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Vibration Analysis of Smart Embedded Shear Deformable Nonhomogeneous Piezoelectric Nanoscale Beams based on Nonlocal Elasticity Theory

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Abstract

Free vibration analysis is presented for a simply-supported, functionally graded piezoelectric (FGP) nanobeam embedded on elastic foundation in the framework of third order parabolic shear deformation beam theory. Effective electro-mechanical properties of FGP nanobeam are supposed to be variable throughout the thickness based on power-law model. To incorporate the small size effects into the local model, Eringen's nonlocal elasticity theory is adopted. Analytical solution is implemented to solve the size-dependent buckling analysis of FGP nanobeams based upon a higher order shear deformation beam theory where coupled equations obtained using Hamilton's principle exist for such beams. Some numerical results for natural frequencies of the FGP nanobeams are prepared, which include the influences of elastic coefficients of foundation, electric voltage, material and geometrical parameters and mode number. This study is motivated by the absence of articles in the technical literature and provides beneficial results for accurate FGP structures design.

Key words: Functionally graded piezoelectric nanobeam, two-parameter elastic foundations, free vibration, nonlocal elasticity, higher-order beam theory

1. Introduction

Functionally graded materials (FGMs) have been regarded as one of the advanced inhomogeneous spatial composite materials which are composed from two or more material constituents including a couple of ceramic and metal in which their volume fractions are supposed to be changed continuously through the arbitrary directions. Due to owning flexible properties, FGMs can be erroneous for various engineering applications [1-7]. Recently, due to formation of micro- and nano- electro-mechanical systems (MEMS/NEMS) from various structural elements including nano-sized beams and plates which have extraordinary mechanical, chemical, and electronic properties, nano-scale engineering structures earned notable interest by various researchers [8-10]. The classical continuum theory have been failed to anticipate the size dependency of small scale structures due to absence of any material length scale parameters. Eringen's nonlocal elasticity theory as one of the non-classical continuum theories has been introduced to attain small size influences in modeling of nanostructures. The nonlocal elasticity theory of Eringen provides a stress state at a reference point as a function of the strain at all neighbor points of the body and hence small size effects are captured.

Till to now, several literatures are devoted to mechanical analyses of size-dependent FG beams. Nonlinear free

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vibration of microbeams made of FGMs has been investigated by Ke et al. [11] based on the modified couple stress theory and von Kármán geometric nonlinearity. Ansari et al. [12] have studied the thermal postbuckling characteristics of microbeams made of FGMs undergoing thermal loads based on the modified strain gradient theory (MSGT). Also, Eltaher et al. [13] have investigated the static and stability responses of FG nanobeams based on nonlocal continuum theory. Şimşek and Yurtcu [14] have recently investigated bending and buckling of FG nanobeam using analytical method by using the TBT and EBT beam models. Sharabiani and Yazdi [15] have studied the nonlinear free vibration of FG nanobeams within the framework of Euler–Bernoulli beam model including the von Kármán geometric nonlinearity. Uymaz [16] has studied the forced vibration analysis of FG nanobeams based on the nonlocal elasticity theory and using Navier method for various shear deformation theories. Also, Rahmani and Pedram [17] have analyzed the size effects on vibration of FG nanobeams based on nonlocal TBT. Zenkour and Abouelregal [18] have studied the vibration of FG nanobeams induced by sinusoidal pulse-heating via a nonlocal thermoelastic model. Most recently Ebrahimi et al. [19] have discussed the applicability of differential transformation method in investigations on vibrational characteristics of FG size-dependent nanobeams. Also, Ebrahimi and Salari [20] have presented a semi-analytical method for vibrational and buckling analysis of FG nanobeams with concept of neutral axis location. Zenkour and Abouelregal [21] have investigated thermoelastic interaction in FG nanobeams subjected to time-dependent heat flux. Ansari et al. [22] have presented an exact solution for the nonlinear forced vibration of FG nanobeams in thermal environment based on surface elasticity theory. Rahmani and Jandaghian [23] have presented the buckling analysis of FG nanobeams based on a nonlocal third-order shear deformation theory. Also, the thermal effect on buckling and free vibration characteristics of FG size-dependent Timoshenko nanobeams subjected to an in-plane thermal loading have been investigated by Ebrahimi and Salari [24]. Zenkour and Sobhy [25] have presented a simplified shear and normal deformations nonlocal theory for bending of nanobeams in thermal environment. Mashat et al. [26] have investigated the vibration and thermal buckling of nanobeams embedded in an elastic medium under various boundary conditions.

Also, in modern technology, the utilization of piezoelectric materials as smart structures is of remarkable interest. Piezoelectric materials produce an electric current when they are placed under mechanical stresses. The piezoelectric process is also reversible, so when an electric current is applied to these materials, they will actually change shape slightly. Therefore, piezoelectrics materials can be used in micro and nano electromechanical systems, resonators, mechanical and chemical sensors. Due to this fact, the mechanical responses of FGP beam structures have aroused more interests in recent years. Doroushi et al. [27] have investigated the free and forced vibration characteristics of an FGPM beam subjected to thermo-electro-mechanical loads using the higher-order shear deformation beam theory. Kiani et al. [28] have analysed the buckling behavior of FGM beams with or without surface-bonded piezoelectric layers subjected to both thermal loading and constant voltage. Komijani et al. [29] have studied the free vibration of functionally graded piezoelectric material (FGPM) beams with rectangular cross sections under in-plane thermal and electrical excitations in pre/post-buckling regimes. Lezgy-Nazargah et al. [30] have suggested an efficient three-nodded beam element model for static, free vibration and dynamic response of FGPM beams. Large amplitude free flexural vibration of shear deformable FGM beams with surface-bonded piezoelectric layers subjected to thermopiezoelectric loadings with random material properties has been presented by Shegokar and Lal [31]. Therefore, it is apparent that an investigation for the vibrational responses of FGP nanobeams embedded on elastic foundation using a parabolic shear deformation beam theory is not yet carried out.

Several higher-order beam theories including a warping of the cross-section have then been suggested (Touratier [32], Soldatos [33], Reddy [34], Aydogdu [8]). Unlike the Timoshenko beam theory, which needs a shear correction factor the higher-order theories are more realistic, since they satisfy zero transverse shear stresses on the top and bottom boundaries of the beam. Therefore, displacement field is usually supposed to be cubic, giving rise to a parabolic shear strain and stress distributions through the thickness.

In the present study, a size-dependent higher-order beam model is presented based on nonlocal elasticity theory for the free vibration analysis of simply-supported FGP nanobeams embedded in elastic foundation. The electro-mechanical material properties of such beams are supposed to be spatially graded according to the power law model. Via Hamilton's principle, the nonlocal equations of motion for the free vibration of higher order FG nanobeams embedded in two-parameter elastic foundation are obtained and are solved using Navier type method. To determine the effects of elastic foundation, external electric voltage, nonlocal parameter, power-law exponent, mode number and slenderness ratio on vibration responses of nonlocal FGP beams, extensive numerical examples are provided.

2. Formulations of the problem

2.1. Material properties of FGP nanobeams

Figure 1 shows the assuming FG nanobeam that composed of PZT-4 and PZT-5H piezoelectric materials exposed to an electric potential $\Phi(x,z,t)$, with length L and uniform thickness h. The effective material properties of the FGPM nanobeam are supposed to change continuously in the z-axis direction (thickness direction) based on the power-law model. So, the effective material properties, P, can be stated in the following form (Komijani et al. [29]):

$$P = P_2V_2 + P_1V_1 \tag{1}$$

where P_1 and P_2 denote the material properties of the bottom and higher surfaces, respectively. Also V_1 and V_2 are the corresponding volume fractions related by:

$$V_2 = \left(\frac{z}{h}+\frac{1}{2}\right)^p, \quad V_1 = 1 - V_2 \tag{2}$$

where p is power-law exponent which is non-negative and estimates the material distribution through the thickness of the nanobeam and z is the distance from the mid-plane of the graded piezoelectric beam. Therefore, according to Eqs. (1) and (2), the effective electro-mechanical material properties of the FGP beam is defined as:

$$P(z) = (P_2 - P_1)\left(\frac{z}{h}+\frac{1}{2}\right)^p + P_1 \tag{3}$$

It must be noted that, the upper surface at z=+h/2 of FGP nanobeam is assumed PZT-4 rich, whereas the bottom surface at z=-h/2 is PZT-5H rich.

2.2. Nonlocal elasticity theory for FGPM nanobeams

Contrary to the constitutive equation of classical elasticity theory, Eringen's nonlocal theory (Eringen [35-37]) notes that the stress state at a point inside a body is regarded to be function of strains of all points in the neighbor regions. For a nonlocal homogeneous piezoelectric solid the basic equations with zero body force may be defined as:

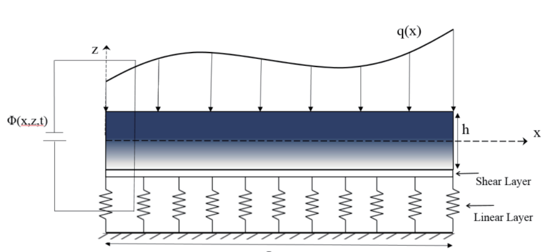

Fig. 1. Configuration of a functionally graded piezoelectric nanobeam.

$$\begin{aligned}\sigma_{ij}(x) &= \iiint_V \varrho(|x-x'|,\tau)\left[C_{ijkl}\varepsilon_{kl}(x') - e_{kij}E_k(x')\right]\mathrm{d}v(x') \\ D_i(x) &= \iiint_V \varrho(|x-x'|,\tau)\left[e_{ikl}\varepsilon_{kl}(x') + \kappa_{ik}E_k(x')\right]\mathrm{d}v(x'),\end{aligned} \tag{4}$$

where σ_{ij}, ε_{ij}, D_i and E_i denote the stress, strain, electric displacement and electric field components, respectively; C_{ijkl}, e_{kij} and κ_{ik} are elastic, piezoelectric and dielectric constants, respectively; $\varrho(|x\text{-}x'|,\tau)$ is the nonlocal kernel function and $|x\text{-}x'|$ is the Euclidean distance. $\tau=e_0a/l$ is defined as scale coefficient, where e_0 is a material constant which is determined experimentally or approximated by matching the dispersion curves of plane waves with those of atomic lattice dynamics; and a and l are the internal and external characteristic length of the nanostructures, respectively. Finally it is possible to represent the integral constitutive relations given by Eq. (4) in an equivalent differential form as:

$$\begin{aligned}\sigma_{ij} - (e_0a)^2\nabla^2\sigma_{ij} &= C_{ijkl}\varepsilon_{kl} - e_{kij}E_k \\ D_i - (e_0a)^2\nabla^2 D_i &= e_{ikl}\varepsilon_{kl} + \kappa_{ik}E_k,\end{aligned} \tag{5}$$

where ∇^2 is the Laplacian operator and e_0a is the nonlocal parameter revealing the size influence on the response of nanostructures.

2.3. Nonlocal higher-order FGP nanobeam model

Based on parabolic third-order beam theory, the displacement components at any point of the beam are supposed to be in the form:

$$\begin{aligned}u_x(x,z) &= u(x) + z\psi(x) - \alpha z^3\left(\psi + \frac{\partial w}{\partial x}\right) \\ u_z(x,z) &= w(x),\end{aligned} \tag{6}$$

where u is the in-plane displacement along the coordinate x and w is the transverse displacement in the mid-plane along the coordinate z, while ψ denotes the total bending rotation of the cross-section.

The distribution of electric potential along the thickness direction should be chosen to satisfy Maxwell's equation in the quasi-static approximation. It is supposed to change as a combination of a cosine and linear variation as follows:

$$\Phi(x,z,t) = -\cos(\xi z)\,\phi(x,t) + \frac{2z}{h}V \tag{7}$$

where $\xi=\pi/h$. Also, V denotes the initial external electric voltage applied to the FGP nanobeam while $\phi(x,t)$ denotes the spatial function of the electric potential in the x-direction. Considering the strain–displacement relationships on the basis of a parabolic beam theory, the non-zero strains may be stated as:

$$\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)} + z^3\varepsilon_{xx}^{(3)}, \quad \gamma_{xz} = \gamma_{xz}^{(0)} + z^2\gamma_{xz}^{(2)} \tag{8}$$

where

$$\varepsilon_{xx}^{(0)} = \frac{\partial u}{\partial x}, \quad \varepsilon_{xx}^{(1)} = \frac{\partial \psi}{\partial x}, \quad \varepsilon_{xx}^{(3)} = -\alpha\left(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}\right)$$
$$\gamma_{xz}^{(0)} = \psi + \frac{\partial w}{\partial x}, \quad \gamma_{xz}^{(2)} = -\beta\left(\psi + \frac{\partial w}{\partial x}\right) \tag{9}$$

According to the defined electric potential in Eq. (7), the non-zero components of electric field (E_x,E_z) can be obtained as:

$$E_x = -\frac{\partial \Phi}{\partial x} = \cos(\xi z)\frac{\partial \phi}{\partial x}$$
$$E_z = -\frac{\partial \Phi}{\partial z} = -\xi \sin(\xi z)\,\phi - \frac{2}{h}V \tag{10}$$

The Hamilton's principle can be stated in the following form to obtain the governing equations of motion:

$$\int_0^t \delta(\Pi_S - \Pi_K + \Pi_W)\mathrm{d}t = 0 \tag{11}$$

where Π_S and Π_K are the is the total strain and kinetic energies while Π_W denotes the work done by the external applied forces. The first variation of the total strain energy Π_S can be expressed as:

$$\delta\Pi_S = \int_0^L \int_{-h/2}^{h/2} (\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{xz}\delta\gamma_{xz} - D_x\delta E_x - D_z\delta E_z)\mathrm{d}z\,\mathrm{d}x \tag{12}$$

Inserting Eqs. (8) and (10) into Eq. (12) yields:

$$\delta\Pi_S = \int_0^L\left(N\delta\varepsilon_{xx}^{(0)} + M\delta\varepsilon_{xx}^{(1)} + P\delta\varepsilon_{xx}^{(3)} + Q\delta\gamma_{xz}^{(0)} + R\delta\gamma_{xz}^{(2)}\right)\mathrm{d}x$$
$$+ \int_0^L \int_{-h/2}^{h/2}\left[-D_x\cos(\xi z)\frac{\partial \delta\phi}{\partial x} + D_z\xi\sin(\xi z)\,\delta\phi\right]\mathrm{d}z\,\mathrm{d}x \tag{13}$$

where *N, M* and *Q* are stress resultants. Relations between the axial force resultant *N*, the bending moment resultant *M*, the shear force resultant *Q* and the additional stress resultants *P* and *R* in one side and the stress components used in Eq. (13) in the other side are given by:

$$\{N, M, P\} = \int_{-h/2}^{h/2}\sigma_{xx}\{1, z, z^3\}\mathrm{d}z$$
$$\{Q, R\} = \int_{-h/2}^{h/2}\sigma_{xz}\{1, z^2\}\mathrm{d}z \tag{14}$$

The kinetic energy Π_K for graded piezoelectric nanobeam is formulated as:

$$\Pi_K = \frac{1}{2}\int_0^L \int_{-h/2}^{h/2}\rho\left[\left(\frac{\partial u_x}{\partial t}\right)^2 + \left(\frac{\partial u_z}{\partial t}\right)^2\right]\mathrm{d}z\,\mathrm{d}x \tag{15}$$

where ρ is the mass density. The first variation of the kinetic energy is presented as:

$$\delta\Pi_K = \int_0^L\Big\{I_0\left(\frac{\partial u}{\partial t}\frac{\partial \delta u}{\partial t} + \frac{\partial w}{\partial t}\frac{\partial \delta w}{\partial t}\right) + I_1\left(\frac{\partial u}{\partial t}\frac{\partial \delta\psi}{\partial t} + \frac{\partial \psi}{\partial t}\frac{\partial \delta u}{\partial t}\right) + I_2\frac{\partial \psi}{\partial t}\frac{\partial \delta\psi}{\partial t}$$
$$-\alpha\Big[I_3\frac{\partial u}{\partial t}\left(\frac{\partial^2 \delta w}{\partial x\partial t} + \frac{\partial \delta\psi}{\partial t}\right) + I_3\frac{\partial \delta u}{\partial t}\left(\frac{\partial^2 w}{\partial x\partial t} + \frac{\partial \psi}{\partial t}\right) + I_4\frac{\partial \psi}{\partial t}\left(\frac{\partial^2 \delta w}{\partial x\partial t} + \frac{\partial \delta\psi}{\partial t}\right)$$
$$+I_4\frac{\partial \delta\psi}{\partial t}\left(\frac{\partial^2 w}{\partial x\partial t} + \frac{\partial \psi}{\partial t}\right) - \alpha I_6\left(\frac{\partial^2 w}{\partial x\partial t} + \frac{\partial \psi}{\partial t}\right)\left(\frac{\partial^2 \delta w}{\partial x\partial t} + \frac{\partial \delta\psi}{\partial t}\right)\Big]\Big\}\mathrm{d}x, \tag{16}$$

where I_j represent the mass inertia

$$I_j = \int_{-h/2}^{h/2}\rho z^j\mathrm{d}z \tag{17}$$

It must also be cited that for homogeneous nanobeams, I_1=I_3=0.

Variation of the work done due to external forces, $\delta\Pi_W$, can be written in the form:

$$\delta\Pi_W = \int_0^L\Big[\left(N_E\frac{\partial w}{\partial x}\frac{\partial}{\partial x} + \alpha P\frac{\partial^2}{\partial x^2} + q - k_W + k_P\frac{\partial^2}{\partial x^2}\right)\delta w$$
$$+f\delta u - N\delta\varepsilon_{xx}^{(0)} - \bar{M}\frac{\partial \delta\psi}{\partial x} - \bar{Q}\delta\gamma_{xz}^{(0)}\Big]\mathrm{d}x, \tag{18}$$

in which $\bar{M} = M - \alpha P$, $\bar{Q} = Q - \beta R$ and $q(x)$ and $f(x)$ denote the transverse and axial distributed loads and k_W and k_P are foundation parameters and N_E is normal forces due to external electric voltage *V* which is given by:

$$N_E = -\int_{-h/2}^{h/2}e_{31}\frac{2}{h}V\mathrm{d}z \tag{19}$$

The nonlocal constitutive relations appeared in Eq. (5) for the FGPM nanobeam exposed to electro-mechanical loading in the one dimensional case may be rewritten as:

$$\sigma_{xx} - (e_0 a)^2\frac{\partial^2\sigma_{xx}}{\partial x^2} = c_{11}\varepsilon_{xx} - e_{31}E_z$$
$$\sigma_{xz} - (e_0 a)^2\frac{\partial^2\sigma_{xz}}{\partial x^2} = c_{55}\gamma_{xz} - e_{15}E_x \tag{20}$$

$$D_x - (e_0 a)^2\frac{\partial^2 D_x}{\partial x^2} = e_{15}\gamma_{xz} + \kappa_{11}E_x$$
$$D_z - (e_0 a)^2\frac{\partial^2 D_z}{\partial x^2} = e_{31}\varepsilon_{xx} + \kappa_{33}E_z \tag{21}$$

Inserting Eqs. (13), (16) and (18) in Eq. (11) and integrating by parts, and gathering the coefficients of δu, δw, $\delta\psi$ and $\delta\phi$, the following governing equations are obtained:

$$\frac{\partial N}{\partial x} + f = I_0\frac{\partial^2 u}{\partial t^2} + \hat{I}_1\frac{\partial^2 \psi}{\partial t^2} - \alpha I_3\frac{\partial^3 w}{\partial x\partial t^2} \tag{22}$$

$$\frac{\partial \bar{M}}{\partial x} - \bar{Q} = \hat{I}_1\frac{\partial^2 u}{\partial t^2} + \hat{I}_2\frac{\partial^2 \psi}{\partial t^2} - \alpha\hat{I}_4\left(\frac{\partial^3 w}{\partial x\partial t^2} + \frac{\partial^2 \psi}{\partial t^2}\right) \tag{23}$$

$$\frac{\partial \bar{Q}}{\partial x} - N_E\frac{\partial^2 w}{\partial x^2} + \alpha\frac{\partial^2 P}{\partial x^2} + q - k_W w + k_P\frac{\partial^2 w}{\partial x^2} \tag{24}$$

$$= I_0\frac{\partial^2 w}{\partial t^2} + \alpha I_3\frac{\partial^3 u}{\partial x\partial t^2} + \alpha I_4\frac{\partial^3 \psi}{\partial x\partial t^2} - \alpha^2 I_6\left(\frac{\partial^4 w}{\partial x^2\partial t^2} + \frac{\partial^3 \psi}{\partial x\partial t^2}\right)$$

$$\int_{-h/2}^{h/2}\left[\cos(\xi z)\frac{\partial D_x}{\partial x} + \xi\sin(\xi z)\,D_z\right]\mathrm{d}z = 0 \tag{25}$$

where $\hat{I}_j = I_j - \alpha I_{j+2}$. By integrating Eqs. (22)-(25) over the beam's cross-section area, the force-strain and the moment-strain of the nonlocal third-order Reddy FGP beam theory can be obtained as follows:

$$N - \mu\frac{\partial^2 N}{\partial x^2} = A_{xx}\frac{\partial u}{\partial x} + (B_{xx} - \alpha E_{xx})\frac{\partial \psi}{\partial x} - \alpha E_{xx}\frac{\partial^2 w}{\partial x^2} + \bar{A}_{31}^e\phi - N_E \tag{26}$$

$$M - \mu\frac{\partial^2 M}{\partial x^2} = B_{xx}\frac{\partial u}{\partial x} + (D_{xx} - \alpha F_{xx})\frac{\partial \psi}{\partial x} - \alpha F_{xx}\frac{\partial^2 w}{\partial x^2} + \bar{B}_{31}^e\phi \tag{27}$$

$$P - \mu\frac{\partial^2 P}{\partial x^2} = E_{xx}\frac{\partial u}{\partial x} + (F_{xx} - \alpha H_{xx})\frac{\partial \psi}{\partial x} - \alpha H_{xx}\frac{\partial^2 w}{\partial x^2} + \bar{E}_{31}^e\phi \tag{28}$$

$$Q - \mu\frac{\partial^2 Q}{\partial x^2} = (A_{xz} - \beta D_{xz})\left(\frac{\partial w}{\partial x} + \psi\right) - \hat{A}_{15}^e\frac{\partial \phi}{\partial x} \tag{29}$$

$$R - \mu\frac{\partial^2 R}{\partial x^2} = (D_{xz} - \beta F_{xz})\left(\frac{\partial w}{\partial x} + \psi\right) - \hat{D}_{15}^e\frac{\partial \phi}{\partial x} \tag{30}$$

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$$\int_{-h/2}^{h/2}\left(D_x-\mu\frac{\partial^2 D_x}{\partial x^2}\right)\cos(\xi z)\,\mathrm{d}z = \left(\hat{A}_{15}^e-\beta\widehat{D}_{15}^e\right)\left(\frac{\partial w}{\partial x}+\psi\right)+\breve{A}_{11}^{\kappa}\frac{\partial\phi}{\partial x} \tag{31}$$

$$\int_{-h/2}^{h/2}\left(D_z-\mu\frac{\partial^2 D_z}{\partial x^2}\right)\xi\sin(\xi z)\,\mathrm{d}z = \bar{A}_{31}^e\frac{\partial u}{\partial x}+(\bar{B}_{31}^e-\alpha\bar{E}_{31}^e)\frac{\partial\psi}{\partial x}-\alpha E_{31}^e\frac{\partial^2 w}{\partial x^2}-\breve{A}_{33}^{\kappa}\phi \tag{32}$$

where $\mu=(e_0a)^2$ and all quantities used in the above equations are defined as:

$$\{A_{xx},B_{xx},D_{xx},E_{xx},F_{xx},H_{xx}\} = \int_{-h/2}^{h/2}c_{11}\{1,z,z^2,z^3,z^4,z^6\}\mathrm{d}z \tag{33}$$

$$\{A_{xz},D_{xz},F_{xz}\}=\int_{-h/2}^{h/2}c_{55}\{1,z^2,z^4\}\mathrm{d}z \tag{34}$$

$$\{\bar{A}_{31}^e,\bar{B}_{31}^e,\bar{E}_{31}^e\}=\int_{-h/2}^{h/2}e_{31}\{1,z,z^3\}\xi\sin(\xi z)\,\mathrm{d}z \tag{35}$$

$$\left\{\hat{A}_{15}^e,\widehat{D}_{15}^e\right\}=\int_{-h/2}^{h/2}e_{15}\{1,z^2\}\cos(\xi z)\,\mathrm{d}z \tag{36}$$

$$\left\{\breve{A}_{11}^{\kappa},\breve{A}_{33}^{\kappa}\right\}=\int_{-h/2}^{h/2}\{\kappa_{11}\cos^2(\xi z),\kappa_{33}\xi^2\sin^2(\xi z)\}\mathrm{d}z \tag{37}$$

The explicit relation of the nonlocal normal force can be derived by substituting for the second derivative of N from Eq. (22) into Eq. (26) as follows:

$$N=A_{xx}\frac{\partial u}{\partial x}+K_{xx}\frac{\partial\psi}{\partial x}-\alpha E_{xx}\frac{\partial^2 w}{\partial x^2}+\bar{A}_{31}^e\phi-N_E +\mu\left(I_0\frac{\partial^3 u}{\partial x\partial t^2}+\hat{I}_1\frac{\partial^3\psi}{\partial x\partial t^2}-\alpha I_3\frac{\partial^4 w}{\partial t^2\partial t^2}-\frac{\partial f}{\partial x}\right) \tag{38}$$

Omitting $\bar{Q}$ from Eqs. (23) and (24), we obtain the following equation:

$$\frac{\partial^2\bar{M}}{\partial x^2}=N_E\frac{\partial^2 w}{\partial x^2}-\alpha\frac{\partial^2 P}{\partial x^2}-q+I_0\frac{\partial^2 w}{\partial t^2}+I_1\frac{\partial^3 u}{\partial x\partial t^2}+I_2\frac{\partial^3\psi}{\partial x\partial t^2} -\alpha I_4\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^3\psi}{\partial x\partial t^2}\right)+k_W w-k_P\frac{\partial^2 w}{\partial x^2}. \tag{39}$$

Also the explicit relation of the nonlocal bending moment can be derived by substituting for the second derivative of $\bar{M}$ from the above equation into Eq. (27) and using Eq. (28) as follows:

$$\bar{M}=K_{xx}\frac{\partial u}{\partial x}+\bar{I}_{xx}\frac{\partial\psi}{\partial x}-\alpha J_{xx}\frac{\partial^2 w}{\partial x^2}+k_W w-k_P\frac{\partial^2 w}{\partial x^2} +(\bar{B}_{31}^e-\alpha\bar{E}_{31}^e)\phi +\mu\left[N_E\frac{\partial^2 w}{\partial x^2}-\alpha\frac{\partial^2 P}{\partial x^2}-q+I_0\frac{\partial^2 w}{\partial t^2}+I_1\frac{\partial^3 u}{\partial x\partial t^2} +I_2\frac{\partial^3\psi}{\partial x\partial t^2}-\alpha I_4\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^3\psi}{\partial x\partial t^2}\right)\right] \tag{40}$$

where

$$K_{xx}=B_{xx}-\alpha E_{xx},\quad I_{xx}=D_{xx}-\alpha F_{xx}, \quad J_{xx}=F_{xx}-\alpha H_{xx},\quad \bar{I}_{xx}=I_{xx}-\alpha J_{xx} \tag{41}$$

By substituting for the second derivative of $\bar{Q}$ from Eq. (24) into Eq. (29) and using Eq. (30), the following expression for the nonlocal shear force is obtained:

$$\bar{Q}=A_{xz}^*\left(\frac{\partial w}{\partial x}+\psi\right)-\left(\hat{A}_{15}^e-\beta\widehat{D}_{15}^e\right)\frac{\partial\phi}{\partial x} +\mu\left[N_E\frac{\partial^3 w}{\partial x^3}-\alpha\frac{\partial^3 P}{\partial x^3}-\frac{\partial q}{\partial x}+k_W\frac{\partial w}{\partial x}-k_P\frac{\partial^3 w}{\partial x^3} +I_0\frac{\partial^3 w}{\partial x\partial t^2}+\alpha I_3\frac{\partial^4 u}{\partial x^2\partial t^2}+\alpha I_4\frac{\partial^4\psi}{\partial x^2\partial t^2} -\alpha^2 I_6\left(\frac{\partial^5 w}{\partial x^3\partial t^2}+\frac{\partial^4\psi}{\partial x^2\partial t^2}\right)\right] \tag{42}$$

where

$$A_{xz}^*=\bar{A}_{xz}-\beta\bar{D}_{xz},\quad \bar{A}_{xz}=A_{xz}-\beta D_{xz}, \quad \bar{D}_{xz}=D_{xz}-\beta F_{xz} \tag{43}$$

In addition, the second derivative of the identity of Eq. (28) may be written as

$$\alpha\frac{\partial^2}{\partial x^2}\left(P-\mu\frac{\partial^2 P}{\partial x^2}\right) =\alpha E_{xx}\frac{\partial^3 u}{\partial x^3}+\alpha J_{xx}\frac{\partial^3\psi}{\partial x^3}-\alpha^2 H_{xx}\frac{\partial^4 w}{\partial x^4}+\alpha\bar{E}_{31}^e\frac{\partial^2\phi}{\partial x^2} \tag{44}$$

Finally, based on the third-order beam theory, the nonlocal equations of motion appeared in Eqs. (22)-(24) for a FG piezoelectric nanobeam can be obtained by substituting for N, $\bar{M}$ and $\bar{Q}$ from Eqs. (38), (40) and (42), respectively, and using Eq. (44) in Eq. (24) as follows:

$$A_{xx}\frac{\partial^2 u}{\partial x^2}+K_{xx}\frac{\partial^2\psi}{\partial x^2}-\alpha E_{xx}\frac{\partial^3 w}{\partial x^3}+\bar{A}_{31}^e\frac{\partial\phi}{\partial x} +f-I_0\frac{\partial^2 u}{\partial t^2}-\hat{I}_1\frac{\partial^2\psi}{\partial t^2}+\alpha I_3\frac{\partial^3 w}{\partial x\partial t^2} -\mu\left(\frac{\partial^2 f}{\partial x^2}-I_0\frac{\partial^4 u}{\partial x^2\partial t^2}-\hat{I}_1\frac{\partial^4\psi}{\partial x^2\partial t^2}+\alpha I_3\frac{\partial^5 w}{\partial x^3\partial t^2}\right)=0 \tag{45}$$

$$K_{xx}\frac{\partial^2 u}{\partial x^2}+\bar{I}_{xx}\frac{\partial^2\psi}{\partial x^2}-\alpha J_{xx}\frac{\partial^3 w}{\partial x^3}-A_{xz}^*\left(\frac{\partial w}{\partial x}+\psi\right) +\bar{\tilde{A}}_{15}^e\frac{\partial\phi}{\partial x}-\hat{I}_1\frac{\partial^2 u}{\partial t^2}-\hat{I}_2\frac{\partial^2\psi}{\partial t^2}+\alpha\hat{I}_4\left(\frac{\partial^3 w}{\partial x\partial t^2}+\frac{\partial^2\psi}{\partial t^2}\right) +\mu\left[\hat{I}_1\frac{\partial^4 u}{\partial x^2\partial t^2}+\hat{I}_2\frac{\partial^4\psi}{\partial x^2\partial t^2}-\alpha\hat{I}_4\left(\frac{\partial^5 w}{\partial x^3\partial t^2}+\frac{\partial^4\psi}{\partial x^2\partial t^2}\right)\right]=0 \tag{46}$$

$$A_{xz}^*\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial\psi}{\partial x}\right)+\alpha E_{xx}\frac{\partial^3 u}{\partial x^3}+\alpha J_{xx}\frac{\partial^3\psi}{\partial x^3} -\alpha^2 H_{xx}\frac{\partial^4 w}{\partial x^4}-N_E\frac{\partial^2 w}{\partial x^2}+q-k_W w+k_P\frac{\partial^2 w}{\partial x^2} -\tilde{A}_{15}^e\frac{\partial^2\phi}{\partial x^2}-I_0\frac{\partial^2 w}{\partial t^2}-\alpha I_3\frac{\partial^3 u}{\partial x\partial t^2}-\alpha I_4\frac{\partial^3\psi}{\partial x\partial t^2} +\alpha^2 I_6\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^3\psi}{\partial x\partial t^2}\right) +\mu\left[N_E\frac{\partial^4 w}{\partial x^4}-\frac{\partial^2 q}{\partial x^2}+k_W\frac{\partial^2 w}{\partial x^2}-k_P\frac{\partial^4 w}{\partial x^4}+I_0\frac{\partial^4 w}{\partial x^2\partial t^2} +\alpha I_3\frac{\partial^5 u}{\partial x^3\partial t^2}+\alpha I_4\frac{\partial^5\psi}{\partial x^3\partial t^2}-\alpha^2 I_6\left(\frac{\partial^6 w}{\partial x^4\partial t^2}+\frac{\partial^5\psi}{\partial x^3\partial t^2}\right)\right]=0 \tag{47}$$

where

$$\tilde{A}_{15}^e=\hat{A}_{15}^e-\alpha E_{31}^e-\beta\widehat{D}_{15}^e, \quad \bar{\tilde{A}}_{15}^e=\hat{A}_{15}^e+\bar{B}_{31}^e-\alpha\bar{E}_{31}^e-\beta\widehat{D}_{15}^e \tag{48}$$

It must be cited that inserting Eq. (25) into Eqs. (31) and (32) does not provide an explicit expressions for D_x and D_z. To overcome this problem, Eq. (25) can be re-expressed in terms of u, w, ψ and ϕ by using Eqs. (31) and (32) as:

$$\bar{A}_{31}^{e}\frac{\partial u}{\partial x}+\tilde{A}_{15}^{e}\frac{\partial^2 w}{\partial x^2}+\bar{\bar{A}}_{15}^{e}\frac{\partial \psi}{\partial x}+\breve{A}_{11}^{\kappa}\frac{\partial^2 \phi}{\partial x^2}-\breve{A}_{33}^{\kappa}\phi=0 \quad (49)$$

3. Solution procedure

In this section, the analytical solution of the governing equations for free vibration of a simply-supported FGP nanobeam has been presented on the basis the Navier's method. The boundary conditions for the present simply-supported FGP beam can be identified as:

$$u(0,t)=0,\quad \left.\frac{\partial u}{\partial x}\right|_{x=L}=0,\quad w(0,t)=w(L,t)=0$$
$$\left.\frac{\partial \psi}{\partial x}\right|_{x=0}=\left.\frac{\partial \psi}{\partial x}\right|_{x=L}=0,\quad \phi(0,t)=\phi(L,t)=0. \quad (50)$$

So, the displacement variables are adopted to satisfy the governing equations of motion and the above simply-supported boundary condition in the form:

$$\begin{Bmatrix} u(x,t)\\ \psi(x,t)\\ w(x,t)\\ \phi(x,t)\end{Bmatrix}=\sum_{n=1}^{\infty}\begin{Bmatrix} U_n\cos(\zeta x)\\ \Psi_n\cos(\zeta x)\\ W_n\sin(\zeta x)\\ \Phi_n\sin(\zeta x)\end{Bmatrix}e^{i\omega_n t} \quad (51)$$

where $\zeta=n\pi/L$, and U_n, W_n, Ψ_n and Φ_n are the unknown Fourier coefficients to be determined for each n value. Substituting Eqs. (51) into Eqs. (45), (46), (47) and (49), leads to the frequency equation

$$([K]-\omega_n^2[M])\{\Delta\}=\{0\} \quad (52)$$

where $\{\Delta\}=\{U_n,\Psi_n,W_n,\Phi_n\}^T$, $[K]$ is the stiffness matrix and $[M]$ is the mass matrix. The coefficients of the symmetric stiffness matrix $k_{ij}=k_{ji}$ are given by

$$\begin{aligned} &k_{11}=\zeta^2A_{xx},\quad k_{12}=\zeta^2K_{xx},\quad k_{13}=\alpha\zeta^3E_{xx},\quad k_{14}=-\zeta\bar{A}_{31}^{e},\\ &k_{22}=A_{xz}^{*}+\zeta^2\bar{I}_{xx},\quad k_{23}=\zeta(A_{xz}^{*}-\alpha\zeta^2J_{xx}),\quad k_{24}=-\zeta\bar{\bar{A}}_{15}^{e},\\ &k_{33}=\zeta^2(A_{xz}^{*}+\alpha^2\zeta^2H_{xx})+(1+\mu\zeta^2)[\zeta^2(k_P-N_E)+k_W],\\ &k_{34}=-\zeta^2\tilde{A}_{15}^{e},\quad k_{44}=-\zeta^2\breve{A}_{11}^{\kappa}-\breve{A}_{33}^{\kappa}.\end{aligned} \quad (53)$$

Also, the coefficients of the symmetric mass matrix $m_{ij}=m_{ji}$ are given by

$$\begin{aligned} &m_{11}=(1+\mu\zeta^2)I_0,\quad m_{12}=(1+\mu\zeta^2)\hat{I}_1,\\ &m_{13}=-\zeta(1+\mu\zeta^2)I_3,\quad m_{14}=0,\\ &m_{22}=(1+\mu\zeta^2)(\hat{I}_2-\alpha\hat{I}_4),\\ &m_{23}=-\zeta\alpha(1+\mu\zeta^2)\hat{I}_4,\quad m_{24}=0,\\ &m_{33}=(1+\mu\zeta^2)(I_0+\alpha^2\zeta^2I_6),\quad m_{34}=m_{44}=0.\end{aligned} \quad (54)$$

4. Numerical results and discussions

In this section, several numerical examples are provided for the electro-mechanical free vibration characteristics of FGPM nanobeams embedded in elastic medium. To achieve this end, the nonlocal FGP beam made of PZT-4 and PZT-5H, with electro-mechanical material properties listed in Table 1, is supposed. The beam geometry has the following dimensions: L (length) = 10 nm and h (thickness) = varied. Also, the following relation is described to calculate the non-dimensional natural frequencies:

$$\bar{\omega}=\omega_nL^2\left.\sqrt{\frac{\rho A}{c_{11}I}}\right|_{\text{PZT-4}}$$
$$\{K_W,K_P\}=\frac{L^2}{c_{11}I|_{\text{PZT-4}}}\{L^2k_W,k_P\} \quad (55)$$

where $I=bh^3/12$ is the moment of inertia of the cross section of the nanobeam and $A=bh$. For verification purpose the frequency results are compared with those of nonlocal FGM Timoshenko beams presented by Rahmani and Pedram [17], due to the fact that any numerical results for the free vibration of FGP nanobeams based on the nonlocal elasticity theory do not exist yet. In this work, the material selection is performed as follows: E_m=70 GPa, ν_m=0.3, ρ_m=7800 kg/m^3 for Steel and E_c=390 GPa, ν_c=0.24, ρ_c=3960 kg/m^3 for Alumina. Therefore, Table 2 presents the fundamental frequency of S-S FG nanobeams in comparison to those of Rahmani and Pedram [17].

Tables 3-5 present influences of various parameters such as elastic foundation parameters (K_W, K_P), external electric voltage (V), power-law index and nonlocal parameter (μ) on the 1st, 2nd and 3rd non-dimensional frequencies of the simply-

Table 1. Electro-mechanical coefficients of material properties for PZT-4 and PZT-5H (Doroushi et al. [27]).

Properties	PZT-4	PZT-5H
c_{11} (GPa)	81.3	60.6
c_{55} (GPa)	25.6	23.0
e_{31} (Cm^{-2})	-10.0	-16.604
e_{15} (C/m^2)	40.3248	44.9046
κ_{11} (C^{2}m^{-2}N^{-1})	0.6712e-8	1.5027e-8
κ_{33} (C^{2}m^{-2}N^{-1})	1.0275e-8	2.554e-8
ρ (kg m^{-3})	7500	7500

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supported nonlocal FGP nanobeams at L/h=20. It is seen that when the of nonlocal parameter increases the natural frequencies of FGP nanobeam decreases due to the fact that presence of nonlocality makes the beam structure more flexible. Also, it is concluded that the reduction in higher modes due to nonlocality effect are more considerable than lower one modes. Contrary to the nonlocal scale parameter, elastic foundation has an increasing influence on the rigidity of the beam and hence it is deduced that with the rise of Winkler and Pasternak parameter the dimensionless frequencies of the FGP nanobeam increase. Another important conclusion is that effect of Pasternak elastic parameter on the presented first three mode results are more than those obtained for the Winkler parameter. So, the

Table 2. Comparison of the non-dimensional fundamental frequency for a S-S FG nanobeam without elastic foundation for various power-law index (L/h=20).

μ (nm)^2	$p=0$		$p=0.5$		$p=1$		$p=5$	
	TBT [17]	Present RBT	TBT [17]	Present RBT	TBT [17]	Present RBT	TBT [17]	Present RBT
0	9.8296	9.829570	7.7149	7.71546	6.9676	6.967613	5.9172	5.916152
1	9.3777	9.377686	7.3602	7.36078	6.6473	6.647300	5.6452	5.644175
2	8.9829	8.982894	7.0504	7.05090	6.3674	6.367454	5.4075	5.406561
3	8.6341	8.634103	6.7766	6.77714	6.1202	6.120217	5.1975	5.196632
4	8.3230	8.323021	6.5325	6.53296	5.8997	5.899708	5.0103	5.009400

Table 3. Influence of elastic foundation and external electric voltage on the 1st non-dimensional frequency of a S-S FGP nanobeam (L/h=20).

K_w	μ		$K_p=0$			$K_p=5$			$K_p=10$		
			$p=0.2$	$p=1$	$p=5$	$p=0.2$	$p=1$	$p=5$	$p=0.2$	$p=1$	$p=5$
0	0	V=-0.5	10.7792	10.5545	10.4459	12.8623	12.6746	12.5843	14.6521	14.4877	14.4087
		V=0	10.1627	9.79425	9.54393	12.3502	12.0488	11.8463	14.2047	13.9435	13.7688
		V=+0.5	9.50628	8.96976	8.54726	11.8160	11.3888	11.0591	13.7428	13.3773	13.0977
	1	V=-0.5	10.3399	10.1381	10.0467	12.4964	12.3300	12.2549	14.3321	14.1871	14.1220
		V=0	9.69548	9.34399	9.10518	11.9687	11.6858	11.4957	13.8743	13.6310	13.4684
		V=+0.5	9.00510	8.47582	8.05440	11.4166	11.0039	10.6827	13.4010	13.0512	12.7815
	2	V=-0.5	9.95814	9.77671	9.70061	12.1824	12.0346	11.9728	14.0591	13.9312	13.8779
		V=0	9.28731	8.95061	8.72186	11.6405	11.3737	11.1946	13.5922	13.3644	13.2123
		V=+0.5	8.56408	8.04008	7.61839	11.0721	10.6719	10.3580	13.1087	12.7725	12.5114
25	0	V=-0.5	11.8802	11.6768	11.5787	13.7981	13.6233	13.5393	15.4801	15.3245	15.2499
		V=0	11.3238	10.9944	10.7720	13.322	13.0431	12.8562	15.0574	14.8112	14.6468
		V=+0.5	10.7386	10.2667	9.89973	12.8283	12.4359	12.1347	14.6224	14.2793	14.0178
	1	V=-0.5	11.4831	11.3018	11.2198	13.4577	13.3033	13.2337	15.1775	15.0407	14.9793
		V=0	10.9065	10.5952	10.3852	12.9691	12.7085	12.5340	14.7461	14.5173	14.3648
		V=+0.5	10.2976	9.83811	9.47746	12.4615	12.0845	11.7928	14.3016	13.9744	13.7229
	2	V=-0.5	11.1406	10.9788	10.9110	13.1667	13.0300	12.9730	14.9201	14.7996	14.7494
		V=0	10.5453	10.2500	10.0509	12.6669	12.4221	12.2583	14.4810	14.2673	14.1250
		V=+0.5	9.91426	9.46529	9.10982	12.1466	11.7830	11.4994	14.0281	13.7144	13.4716
50	0	V=-0.5	12.8875	12.7002	12.6101	14.6743	14.5101	14.4313	16.2660	16.1180	16.0471
		V=0	12.3765	12.0758	11.8737	14.2276	13.9668	13.7925	15.8642	15.6307	15.4751
		V=+0.5	11.8435	11.4173	11.0884	13.7664	13.4015	13.1225	15.4519	15.1277	14.8811
	1	V=-0.5	12.5224	12.3563	12.2814	14.3547	14.2101	14.1450	15.9783	15.8484	15.7901
		V=0	11.9958	11.7136	11.5240	13.8978	13.6549	13.4926	15.5691	15.3526	15.2084
		V=+0.5	11.4451	11.0335	10.7131	13.4253	13.0761	12.8070	15.1488	14.8402	14.6036
	2	V=-0.5	12.2091	12.0616	12.000	14.0823	13.9545	13.9013	15.7340	15.6197	15.5722
		V=0	11.6684	11.4022	11.2236	13.6162	13.3887	13.2369	15.3182	15.1164	14.9821
		V=+0.5	11.1014	10.7024	10.3893	13.1335	12.7980	12.5373	14.8908	14.5957	14.3678

shear layer of elastic foundation or Pasternak foundation has a significant influence on the frequency results. Moreover, it is observable that for all values of Winkler and Pasternak constants, negative electric voltages create larger frequencies than positive voltages.

The variations of the first fundamental frequency of FGP nanobeams versus the Winkler and Pasternak parameters for various electric voltages and nonlocal parameters at L/h=20 and p=1 are illustrated in Figs. 2-3, respectively. It is seen from these figures that regardless of the sign and magnitude of electric voltage, the non-dimensional natural frequency arises with the increase of Winkler and Pasternak parameters, because of the increment in stiffens of the FGP nanobeam. As a more exact conclusion, it must be mentioned that at a constant electric voltage the increase of non-dimensional natural frequency with Pasternak parameter occurs with a higher rate than those of Winkler parameter.

The effects of Winkler and Pasternak parameters on the variations of the first non-dimensional frequency of the simply-supported FG nanobeams versus power-law exponent for different electric voltages at L/h=20 and p=1 are presented in Figs. 4 and 5, respectively. It is seen that for all values of elastic foundation constants the non-dimensional frequency reduces with the increase of power-law exponent. But, this reduction is more sensible according to the positive values of external electric voltage, due to the fact that the reduction in the frequency occurs with a higher slope.

Figs. 6-7 show the variations the dimensionless frequency of nonlocal FGP beams with respect to external voltage at L/h=20 and p=1 for various nonlocal parameters with changing of Winkler and Pasternak constants. It is seen that although Winkler and Pasternak parameters cause increment in natural frequency, but external electric voltage has a reducing influence on the natural frequencies of FG nanobeams when

Table 4. Influence of elastic foundation and external electric voltage on the 2nd non-dimensional frequency of a S-S FGP nanobeam (L/h=20).

K_w	μ		$K_p=0$			$K_p=5$			$K_p=10$		
			$p=0.2$	$p=1$	$p=5$	$p=0.2$	$p=1$	$p=5$	$p=0.2$	$p=1$	$p=5$
0	0	V=-0.5	41.0434	39.7276	38.8856	43.3634	42.1201	41.3269	45.5655	44.3838	43.6318
		V=0	40.4132	38.9457	37.9525	42.7674	41.3834	40.4502	44.9986	43.6853	42.8023
		V=+0.5	39.7729	38.1477	36.9959	42.1629	40.6334	39.5540	44.4246	42.9755	41.9565
	1	V=-0.5	34.9612	33.8965	33.2325	37.6579	36.6714	36.0586	40.1739	39.2507	38.6787
		V=0	34.2192	32.9766	32.1357	36.9700	35.8228	35.0503	39.5298	38.4590	37.7405
		V=+0.5	33.4606	32.0303	31.0001	36.2690	34.9537	34.0122	38.8750	37.6508	36.7784
	2	V=-0.5	31.0479	30.1509	29.6071	34.0558	33.2400	32.7476	36.8188	36.0655	35.6122
		V=0	30.2099	29.1129	28.3705	33.2936	32.3014	31.6339	36.1150	35.2023	34.5909
		V=+0.5	27.5679	28.0365	27.0774	30.4797	31.3347	30.4797	35.3971	34.3174	33.5385
25	0	V=-0.5	41.3444	40.0385	39.2032	43.6485	42.4135	41.6259	45.8369	44.6623	43.9151
		V=0	40.7189	39.2628	38.2779	43.0564	41.6820	40.7556	45.2734	43.9683	43.0911
		V=+0.5	40.0835	38.4714	37.3296	42.4560	40.9374	39.8663	44.7028	43.2631	42.2510
	1	V=-0.5	35.3142	34.2604	33.6036	37.9857	37.0080	36.4008	40.4814	39.5653	38.9980
		V=0	34.5797	33.3505	32.5193	37.3039	36.1674	35.4023	39.8423	38.7801	38.0677
		V=+0.5	33.8292	32.4151	31.3975	36.6093	35.3067	34.3748	39.1927	37.9787	37.1140
	2	V=-0.5	31.4448	30.5594	30.0230	34.4180	33.6110	33.1241	37.1541	36.4077	35.9587
		V=0	30.6176	29.5357	28.8043	33.6640	32.6831	32.0236	36.4567	35.5528	34.9475
		V=+0.5	29.7674	28.4753	27.5316	32.8927	31.7280	30.8838	35.7457	34.6769	33.9063
50	0	V=-0.5	41.6433	40.3470	39.5183	43.9317	42.7049	41.9227	46.1066	44.9391	44.1966
		V=0	41.0223	39.5774	38.6005	43.3434	41.9784	41.0587	45.5465	44.2494	43.3779
		V=+0.5	40.3917	38.7924	37.6603	42.7471	41.2392	40.1762	44.9794	43.5488	42.5435
	1	V=-0.5	35.6636	34.6204	33.9706	38.3108	37.3416	36.7399	40.7866	39.8775	39.3147
		V=0	34.9364	33.7203	32.8984	37.6348	36.5086	35.7509	40.1523	39.0986	38.3920
		V=+0.5	34.1938	32.7954	31.7900	36.9465	35.6562	34.7337	39.5079	38.3038	37.4467
	2	V=-0.5	31.8367	30.9625	30.4333	34.7765	33.9779	33.4964	37.4864	36.7467	36.3019
		V=0	31.0200	29.9527	29.2316	34.0304	33.0603	32.4085	36.7953	35.8999	35.3006
		V=+0.5	30.1811	28.9075	27.9784	33.2675	32.1165	31.2828	36.0909	35.0327	34.2700

it varies from negative values to positive one.

Also, Fig. 8 demonstrates the variations the non-dimensional natural frequency of FGP nanobeams versus electric voltage at L/h=20 and μ=2 (nm)^2 for different material composition with and without elastic foundation. It is observable that as the external voltage increases from negative to positive values the natural frequency reduces; also the reduction for the higher values of power-law index is more significant. Moreover, difference between the frequency results increases with the rise of electric voltage from -1V to +1V, so the frequencies for negative voltages are more close to each other and hence it is concluded that regardless of magnitude of electric voltage its sign has a remarkable influence on the frequencies.

Figure 9 depicts the variations of the non-dimensional natural frequency of piezoelectric FG nanobeam with respect to slenderness ratio for power-law exponent p=0.2 and nonlocal scale parameter μ=2 (nm)^2. It is shown that, external electric voltage has an increasing influence on natural frequencies of FGP nonlocal beams for negative values of external voltage as well as a reducing effect for positive voltages. This is due to the axial compressive and tensile forces produced in the FGP nanobeams via the applied positive and negative voltages, respectively. Also, it must be mentioned that curvature of the lines for higher values of electric voltage is more sensible compared to lower voltages. In addition, it is clearly observed that the dimensionless natural frequency is approximately independent of slenderness ratio for zero electric voltages (V=0). The influence of nonlocal parameter as well as mode number on the non-dimensional frequencies of nonlocal FGP beams with and without elastic foundation at power-

Table 5. Influence of elastic foundation and external electric voltage on the 3rd non-dimensional frequency of a S-S FGP nanobeam (L/h=20). p=1,L/h=20,K_W=25

K_W	μ		$K_P=0$			$K_P=5$			$K_P=10$		
			$p=0.2$	$p=1$	$p=5$	$p=0.2$	$p=1$	$p=5$	$p=0.2$	$p=1$	$p=5$
0	0	V=-0.5	90.7489	87.6146	85.5572	93.1229	90.0712	88.0712	95.4380	92.4625	90.5154
		V=0	90.1162	86.8285	84.6181	92.5065	89.3068	87.1592	94.8365	91.7180	89.6282
		V=+0.5	89.4790	86.0353	83.6684	91.8858	88.5357	86.2375	94.2313	90.9674	88.7322
	1	V=-0.5	66.4467	64.2634	62.8631	69.6541	67.5743	66.2440	72.7201	70.7303	69.4606
		V=0	65.5800	63.1875	61.5789	68.8277	66.5519	65.0266	71.9290	69.7543	68.3005
		V=+0.5	64.7016	62.0930	60.2672	67.9913	65.5136	63.7859	71.1290	68.7643	67.1203
	2	V=-0.5	55.1297	53.4084	52.3321	58.9560	57.3494	56.3485	62.5486	61.0364	60.0970
		V=0	54.0819	52.1088	50.7823	57.9773	56.1412	54.9121	61.6270	59.9026	58.7523
		V=+0.5	53.0133	50.7761	49.1835	56.9819	54.9064	53.4371	60.6915	58.7470	57.3761
25	0	V=-0.5	90.8842	87.7547	85.7007	93.2548	90.2075	88.2106	95.5666	92.5953	90.6510
		V=0	90.2524	86.9699	84.7631	92.6392	89.4442	87.3000	94.9660	91.8519	89.7652
		V=+0.5	89.6162	86.1780	83.8151	92.0195	88.6744	86.3798	94.3616	91.1024	88.8705
	1	V=-0.5	66.6314	64.4543	63.0583	69.8303	67.7558	66.4292	72.8889	70.9038	69.6372
		V=0	65.7670	63.3816	61.7780	69.0060	66.7363	65.2152	72.0996	69.9302	68.4801
		V=+0.5	64.8912	62.2905	60.4707	68.1718	65.7009	63.9782	71.3016	68.9428	67.3031
	2	V=-0.5	55.3521	53.6379	52.5664	59.1640	57.5632	56.5661	62.7447	61.2374	60.3011
		V=0	54.3085	52.3441	51.0236	58.1888	56.3596	91.7180	61.8261	60.1074	58.9610
		V=+0.5	53.2445	91.7180	49.4327	57.1971	55.1296	53.6665	60.8936	58.9557	57.5898
50	0	V=-0.5	91.0192	87.8946	85.8439	93.3864	90.3436	88.3497	95.6951	92.7279	90.7864
		V=0	90.3884	87.1111	84.9079	92.7717	89.5815	87.4406	95.0953	91.9855	89.9019
		V=+0.5	89.7531	86.3204	83.9615	92.1529	88.8128	86.5219	94.4917	91.2371	89.0087
	1	V=-0.5	66.8155	64.6446	63.2528	70.0060	67.9369	66.6139	73.0572	71.0768	69.8134
		V=0	65.9536	63.5752	61.9766	69.1838	66.9201	65.4033	72.2698	70.1056	68.6593
		V=+0.5	65.0802	62.4874	60.6736	68.3518	65.8876	64.1699	71.4737	69.1207	67.4854
	2	V=-0.5	55.5736	53.8665	52.7996	59.3713	57.7762	56.7829	62.9402	61.4377	60.5045
		V=0	54.5343	52.5783	51.2638	58.3996	56.5771	55.3577	62.0245	60.3114	59.1690
		V=+0.5	53.4748	51.2577	49.6806	57.4115	55.3520	53.8949	61.0950	59.1637	57.8028

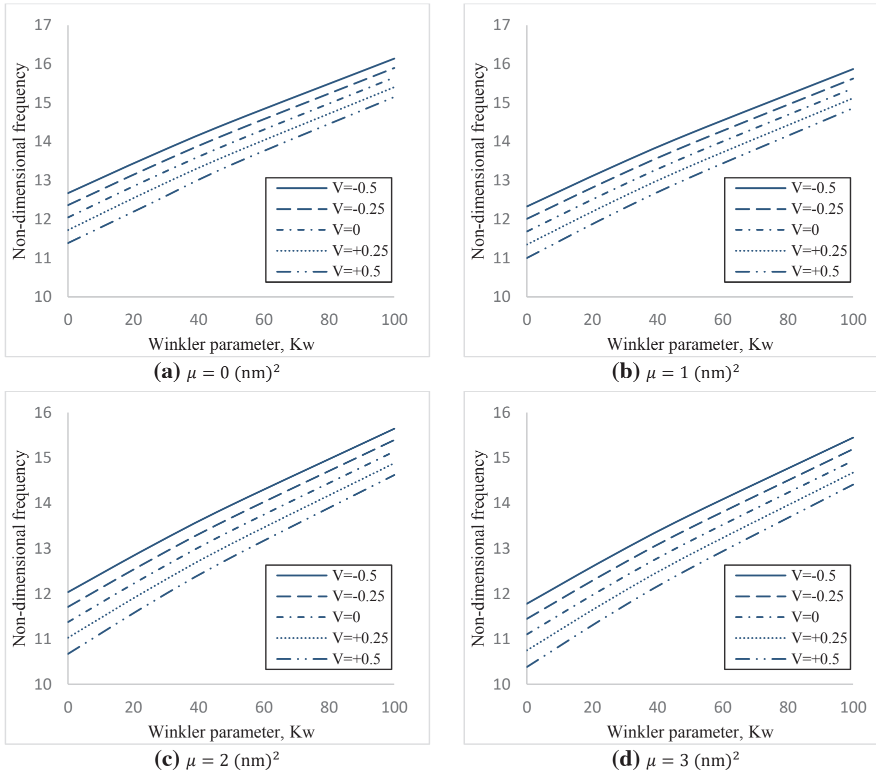


Fig. 2. Effect of electric voltage on the variation of non-dimensional frequency of the S-S FGP nanobeam with respect to Winkler parameter for different values nonlocal parameter (p=1,L/h=20,K_P=5).

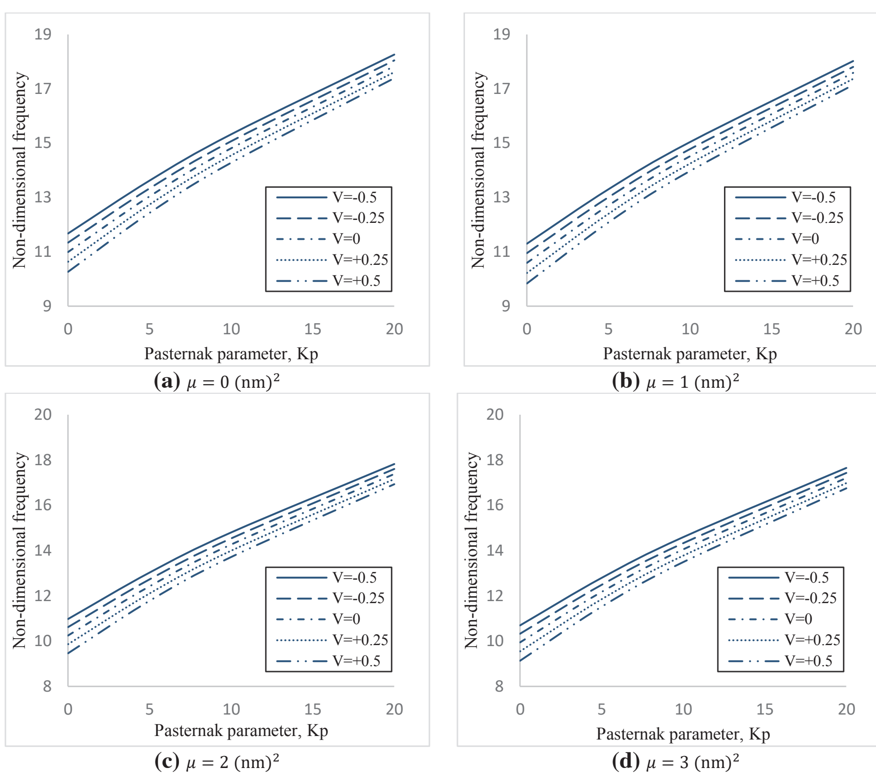


Fig. 3. Effect of electric voltage on the variation of non-dimensional frequency of the S-S FGP nanobeam with respect to Pasternak parameter for different values nonlocal parameter (p=1,L/h=20,K_W=25).

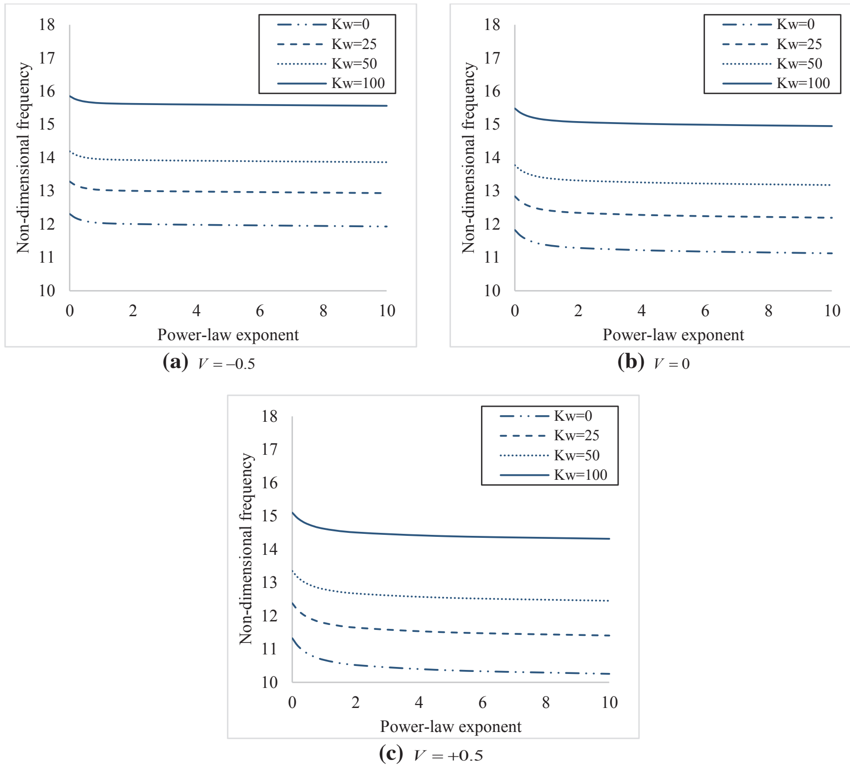


Fig. 4. Effect of Winkler parameter on the variation of non-dimensional frequency of the S-S FGP nanobeam with respect to power-law index for different values of electric voltage (μ=2,L/h=20,K_P=5).

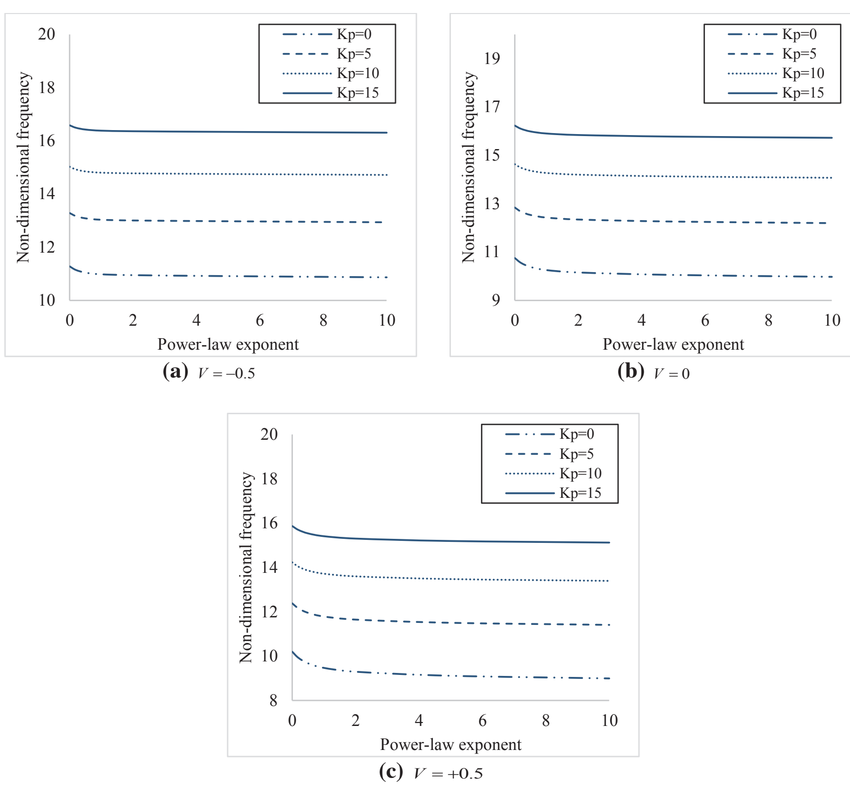


Fig. 5. Effect of Pasternak parameter on the variation of non-dimensional frequency of the S-S FGP nanobeam with respect to power-law index for different values of electric voltage (μ=2,L/h=20,K_W=25).

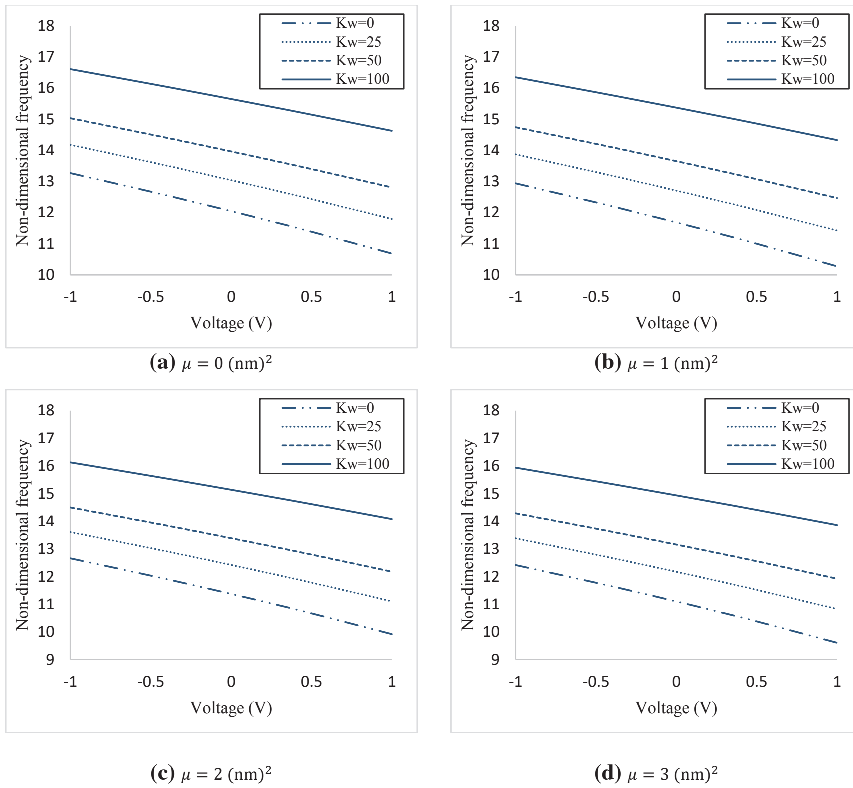


Fig. 6. Effect of Winkler parameter on the variation of non-dimensional frequency of the S-S FGP nanobeam with respect to electric voltage for different values nonlocal parameter (p=1,L/h=20,K_P=5).

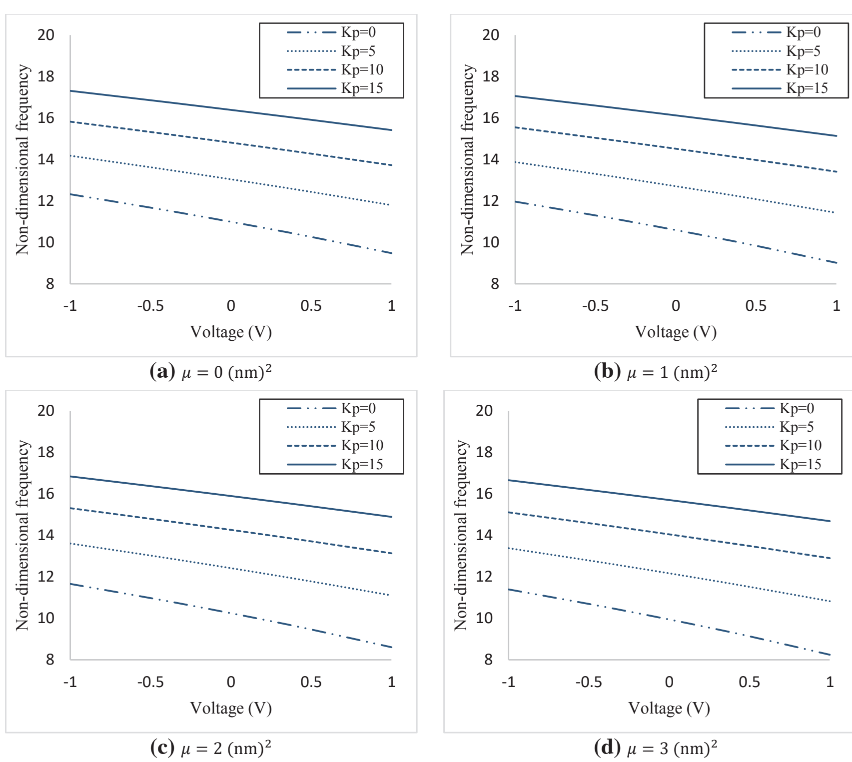
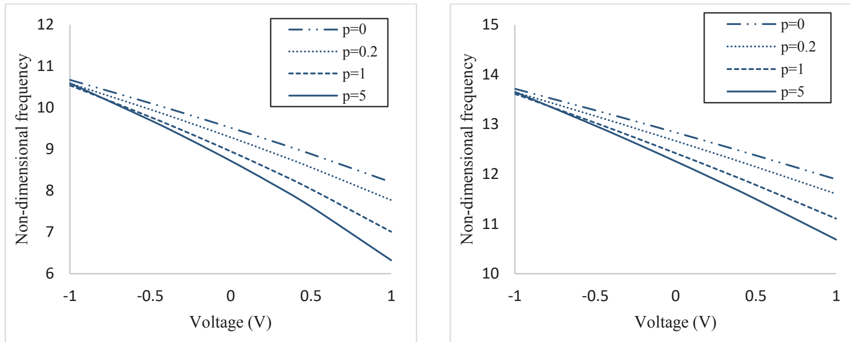
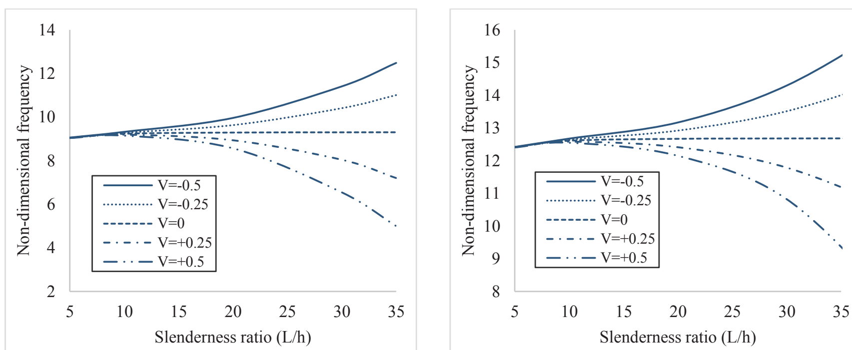


Fig. 7. Effect of Pasternak parameter on the variation of non-dimensional frequency of the S-S FGP nanobeam with respect to electric voltage for different values nonlocal parameter (p=1,L/h=20,K_W=25).



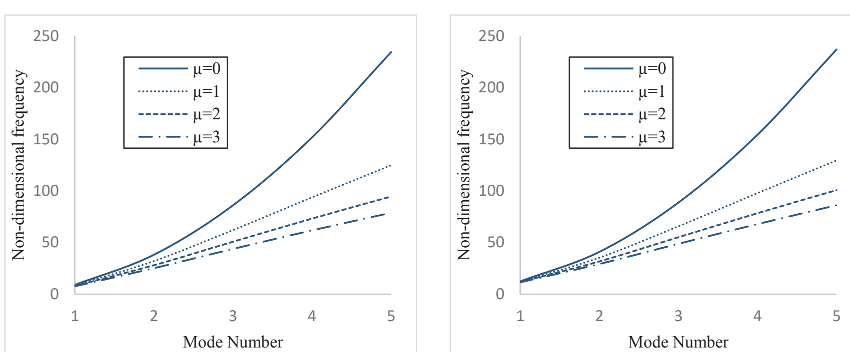
(a) $K_W = K_P = 0$ **(b)** $K_W = 25, K_P = 5$

Fig. 8. Effect of material composition on the variation of non-dimensional frequency of FGP nanobeam versus electric voltage with and without elastic foundation (*L/h*=20,*μ*=2).



(a) $K_W = K_P = 0$ **(b)** $K_W = 25, K_P = 5$

Fig. 9. Effect of slenderness ratio on the variation of non-dimensional frequency of the S-S FGP nanobeam for different values of electric voltage with and without elastic foundation (*L/h*=20,*μ*=2,*p*=0.2).



(a) $K_W = K_P = 0$ **(b)** $K_W = 25, K_P = 5$

Fig. 10. Effect of mode number on the dimensionless frequency of the S-S FGP nanobeam for different values of external voltages (*p*=1,*L/h*=20,*V*=+0.5).

law exponent *p*=1, slenderness ratio *L/h*=20 and electric voltage *V*=+0.5 is plotted in Fig. 10. As a consequence, the influence of nonlocal parameter on the higher modes of FGP nanobeams is more prominent than lower modes. Therefore, as a mode number raises the difference between local and nonlocal frequency results of FGP nanobeams increase.

5. Conclusions

This article studies free vibration behavior of piezoelectric FG nanobeams embedded in elastic medium based on nonlocal higher order beam theory. Adopting Eringen's nonlocal elasticity theory to capture the small size effects, the nonlocal governing equations are derived and solved using analytical method. Electro-mechanical properties of the FGP nanobeams are supposed to be position dependent based on power-law model. Various numerical examples demonstrate the influences of elastic foundation parameters, external electric voltage, gradient index, nonlocal parameter, slenderness ratio and mode number on the natural frequencies of FGP nanobeams. It is observed that existence of nonlocality yields in reduction in both rigidity of the beam and natural frequencies. Unlike the nonlocal parameter, with the increase of Winkler or Pasternak constants the rigidity of the nonlocal FGP beams and the frequency results rise. Also, according to the sign and magnitude of the electric voltage, it shows both reducing and increasing influence on the fundamental frequencies.

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