcode of $\mathcal{C}$. According to the proof of Lemma 2.3, there exists an invertible matrix $Y_{t\times t}$ such that

$$Y_{t\times t}X_{t\times k} = \begin{pmatrix} X'_{t_1\times k_1} & 0_{t_1\times(k_2-k_1)} & 0_{t_1\times(k-k_2)} \\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k-k_2)} \\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)} \end{pmatrix},$$

with

$$\text{rank}(X'_{t_1\times k_1}) = t_1, \text{rank}(X''_{(t_2-t_1)\times(k_2-k_1)}) = t_2 - t_1 \text{ and}$$
$$\text{rank}(X''_{(t-t_2)\times(k-k_2)}) = t - t_2.$$

Next, we can find another invertible matrix $Z_{t\times t}$ such that

$$Z_{t\times t}Y_{t\times t}X_{t\times k}=\begin{pmatrix} X'''_{t_1\times k_1} & X'''_{t_1\times(k_2-k_1)} & X'''_{t_1\times(k-k_2)} \\ X'''_{(t_2-t_1)\times k_1} & X'''_{(t_2-t_1)\times(k_2-k_1)} & X'''_{(t_2-t_1)\times(k-k_2)} \\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)} \end{pmatrix},$$

with each row of $X'''_{t_1\times(k-k_2)}$ and $X'''_{(t_2-t_1)\times(k-k_2)}$ is the same as the last row of $X''_{(t-t_2)\times(k-k_2)}$ and each row of $X'''_{t_1\times(k_2-k_1)}$ equals to the last row of

$$X''_{(t-t_2)\times(k_2-k_1)}.$$

Denote $c_1,\dots,c_t$ the rows of matrix $Z_{t\times t}Y_{t\times t}X_{t\times k}G$ (as the new t linearly independent codewords). Then, we can conclude that these t linearly independent codewords have intersection

$$\begin{aligned} inter &= inter_3-inter_2-inter_1\\ &=(\frac{1}{2})^{t-1}m_1 2^{k-1}-(\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1}\\ &\quad-(\frac{1}{2})^{t-t_1-1}(m_1-m_2)2^{k-k_1-1}\\ &=(\frac{1}{2})^{t-1}\omega_1-(\frac{1}{2})^{t-t_2-1}(\omega_2-\omega_3)-(\frac{1}{2})^{t-t_1-1}(\omega_1-\omega_2). \end{aligned}$$

So, all t linearly independent codewords with property that their generating subspace is a relative $(t,t_1,t_2)(t_1\le t_2<t)$ subcode have the minimum intersection

$$(\frac{1}{2})^{t-1}\omega_1-(\frac{1}{2})^{t-t_2-1}(\omega_2-\omega_3)-(\frac{1}{2})^{t-t_1-1}(\omega_1-\omega_2).$$

Thus, for all parameters t_1 and t_2, we get the t-wise intersection of binary relative three-weight codes in case $\omega_1>\omega_2>\omega_3$, that is,

$$\min_{(t_1,t_2)} inter=\begin{cases}(\frac{1}{2})^{t-1}\omega_1-(\frac{1}{2})^{t-t_2^{\max}-1}(\omega_2-\omega_3)-(\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1-\omega_2), & t_2^{\max}<t\\ (\frac{1}{2})^{t-1}\omega_1-(\omega_2-\omega_3)-(\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1-\omega_2), & t_2^{\max}=t \text{ and } t_1^{\max}<t\\ (\frac{1}{2})^{t-1}\omega_1-(\omega_1-\omega_3), & t_2^{\max}=t_1^{\max}=t.\end{cases}$$

(Case iii) From $\omega_1>\omega_3>\omega_2$, we deduce $m_1>m_2$ and $m_3>m_2$. For the generator matrix G of code $\mathcal{C}$, there are three matrices G_1,G_2,G_3 with following properties.

$$(G,G_1)=(G_2,G_3),$$

where the block matrix G_1 is made up of all points in $S_2\cup S_3$ and each point repeats m_1-m_2 times. G_2 consists of all points in $PG(k-1,2)$ and each point appears m_1 times. All points in S_3 constitute columns of G_3 and each point occurs m_3-m_2 times. Thus, G_1 generates a $(k-k_1)$-dimensional constant-weight code $\mathcal{C}'$ with weight $(m_1-m_2)2^{k-k_1-1}$ and length $l_1=(m_1-m_2)(2^{k-k_1}-1)$. G_2 generates a k-dimensional constant-weight code $\mathcal{C}''$ with weight $m_1 2^{k-1}$ and length $l_2=m_1(2^k-1)$. G_3 generates a $(k-k_2)$-dimensional constant-weight code $\mathcal{C}'''$ with weight $(m_3-m_2)2^{k-k_2-1}$ and length $l_3=(m_3-m_2)(2^{k-k_2}-$

1). Assume that $c_1, \dots, c_t$ with the matrix form $\begin{pmatrix} c_1 \\ \vdots \\ c_t \end{pmatrix} = X_{t\times k}G$ are any t linearly independent codewords in $\mathcal{C}$. Obviously, for any $i \in \{1, 2, \dots, t\}$, we have $(c_i, c_i') = (c_i'', c_i''')$ where $c_i' \in \mathcal{C}', c_i'' \in \mathcal{C}''$ and $c_i''' \in \mathcal{C}'''$. Additionally, $\mathrm{rank}(c_1'', \dots, c_t'') = t$. Based on Lemma 2.3, we have $\mathrm{rank}(c_1', \dots, c_t') = t - t_1$ and $\mathrm{rank}(c_1''', \dots, c_t''') = t - t_2$. Furthermore, we have $inter = inter_2 + inter_3 - inter_1$ with $inter_2 = (\frac{1}{2})^{t-1} m_1 2^{k-1}$, $0 \le inter_1 \le (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$ and $0 \le inter_3 \le (\frac{1}{2})^{t-t_2-1}(m_3 - m_2)2^{k-k_2-1}$ by Lemma 2.1.

Next, we state that both $inter_3 = 0$ and $inter_1 = (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$ are reachable when $\langle c_1, \dots, c_t \rangle$ is a relative $(t, t_1, t_2)(t_1 < t_2 \le t)$ subcode. For any t linearly independent codewords $c_1, \dots, c_t$ with $\langle c_1, \dots, c_t \rangle$ being a relative (t, t_1, t_2) subcode, we can always find two invertible matrices $Y_{t\times t}$ and $Z_{t\times t}$ such that

$$Z_{t\times t} Y_{t\times t} X_{t\times k} = \begin{pmatrix} X'''_{t_1\times k_1} & X'''_{t_1\times(k_2-k_1)} & X'''_{t_1\times(k-k_2)} \\ X''_{(t_2-t_1)\times k_1} & X''_{(t_2-t_1)\times(k_2-k_1)} & 0_{(t_2-t_1)\times(k-k_2)} \\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)} \end{pmatrix},$$

where each row of $X'''_{t_1\times(k-k_2)}$ equals to the last row of matrix $X''_{(t-t_2)\times(k-k_2)}$ and each row of $X'''_{t_1\times(k_2-k_1)}$ is same as the last row of $X''_{(t-t_2)\times(k_2-k_1)}$. Then taking the rows of matrix $Z_{t\times t} Y_{t\times t} X_{t\times k} G$ to be new t linearly independent codewords and still denoting them by $c_1, \dots, c_t$, we can infer that $inter_3 = 0$ and $inter_1 = (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1}$. Hence, all the t linearly independent codewords such that their generating subspace are relative $(t, t_1, t_2)(t_1 < t_2 \le t)$ subcodes have the minimum intersection

$$\begin{aligned} inter &= (\frac{1}{2})^{t-1} m_1 2^{k-1} - (\frac{1}{2})^{t-t_1-1}(m_1 - m_2)2^{k-k_1-1} \\ &= (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_1-1}(\omega_1 - \omega_2). \end{aligned}$$

Therefore, the t-wise intersection of binary relative three-weight codes in case $\omega_1 > \omega_3 > \omega_2$ is

$$\min_{(t_1,t_2)} inter = \begin{cases} (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1 - \omega_2), & t_1^{\max} < t \\ (\frac{1}{2})^{t-1}\omega_1 - (\omega_1 - \omega_2), & t_1^{\max} = t. \end{cases}$$

(Case iv) In this case, we can infer that $m_1 > m_2$ and $m_3 > m_2$. Then, according to the same analysis procedure in (Case iii), one gets that the minimum intersection of all t linearly independent codewords with property that the subspaces they generate are relative $(t, t_1, t_2)(t_1 < t_2 \le t)$ subcodes of $\mathcal{C}$ is

$$inter = (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_1-1}(\omega_1 - \omega_2).$$

Thus, the t-wise intersection of binary relative three-weight codes in case $\omega_3 > \omega_1 > \omega_2$ is

$$\min_{(t_1,t_2)} inter = \begin{cases} (\frac{1}{2})^{t-1}\omega_1 - (\frac{1}{2})^{t-t_1^{\max}-1}(\omega_1-\omega_2), & t_1^{\max} < t \\ (\frac{1}{2})^{t-1}\omega_1 - (\omega_1-\omega_2), & t_1^{\max} = t. \end{cases}$$

(Case v) According to Lemma 3.1, for any t linearly independent codewords with property that their generating subspace is relative (t, t_1, t_2) subcode of $\mathcal{C}$, if the corresponding $inter_3 \neq 0$, we have

$$inter = inter_1 + inter_2 - inter_3,$$

with $inter_1 = (\frac{1}{2})^{t-1}m_1 2^{k-1}$, $inter_2 = (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1}$. Then, while $inter_3 = (\frac{1}{2})^{t-t_2-1}(m_2-m_3)2^{k-k_2-1}$ is reachable, $inter$ has its minimum value.

For any t given codewords with aforementioned properties, there exist two invertible matrices $Y_{t\times t}$ and $Z_{t\times t}$ such that

$$Z_{t\times t}Y_{t\times t}X_{t\times k} = \begin{pmatrix} X'''_{t_1\times k_1} & X'''_{t_1\times(k_2-k_1)} & X'''_{t_1\times(k-k_2)} \\ X'''_{(t_2-t_1)\times k_1} & X'''_{(t_2-t_1)\times(k_2-k_1)} & X'''_{(t_2-t_1)\times(k-k_2)} \\ X''_{(t-t_2)\times k_1} & X''_{(t-t_2)\times(k_2-k_1)} & X''_{(t-t_2)\times(k-k_2)} \end{pmatrix},$$

with each row of $X'''_{t_1\times(k-k_2)}$ and $X'''_{(t_2-t_1)\times(k-k_2)}$ being the same as the last row of $X''_{(t-t_2)\times(k-k_2)}$ and each row of $X'''_{t_1\times(k_2-k_1)}$ being equal to the last row of $X''_{(t-t_2)\times(k_2-k_1)}$.

Let $c_1, \ldots, c_t$ be rows of matrix $Z_{t\times t}Y_{t\times t}X_{t\times k}G$. Then, $c_1, \ldots, c_t$ constitute new t linearly independent codewords. Then we have $inter = (\frac{1}{2})^{t-1}m_1 2^{k-1} + (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1} - (\frac{1}{2})^{t-t_2-1}(m_2 - m_3)2^{k-k_2-1}$.

Additionally, if $inter_3 = 0$, we have that

$$inter = inter_1 + inter_2 - inter_3,$$

with $inter_1 = (\frac{1}{2})^{t-1}m_1 2^{k-1}$ and $inter_3 = 0$. Thus, the minimum value of $inter$ is $(\frac{1}{2})^{t-1}m_1 2^{k-1}$ when $inter_2 = 0$. Next, we state that $inter_2 = 0$ can be reached. Since $\dim(\mathcal{C}_1) \geq 1$, we choose a nonzero codeword c_1 in $\mathcal{C}_1$ and expand it to t linearly independent codewords $c_1, \ldots, c_t$. It can be checked that $inter_2 = inter_3 = 0$. Thus, if $inter_3 = 0$, the minimum intersection of t linearly independent codewords is $inter = (\frac{1}{2})^{t-1}m_1 2^{k-1}$.

Summarizing the above discussion, we have that all t linearly independent codewords $(c_1, \ldots, c_t)$ with $\langle c_1, \ldots, c_t\rangle$ being a (t, t_1, t_2) $(t_1 \leq t_2 < t)$ subcodes of $\mathcal{C}$ have the minimum intersection

$$inter = \min\{ (\frac{1}{2})^{t-1}m_1 2^{k-1} + (\frac{1}{2})^{t-t_1-1}(m_2 - m_1)2^{k-k_1-1} - (\frac{1}{2})^{t-t_2-1}(m_2 - m_3)2^{k-k_2-1}; (\frac{1}{2})^{t-1}m_1 2^{k-1}\}$$

$$=\min\{\ (\frac{1}{2})^{t-1}\omega_1+(\frac{1}{2})^{t-t_1-1}(\omega_2-\omega_1)-(\frac{1}{2})^{t-t_2-1}(\omega_2-\omega_3);(\frac{1}{2})^{t-1}\omega_1\}.$$

Hence, the t-wise intersection of binary relative three-weight codes in case $\omega_2 > \omega_3 > \omega_1$ is

$$\min_{(t_1,t_2)} inter = \begin{cases} \min\{(\frac{1}{2})^{t-1}\omega_2-(\frac{1}{2})^{t-t_2^{\max}-1}(\omega_2-\omega_3);(\frac{1}{2})^{t-1}\omega_1\}, & t_2^{\max}<t \\ \min\{(\frac{1}{2})^{t-1}\omega_2-(\omega_2-\omega_3);(\frac{1}{2})^{t-1}\omega_1\}, & t_2^{\max}=t. \end{cases}$$

(Case vi) Similarly to the proof of (Case v), we can obtain the result in (vi) (the detailed proof is omitted). □

Remark 3.1. In fact, by generalizing the proof of Theorem 3.1 slightly, we may obtain the t-wise intersection of any q-ary ($q > 2$) relative three-weight code for the first four cases. For the last two cases, however, the generalization to q-ary ($q > 2$) codes is more difficult. The reason, as we have observed in the proof of Lemma 3.1, is that the element at the common support coordinate position of the codewords for $q = 2$ is explicit, that is, the unique nonzero element "1" in $GF(2)$, whereas for $q > 2$, this is not the case. Thus, for $q > 2$, we are not able to obtain a similar result as in Lemma 3.1 which can be used to prove the last two cases in Theorem 3.1.

4. Another application of the t-wise intersection

The t-wise intersection of a linear code is the minimal size of the support intersection of all the t linearly independent codewords. Our method to compute the t-wise intersection of relative three-weight codes is to use the geometric structure of relative three-weight codes given in (1). Note that the method is not only to obtain the t-wise intersection of relative three-weight codes but also to be able to determine the coordinate positions corresponding to the t-wise intersection positions, and thus we may locate the columns of the generator matrix G corresponding to the t-wise intersection positions after we write these t linearly independent codewords in matrix form by using G. Then, we may get a new matrix G' by puncturing those columns of G aforementioned, and it is possible to preserve the first k_2 (and thus the first k_1 rows) of G' still to be independent by making use of the geometric structure of a relative three-weight code given in (1). Such a puncturing operation above obviously produces a new t-wise intersecting code generated by G' only if some columns of G corresponding to the t-wise intersection positions are preserved. Different puncturing operations produce different value assignments, and thus produce different t-wise intersecting codes. Thus, we may get many t-wise intersecting codes in such a way. Let's illustrate our method as follows.

Example 4.1. For $q = 2$, let $k = 5, k_1 = 2, k_2 = 3, t = 3$. Give a value function as follows:

$$m(p) = \begin{cases} 1, & p \in S_1 \\ 2, & p \in S_2 \\ 3, & p \in S_3. \end{cases} \tag{7}$$

Then we have $\omega_1 = m_1 2^{k-1} = 16$, $\omega_2 = \omega_1 - 2^{k-k_1-1}(m_1 - m_2) = 20$ and $\omega_3 = \omega_2 - 2^{k-k_2-1}(m_2 - m_3) = 22$. Thus, $\omega_3 > \omega_2 > \omega_1$ which is corresponding to Case (i) in Theorem 3.1. So, the 3-wise intersection of binary relative three-weight codes is $(\frac{1}{2})^{t-1}\omega_1 = 4$. Besides, according to (7), a generator matrix G of $\mathcal{C}$ is

$$\left(\begin{array}{cccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&0&0&1&0&1&0&1&1&1&0&0&0&1&0&1&1&1&0&0\\
0&1&1&0&1&0&1&0&0&1&1&0&1&0&1&0&0&1&1&0&1&0&1&0\\
0&1&1&1&0&1&0&0&0&1&1&1&0&0&0&1&0&1&1&1&0&0&0&1
\end{array}\right.$$

$$\left.\begin{array}{ccccccccccccccccc}
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&1&1&0&0&1&1&0&0&1&1&0&1&1&0&1&1\\
0&1&0&1&0&1&0&1&1&0&1&1&0&1&1&0&1
\end{array}\right).$$

As in the proof of Case (i) in Theorem 3.1, we select an arbitrary nonzero codeword c_1 from $\mathcal{C}_1$ and expand it to 3 linearly independent codewords c_1, c_2, c_3 in $\mathcal{C}$. Then, the intersection of these codewords should achieve the minimum, that is, $(\frac{1}{2})^{t-1}\omega_1 = 4$. Choose such codewords c_1, c_2 and c_3 satisfying

$$\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0 \end{pmatrix} G$$

$$= \left(\begin{array}{ccccccccccccccccccccc}
1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&1&1&1&1&1\\
0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1\\
0&1&0&1&1&0&0&1&0&1&0&1&1&1&0&0&0&1&0&1&1
\end{array}\right.$$

$$\left.\begin{array}{cccccccccccccccccccc}
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0
\end{array}\right).$$

Observe that the intersecting coordinate positions of c_1, c_2 and c_3 are columns 18, 20, 21 and 22. According to our method, we may construct new 3-intersecting codes by puncturing part of columns 18, 20, 21 and 22 of the matrix G. As a first step, we puncture column 18 of G and denote the new matrix by G_0. Then, it can be checked that the rows of G_0 are still independent, and thus G_0 generates a 5-dimensional code, $\mathcal{C}_0$, say. Observe that $\mathcal{C}_0$ is 3-wise intersecting since its 3-wise intersection is equal to 3 (> 0). We may similarly proceed such puncturing steps, and consider to puncture two of columns 18, 20, 21 and 22 of G, and further three of columns 18, 20, 21 and 22 of G, to obtain new matrices. Then, the codes generated by the new matrices remain to be 5-dimensional 3-wise intersecting codes since they all have nonzero 3-wise intersections.

The t-wise intersection of a linear constant-weight code and a relative two-weight code are determined in [8]. Note that our puncturing method also applies to the results in [8] to obtain new t-wise intersecting codes.

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