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LINEAR PRESERVERS OF SYMMETRIC ARCTIC RANK OVER THE BINARY BOOLEAN SEMIRING

LeRoy B. Beasley and Seok-Zun Song

Abstract. A Boolean rank one matrix can be factored as $\mathbf{u}\mathbf{v}^t$ for vectors $\mathbf{u}$ and $\mathbf{v}$ of appropriate orders. The perimeter of this Boolean rank one matrix is the number of nonzero entries in $\mathbf{u}$ plus the number of nonzero entries in $\mathbf{v}$. A Boolean matrix of Boolean rank k is the sum of k Boolean rank one matrices, a rank one decomposition. The perimeter of a Boolean matrix A of Boolean rank k is the minimum over all Boolean rank one decompositions of A of the sums of perimeters of the Boolean rank one matrices. The arctic rank of a Boolean matrix is one half the perimeter. In this article we characterize the linear operators that preserve the symmetric arctic rank of symmetric Boolean matrices.

1. Introduction and preliminaries

The *binary Boolean algebra* consists of the set $\mathbb{B} = \{0, 1\}$ equipped with two binary operations, addition and multiplication. The operations are defined as usual except that $1 + 1 = 1$.

Let $\mathcal{M}_{m,n}(\mathbb{B})$ denote the set of all $m \times n$ Boolean matrices with entries in $\mathbb{B}$. If $m = n$, we use the notation $\mathcal{M}_n(\mathbb{B})$ instead of $\mathcal{M}_{n,n}(\mathbb{B})$. The matrix I_n is the $n \times n$ identity matrix, $J_{m,n}$ is the $m \times n$ matrix of all ones, $O_{m,n}$ is the $m \times n$ zero matrix, and we write J_n for $J_{n,n}$ and O_n for $O_{n,n}$. We omit the subscripts when the order is obvious from the context and we write I, J and O, respectively. For matrices A and B, $A \oplus B$ is the direct sum of A and B so that $A \oplus B = \left[\begin{smallmatrix} A & O \\ O & B \end{smallmatrix}\right]$.

Note that any subset of matrices $\mathcal{L}$ in $\mathcal{M}_{m,n}(\mathbb{B})$ that is closed under addition (and scalar multiplication) has a unique "basis", that is a set of matrices such that any member $A \in \mathcal{L}$ is a linear combination of elements of that set and no member of that set is a linear combination of the remaining members of that

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set. Further, the linear combination of elements of a basis that equals A is unique, no other different linear combination of elements of the basis equals A (See [4] or [8]). The elements of a basis are called *base elements*.

A matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ is called a *cell* if it has exactly one nonzero entry, that being a 1. We denote the cell whose nonzero entry is in the $(i,j)^{th}$ position by $E_{i,j}$. Let $\mathcal{E} = \{E_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$. Then, $\mathcal{E}$ is the unique basis for $\mathcal{M}_{m,n}(\mathbb{B})$.

We let $\mathcal{S}_n(\mathbb{B})$ denote the set of all $n \times n$ *symmetric* Boolean matrices. For $1 \leq i < j \leq n$ let $D_{i,j} = E_{i,j} + E_{j,i}$. The matrix $D_{i,j}$ is called a *digon*. In $\mathcal{M}_{m,n}(\mathbb{B})$ *base elements* are all the cells in $\mathcal{E}$, whereas in $\mathcal{S}_n(\mathbb{B})$ the base elements are digons and diagonal cells. For convenience of notation, we occasionally use $D_{i,i}$ to represent $E_{i,i}$.

For $A \in \mathcal{M}_{m,n}(\mathbb{B})$ let $|A|$ denote the number of nonzero entries in A. That is $|\cdot| : \mathcal{M}_{m,n}(\mathbb{B}) \to \mathbb{Z}_+$ is the function such that $|A|$ is the number of nonzero entries in A, where $\mathbb{Z}_+$ is the set of nonnegative integers.

For $X,\ Y \in \mathcal{M}_{m,n}(\mathbb{B})$, X *dominates* Y, written $X \sqsupseteq Y$, if $y_{i,j} \neq 0$ implies $x_{i,j} \neq 0$ for all i and j. Thus, for $A \in \mathcal{M}_{m,n}(\mathbb{B})$, $|A|$ is the number of base elements of $\mathcal{M}_{m,n}(\mathbb{B})$ that are dominated by A. If $X \sqsupseteq Y$, then $X \setminus Y = Z$ is the matrix such that $z_{i,j} = x_{i,j}$ if $y_{i,j} = 0$ and is 0 otherwise.

For $A \in \mathcal{S}_n(\mathbb{B})$, we let $\#(A)$ denote the number of base elements of $\mathcal{S}_n(\mathbb{B})$ that A dominates. That is $\# : \mathcal{S}_n(\mathbb{B}) \to \mathbb{Z}_+$ is the function such that $\#(A)$ is the number of base elements of $\mathcal{S}_n(\mathbb{B})$ that A dominates. Note that if A is not symmetric, then $\#(A)$ is undefined.

Example 1.1. Since $D_{i,j} \in \mathcal{S}_n(\mathbb{B})$ and $D_{i,j}$ is also in $\mathcal{M}_n(\mathbb{B})$, we have that for $i \neq j$ $|D_{i,j}| = 2$ while $\#(D_{i,j}) = 1$. For another example, let $K_{v,v} = \left[\begin{smallmatrix} I_v & J_v \\ J_v & I_v \end{smallmatrix}\right]$. Then, $|K_{v,v}| = 2v^2 + 2v$ and $\#(K_{v,v}) = v^2 + 2v$.

For $A,\ B \in \mathcal{M}_{m,n}(\mathbb{B})$ the *Hadamard* or *Schur product* of A and B is the matrix C if the $(i,j)^{th}$ entry of C is $c_{i,j} = a_{i,j}b_{i,j}$, and we write $A \circ B = C$.

The (*Boolean*) *rank* or *factor rank*, $b(A)$, of a nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is defined as the least integer k for which there exist $B \in \mathcal{M}_{m,k}(\mathbb{B})$ and $C \in \mathcal{M}_{k,n}(\mathbb{B})$ such that $A = BC$. The rank of the zero matrix is zero.

The rank of $A \in \mathcal{M}_{m,n}(\mathbb{B})$ is 1 if and only if there exist nonzero (Boolean) vectors $\mathbf{b} \in \mathbb{B}^m = \mathcal{M}_{m,1}(\mathbb{B})$ and $\mathbf{c} \in \mathbb{B}^n = \mathcal{M}_{n,1}(\mathbb{B})$ such that $A = \mathbf{b}\mathbf{c}^t$. It is easy to verify that these vectors $\mathbf{b}$ and $\mathbf{c}$ are uniquely determined by A. It is well known ([4]) that $b(A)$ is the least k such that A is the sum of k matrices of rank 1 . It follows that $0 \leq b(A) \leq m$ for all nonzero $A \in \mathcal{M}_{m,n}(\mathbb{B})$.

The *perimeter* ([4]) of the rank-1 matrix $A = \mathbf{b}\mathbf{c}^t \in \mathcal{M}_{m,n}(\mathbb{B})$, $per(A)$, is $|\mathbf{b}| + |\mathbf{c}|$ the sum of the number of nonzero entries in $\mathbf{b}$ plus the number of nonzero entries in $\mathbf{c}$.

For $A \in \mathcal{M}_{m,n}(\mathbb{B})$, let $\mathcal{F}(A)$ be the set of ordered pairs of matrices that factor A. That is,

$$\mathcal{F}(A) = \{(B,C) \mid B \in \mathcal{M}_{m,k}(\mathbb{B}), C \in \mathcal{M}_{k,n}(\mathbb{B}) \text{ for some } k \text{ such that } A = BC\}.$$

Then $per(A) = \min_{(B,C)\in\mathcal{F}(A)} \{|B| + |C|\}$. That is, the *perimeter* of a matrix is a measure of the minimum number of nonzero entries in a rank k factorization, for any k for which there is such a factorization. An easy observation is that every matrix in $\mathcal{M}_{m,n}(\mathbb{B})$ whose perimeter is either 2 or 3 has rank 1. The *arctic rank* of A, $Arc(A)$ is one half the perimeter. Note that $Arc(A)$ may not be an integer. (See [3]).

Let $\mathcal{F}_{sym}$ denote the matrices in $\mathcal{S}_n(\mathbb{B})$ that have a symmetric factorization, that is if $A \in \mathcal{F}_{sym}$, then for some k, there exists $B \in \mathcal{M}_{n,k}(\mathbb{B})$ such that $A = BB^t$. In this case, $A = \mathbf{b}_1\mathbf{b}_1^t + \mathbf{b}_2\mathbf{b}_2^t + \cdots + \mathbf{b}_k\mathbf{b}_k^t$ where $\mathbf{b}_j$ is the j^{th} column of B. Then, the *symmetric perimeter* of A, $sper(A)$, is the minimum number, $|B| + |B^t| = 2|B|$, over all symmetric factorizations of $A = BB^t$. The *symmetric arctic rank* of A, $\rho_{\mathbf{sa}}(A)$, is one half the symmetric perimeter of A, so $\rho_{\mathbf{sa}}(A) = \frac{1}{2}sper(A)$, the minimum number of nonzero entries in B for a symmetric factorization of $A = BB^t$. Note that this symmetric arctic rank may be much larger than the order of the given symmetric matrix. For example, consider a 3×3 matrix

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then $\rho_{\mathbf{sa}}(A) = 4$ which is larger than 3.

Note that not all members of $\mathcal{S}_n(\mathbb{B})$ have a symmetric factorization, for example, $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$ has rank two, but no two symmetric rank one matrices sum to $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$. In fact, the product of a symmetric 2×2 matrix and its transpose can never equal $\left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]$, since if B has a nonzero entry, B^tB has a nonzero entry on the main diagonal. If $A \in \mathcal{S}_n(\mathbb{B}) \setminus \mathcal{F}_{sym}$, then we let $sper(A) = \rho_{\mathbf{sa}}(A) = \infty$. In fact, we can characterize all symmetric Boolean matrices that have a symmetric factorization.

Recall that the product of two matrices can be calculated in two ways: First by considering each entry of the product as an inner product of rows of the first factor times columns of the second (inner products); and second by considering the product as a sum of rank one matrices (outer products of columns of the first and rows of the second). Let $\mathbf{x}_i$ represent the i^{th} column of X and $\mathbf{x}^j$ the j^{th} row of X. If $A = BC$, with B an $n \times k$ matrix and C a $k \times m$ matrix, then $a_{i,j} = \mathbf{b}^i\mathbf{c}_j$ using the first method, and $A = \sum_{\ell=1}^{k} \mathbf{b}_\ell \mathbf{c}^\ell$ using the second method. The second method will be used in the proof that follows.

Theorem 1.2. *Suppose $A \in \mathcal{S}_n(\mathbb{B})$. Then $A \in \mathcal{F}_{sym}$ if and only if either $A = O$ or for some $0 \le s \le n-1$ there exists a permutation matrix P such that $PAP^t = B \oplus O_s$ where $b_{i,i} = 1$ for all $i = 1, \ldots, n-s$.*

Proof. Suppose that $A \neq O$ and that $A = XX^t$ for $X \in \mathcal{M}_{n,q}(\mathbb{B})$. In this case, $A = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{y}_i\mathbf{y}_j^t$ where $\mathbf{y}_i$ is the i^{th} row of X. If $a_{i,j} = 1$, then since $a_{i,j} = \mathbf{y}_i\mathbf{y}_j^t$, we have that $\mathbf{y}_i \neq \mathbf{0}$, and hence $a_{i,i} = \mathbf{y}_i\mathbf{y}_i^t = 1$. Thus any row of A that contains a nonzero entry has a nonzero entry on the diagonal. Let P be a permutation matrix which permutes the nonzero rows of A to the first $n-s$ rows, if there are $n-s$ such rows. Then PAP^t has nonzero entries in the first $n-s$ rows/columns and only zero entries in the last s rows/columns. That is $PAP^t = B \oplus O_s$. Since any row of A that has a nonzero entry has a nonzero diagonal entry, $b_{i,i} = 1$ for all $i = 1, \ldots, n-s$.

For the converse, if $A = O$, then $A = O_{n,1}O_{n,1}^t$ so that A has a symmetric factorization,

Now suppose that $A \neq O$ and for some $0 \leq s \leq n-1$ and for some permutation matrix P, $PAP^t = B \oplus O_s$ such that $b_{i,i} = 1$ for $1 \leq i \leq n-s$. Then, A has a nonzero diagonal entry in every row which has a nonzero entry. Let $A_{i,i} = a_{i,i}E_{i,i}$ and for $1 \leq i < j \leq n$ let $A_{i,j} = a_{i,j}(E_{i,i}+E_{j,j}+D_{i,j})$. Note that $A_{i,i} = a_{i,i}\mathbf{e}_i\mathbf{e}_i^t$ and if $i < j$, $A_{i,j} = a_{i,j}[(\mathbf{e}_i + \mathbf{e}_j)(\mathbf{e}_i + \mathbf{e}_j)^t]$, so for $1 \leq i \leq j \leq n$, either $A_{i,j} = O$ or $A_{i,j}$ is a rank 1 matrix. Further, $A = \sum_{1 \leq i \leq j \leq n} A_{i,j}$. Let $\mathbf{x}_{i,i} = a_{i,i}\mathbf{e}_i$ and for $i < j$ let $\mathbf{x}_{i,j} = a_{i,j}(\mathbf{e}_i + \mathbf{e}_j)$. Observe that either $\mathbf{x}_{i,j}$ is the zero vector or a vector with one or two nonzero entries. Consider the $n \times \frac{n(n+1)}{2}$ matrix $X = [\mathbf{x}_{1,1}, \mathbf{x}_{1.2}, \ldots, \mathbf{x}_{1,n}, \mathbf{x}_{2,2}, \ldots, \mathbf{x}_{2,n}, \ldots, \mathbf{x}_{i,i}, \mathbf{x}_{i,i+1}, \ldots, \mathbf{x}_{i,n}, \ldots, \mathbf{x}_{n,n}]$. Then,

$$XX^t = \sum_{i=1}^{n}\sum_{j=i}^{n} \mathbf{x}_{i,j}x_{i,j}^t = \sum_{i=1}^{n}\sum_{j=i}^{n} a_{i,j}^2(\mathbf{e}_i + \mathbf{e}_j)(\mathbf{e}_i + \mathbf{e}_j)^t = \sum_{i=1}^{n}\sum_{j=i}^{n} A_{i,j}$$

since $a_{i,j}^2 = a_{i,j}$ and $\mathbf{e}_i + \mathbf{e}_i = \mathbf{e}_i$ over the Boolean semiring $\mathbb{B}$. That is, $XX^t = A$. (Note that the fact that X may have many zero columns does not change the fact that A has a symmetric factorization.) □

Let SA_k denote the set of all matrices in $\mathcal{M}_n(\mathbb{B})$ whose symmetric arctic rank is k and note that SA_∞ is the set of all matrices in $\mathcal{S}_n(\mathbb{B})$ that do not have a symmetric factorization.

A matrix of special interest in our investigations is $D_{i,j}^+ = E_{i,i} + E_{j,j} + D_{i,j}$.

Note that:

If $\rho_{\mathbf{sa}}(A) = 1$, then A is a diagonal cell. That is, SA_1 is the set of all diagonal cells;

If $\rho_{\mathbf{sa}}(A) = 2$, then for some $i \neq j$, $A = D_{i,j}^+ = E_{i,i} + E_{j,j} + D_{i,j}$ or $A = E_{i,i} + E_{j,j}$; and

If $\rho_{\mathbf{sa}}(A) = 3$, then up to permutational similarity, $A = J_3 \oplus O$, $A = I_3 \oplus O + D_{1,2} = J_2 \oplus [1] \oplus O$, or $A = I_3 \oplus O$. Note that when $\rho_{\mathbf{sa}}(A) = 3$, $\#(A) = 3, 4$ or 6 only.

A mapping $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ is called a (*Boolean*) *linear operator* if $T(A+B) = T(A)+T(B)$ for all $A, B \in \mathcal{S}_n(\mathbb{B})$, and $T(O) = O$. A linear operator $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ is called a (P, P^t)-*operator* if there exists a permutation matrix P such that $T(X) = PXP^t$ for all $X \in \mathcal{S}_n(\mathbb{B})$, where X^t denotes the transpose of X. A linear operator T is said to *preserve* a set $\mathcal{Q}$ if $A \in \mathcal{Q}$ implies $T(A) \in \mathcal{Q}$. Also, T *strongly preserves* the set $\mathcal{Q}$ if $A \in \mathcal{Q}$ if and only if $T(X) \in \mathcal{Q}$.

Lately there have been many articles on linear preserver problems. For an excellent survey see [6, 7]. In [10], the linear operators that preserve perimeters of matrices over semirings were characterized. In this article we investigate the linear operators that preserve symmetric arctic rank of symmetric matrices over the binary Boolean semirings.

It is well known that the adjacency matrix of an undirected graph is a symmetric (Boolean) (0,1)-matrix and that the adjacency matrix of a clique (a graph whose nonisolated vertices induce a complete graph) is a symmetric rank one Boolean matrix. So the symmetric arctic rank is the minimum over all decompositions of the graph into a union of cliques of the sum of the orders of the cliques. Let A be the adjacency matrix of the complete balanced bipartite graph on $n = 2\ell$ vertices, that is $A = A(K_{\ell,\ell})$, and let $Z = I + A$, the adjacency matrix of $K_{\ell,\ell}$ with loops added at each vertex. Then the factor rank of Z, $b(Z)$, is n, the symmetric factor rank of Z (the smallest number of symmetric rank one matrices whose sum is Z) is $\frac{n^2}{4}$, and the symmetric arctic rank of Z, $\rho_{\mathbf{sa}}(A)$, is $\frac{n^2}{2}$. So the symmetric rank of a Boolean matrix may be much larger than any other rank. For more on the connection between clique covers of graphs and the ranks of Boolean matrices see [2].

2. Preservers of symmetric arctic ranks

We begin this section with an example to show that preservers of the three sets SA_1, SA_2, and SA_3 do not have the same characterization as, for example, strong preservers of SA_4.

Example 2.1. Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be defined by $T(E_{i,i}) = E_{i,i}$, $T(D_{i,j}) = E_{i,i} + E_{j,j}$ and extend linearly. Then T strongly preserves SA_1, T strongly preserves SA_2 and T preserves (not strongly) SA_3. Further T does not preserve any other SA_k, $4 \leq k \leq n$, since, for example, for $A = \begin{bmatrix} 1&1&1\\1&1&0\\1&0&1 \end{bmatrix} \oplus I_{k-4} \oplus O_{n-k+1} = \begin{bmatrix} 1&1\\1&0\\0&1 \end{bmatrix} \begin{bmatrix} 1&1&0\\1&0&1 \end{bmatrix} \oplus I_{k-4} \oplus O_{n-k+1}$, $\rho_{\mathbf{sa}}(A) = k$ while $\rho_{\mathbf{sa}}(T(A)) = k-1$, since $T(A) = I_{k-1} \oplus O_{n-k+1}$.

An easy observation is that if a linear operator $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ preserves SA_1, then the image of a diagonal cell is a diagonal cell.

Lemma 2.2. *If a linear operator $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ preserves SA_1 and SA_k for any $k, 2 \leq k \leq n$, then T maps the set of diagonal cells bijectively onto the set of diagonal cells. That is T is bijective on SA_1.*

Proof. Since T preserves SA_1, the image of a diagonal cell is a diagonal cell. Suppose that the image of two diagonal cells is the same diagonal cell. Without loss of generality assume that $T(E_{1,1}) = T(E_{2,2})$. Then $T(I_k \oplus O_{n-k})$ is the sum of at most $k-1$ diagonal cells and hence $\rho_{\mathbf{sa}}(T(I_k \oplus O_{n-k})) \leq k-1$, a contradiction, since T preserves SA_k. Since SA_1 is finite, T is bijective on SA_1. $\square$

Theorem 2.3. *Let $L : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then, L preserves SA_1 and SA_k for some $4 \le k \le n-1$ if and only if there exist scalars $\alpha_{i,j},\ \beta_{i,j} \in \mathbb{B}$ for $1 \le i < j \le n$ and a permutation matrix P such that*

$$L(X) = P\left(X + \sum_{1\le i<j\le n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$$

for all $X \in \mathcal{S}_n(\mathbb{B})$.

Proof. Since L preserves both SA_1 and SA_k for some $4 \le k \le n-1$, by Lemma 2.2, L is bijective on the set of diagonal cells. Permute by P so that $T(X) = P^tL(X)P$ for all X and such that $T(E_{i,i}) = E_{i,i}$ for all i. Since permutational similarity preserves all symmetric arctic ranks we also have that T preserves SA_1 and SA_k.

Now suppose that $T(D^+_{i,j})$ dominates at least three diagonal cells. Let I' be the sum of $k-2$ diagonal cells such that $\rho_{\mathbf{sa}}(D^+_{i,j} + I') = k$ and such that $T(D^+_{i,j} + I')$ dominates $k+1$ diagonal cells. This is always possible since T is bijective on the set of diagonal cells. Then, $\rho_{\mathbf{sa}}(T(D^+_{i,j} + I')) \ge k+1$, a contradiction since $\rho_{\mathbf{sa}}(D^+_{i,j} + I') = k$.

Now suppose that $T(D_{i,j})$ dominates two digons. Then there is some k such that $T(D^+_{i,j})$ dominates $D_{k,\ell}$ for some ℓ, but does not dominate $E_{k,k}$. Let I' be the sum of $k-2$ diagonal cells whose image does not dominate $E_{k,k}$ and such that $\rho_{\mathbf{sa}}(D^+_{i,j} + I') = k$. We then have arrived at a contradiction since $\rho_{\mathbf{sa}}(T(D^+_{i,j} + I')) = \infty$, since $T(D^+_{i,j} + I') \notin \mathcal{F}_{sym}$. Thus, for each (i,j) there is some (p,q) such that $T(D^+_{i,j}) \sqsubseteq D^+_{p.q}$.

Now suppose that $T(D^+_{i,j}) \sqsubseteq D^+_{p,q}$ and $T(D^+_{k,\ell}) \sqsubseteq D^+_{p.q}$ for $(i,j) \ne (k,\ell)$. Let I' be the sum of either $k-3$ or $k-4$ diagonal cells such that $\rho_{\mathbf{sa}}(D^+_{i,j} + D^+_{k,\ell} + I') = k$. Then, $T(D^+_{i,j} + D^+_{k,\ell} + I') \sqsubseteq D^+_{p,q} + T(I')$ which has symmetric arctic rank at most k-1, a contradiction.

We now have that $T(D^+_{i,j}) \sqsubseteq D^+_{i,j}$ for all (i,j). Suppose that $T(D^+_{i,j}) \ne D^+_{i,j}$. Then, $T(D^+_{i,j}) = E_{i,i} + E_{j,j}$. Without loss of generality, assume that $T(D^+_{1,2}) = E_{1,1} + E_{2,2}$. Then, $\rho_{\mathbf{sa}}(D^+_{1,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{k-1,k-1}) = k$ (See Example 2.1), while $\rho_{\mathbf{sa}}(E_{2,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{k-1,k-1}) = k-1$, but $T(D^+_{1,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{k-1,k-1}) = T(E_{2,2} + D^+_{1,3} + E_{4,4} + \cdots + E_{k-1,k-1})$, a contradiction. Thus, $T(D^+_{i,j}) = D^+_{i,j}$ for all (i,j).

Further, $T(D_{i,j}) \sqsubseteq D^+_{i,j}$, so let $\alpha_{i,j}$ and $\beta_{i,j}$ be defined by $T(D_{i,j}) = D_{i,j} + \alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j}$. Since to get T we permuted by P, it follows that $L(X) = P\left(X + \sum_{1\le i<j\le n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$ for all $X \in \mathcal{S}_n(\mathbb{B})$.

For the converse, if $L(X) = P\left(X + \sum_{1\le i<j\le n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$ for all $X \in \mathbb{B}$, then L preserves all symmetric arctic ranks except $\rho_{\mathbf{sa}}(A) = \infty$. □

Theorem 2.4. *Let $L : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then, L preserves SA_2 and SA_k for some $4 \le k \le n-1$ if and only if there exist scalars $\alpha_{i,j},\ \beta_{i,j} \in \mathbb{B}$ for $1 \le i < j \le n$ and a permutation matrix P such that*

$$T(X) = P\left(X + \sum_{1\le i<j\le n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$$

for all $X \in \mathcal{S}_n(\mathbb{B})$.

Proof. We first show that T preserves SA_1.

Suppose that for some i, $\rho_{\mathbf{sa}}(T(E_{i,i})) \ne 1$, and assume without loss of generality that $i = 1$. Either $T(E_{1,1}) \sqsupseteq D_{i,j}$ for some $i \ne j$ or $T(E_{1,1}) \sqsubseteq I_n$ and $\#(T(E_{1,1})) \ge 2$.

If $T(E_{1,1}) \sqsupseteq D_{i,j}$, then $T(E_{1,1}) \sqsubseteq D^+_{i,j}$, and for any $q, 1 \le q \le n$, $T(E_{q,q}) \sqsubseteq T(E_{11} + E_{q,q}) \sqsubseteq D^+_{i,j}$ since $E_{11} + E_{j,j}$ is in SA_2 and the only member of SA_2 that dominates $D_{i,j}$ is $D^+_{i,j}$. But then, $T(I_k \oplus O_{n-k}) \sqsubseteq D^+_{i,j}$, a contradiction since $\rho_{\mathbf{sa}}(I_k \oplus O_{n-k}) = k$ and $\rho_{\mathbf{sa}}(D^+_{i,j}) = 2$. Thus, $T(E_{i,i}) \sqsubseteq I_n$ for all i, and for some i $\#(T(E_{i,i})) \ge 2$. But then without loss of generality we may assume that $E_{1,1} + E_{2,2} \sqsubseteq T(D^+_{1,2})$, and hence, we must have that $T(E_{1,1}) \sqsupseteq T(I_2 \oplus O_{n-2})$ so that we have that $T(E_{1,1}) = E_{1,1} + E_{2,2}$. But then, $T(E_{j,j}) \sqsubseteq T(E_{1,1} + E_{j,j}) \sqsubseteq I_2 \oplus O_{n-2}$, for all j since $\rho_{\mathbf{sa}}(E_{1,1} + E_{j,j}) = 2$. We now must have that $T(I_k) \sqsubseteq I_2 \oplus O_{n-2}$, and as above we have a contradiction. That is, T preserves SA_1.

By Theorem 2.3 the theorem follows. □

3. Bijective preservers of symmetric arctic ranks

In this section we shall classify all bijective linear operators on $\mathcal{S}_n(\mathbb{B})$ that preserve the set of matrices of symmetric arctic rank k. We begin with some lemmas.

Lemma 3.1. *If $A \in \mathcal{F}_{sym}$ and $|A \circ I| \ge k$, then $\rho_{\mathbf{sa}}(A) \ge k$. Further if $|A \circ I| > k$, then $\rho_{\mathbf{sa}}(A) > k$.*

Proof. Let $A = A_1 + A_2 + \cdots + A_\ell$ be a rank one decomposition of A such that $\rho_{\mathbf{sa}}(A) = \sum_{i=1}^{\ell} \rho_{\mathbf{sa}}(A_i)$ and let $\alpha_i = |A_i \circ I|$ so that $\alpha_i = \rho_{\mathbf{sa}}(A_i)$. Then $|A \circ I| = |(\sum_{i=1}^{\ell} A_i) \circ I| \le \sum_{i=1}^{\ell} |A_i \circ I| = \sum_{i=1}^{\ell} \alpha_i = \rho_{\mathbf{sa}}(A)$. That is, $k \le \rho_{\mathbf{sa}}(A)$, and if $|A \circ I| > k$, then $\rho_{\mathbf{sa}}(A) > k$. □

Lemma 3.2. *If $A \in \mathcal{F}_{sym}$ and $\#(A) > \frac{k^2+k}{2}$, then $\rho_{\mathbf{sa}}(A) > k$.*

Proof. Suppose $A \in \mathcal{F}_{sym}$ and $\#(A) > \frac{k^2+k}{2}$. If $|A \circ I| \le k$, then all the nonzero entries of A lie in the intersection of $|A \circ I|$ columns and the same $|A \circ I|$ rows of A, so that $\#(A) \le \frac{k^2+k}{2}$, a contradiction. Thus $|A \circ I| > k$, so by Lemma 3.1, $\rho_{\mathbf{sa}}(A) > k$. □

Lemma 3.3. *If $A \in \mathcal{F}_{sym}$, $\rho_{\mathbf{sa}}(A) = k$ and $\#(A) = \frac{k^2+k}{2}$, then there exists a permutation matrix P such that $A = P^t(J_k \oplus O_{n-k})P$.*

Proof. Let P be a permutation matrix such that $P(A \circ I)P^t = I_\ell \oplus O_{n-\ell}$. Since $\rho_{\mathbf{sa}}(A) = k$, $\ell \leq k$ by Lemma 3.1. Thus, since $A \in \mathcal{F}_{sym}$, $A \sqsubseteq P^t(J_k \oplus O_{n-k})P$. But $\#(J_k \oplus O_{n-k}) = \frac{k^2+k}{2}$, so we must have that $A = P^t(J_k \oplus O_{n-k})P$. □

Let $Z \in \mathcal{S}_n(\mathbb{B})$. In the following lemma we use the notation $Z[1, \ldots, k|1, \ldots, k]$ to denote the submatrix of Z consisting of the intersection of the first k rows and the first k columns. We use $Z(1, \ldots, k|1, \ldots, k)$ to denote the submatrix of Z consisting of the intersection of the last $n-k$ rows and the last $n-k$ columns.

Lemma 3.4. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_k for some $3 \leq k \leq n$, then T preserves SA_1.*

Proof. Since T is bijective on $\mathcal{S}_n(\mathbb{B})$, T is bijective on the set of base elements. Suppose that $T(E_{i,i}) = D_{r,s}$ for some $r \neq s$. We may assume that $i = 1$.

Consider $T(J_k \oplus O_{n-k})$. Since T is bijective on the set of base elements, $\#(T(J_k \oplus O_{n-k})) = \#(J_k \oplus O_{n-k}) = \frac{k^2+k}{2}$ and since $\rho_{\mathbf{sa}}(T(J_k \oplus O_{n-k})) = k$, by Lemma 3.3 there is some permutation matrix P such that $T(J_k \oplus O_{n-k}) = P^t(J_k \oplus O_{n-k})P$. Without loss of generality, by permuting, we may assume that $T(J_k \oplus O_{n-k}) = (J_k \oplus O_{n-k})$.

As in the paragraph above, there is some permutation matrix Q such that $Q^t T(O_1 \oplus J_k \oplus O_{n-k-1})Q = J_k \oplus O_{n-k}$ so that we have that $T(O_1 \oplus J_k \oplus O_{n-k-1})$ is a rank one matrix.

Let $Z = T(O_1 \oplus J_k \oplus O_{n-k-1})$ and let P_1 be a permutation matrix such that $P_1 Z[1, \ldots, k|1, \ldots, k]P_1^t = O_{k-\ell} \oplus J_\ell$. Let Q_1 be a permutation matrix such that $Q_1 Z(1, \ldots, k|1, \ldots, k)Q_1^t = J_{k-\ell} \oplus O_{n-2k+\ell}$. Then,

$$(P_1 \oplus Q_1)Z(P_1 \oplus Q_1)^t = \begin{bmatrix} O_{k-\ell} & O & A & B \\ O & J_\ell & C & D \\ A^t & C^t & J_{k-\ell} & O \\ B^t & D^t & O & O_{n-2k+\ell} \end{bmatrix}.$$

Note that if $X \in \mathcal{F}_{sym}$ and the i^{th} row or column of X has a nonzero entry, then $x_{i,i} \neq 0$. Thus, A, B, and D are all zero matrices. Since Z is rank one, we must have that $C = J_{\ell.k-\ell}$. It now follows that $(P_1 \oplus Q_1)Z(P_1 \oplus Q_1)^t = O_{k-\ell} \oplus J_k \oplus O_{n-2k+\ell}$. Further, $(P_1 \oplus Q_1)(J_k \oplus O_{n-k})(P_1 \oplus Q_1)^t = P_1 J_k P_1^t \oplus Q_1 O_{n-k} Q_1^t = J_k \oplus O_{n-k}$ so that $(P_1 \oplus Q_1)T(J_k \oplus O_{n-k})(P_1 \oplus Q_1)^t = J_k \oplus O_{n-k}$.

Let $L : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be defined by $L(X) = (P_1 \oplus Q_1)T(X)(P_1 \oplus Q_1)^t$. Then, L is bijective on the set of base elements, $L(E_{1,1}) = D_{p,q}$ for some $p \neq q$, $L(J_k \oplus O_{n-k}) = J_k \oplus O_{n-k}$, and $L(O_1 \oplus J_k \oplus O_{n-k-1}) = O_{k-\ell} \oplus J_k \oplus O_{n-2k+\ell}$. Since $O_1 \oplus J_{k-1} \oplus O_{n-k} \sqsubseteq J_k \oplus O_{n-k}$ and $O_1 \oplus J_{k-1} \oplus O_{n-k} \sqsubseteq O_1 \oplus J_k \oplus O_{n-k-1}$, we must have that $L(O_1 \oplus J_{k-1} \oplus O_{n-k}) \sqsubseteq L(J_k \oplus O_{n-k}) = J_k \oplus O_{n-k}$ and $L(O_1 \oplus J_{k-1} \oplus O_{n-k}) \sqsubseteq L(O_1 \oplus J_k \oplus O_{n-k-1}) = O_{k-\ell} \oplus J_k \oplus O_{n-2k+\ell}$. Since

$\#(O_1 \oplus J_{k-1} \oplus O_{n-k}) = \frac{(k-1)^2+(k-1)}{2}$ and L is bijective on the base elements, it follows that $\ell = k-1$. That is, $L(O_1 \oplus J_{k-1} \oplus O_{n-k}) = (O_1 \oplus J_{k-1} \oplus O_{n-k})$.

We now have that $L(E_{1,1}+(O_1 \oplus J_{k-1} \oplus O_{n-k})) = D_{p,q}+(O_1 \oplus J_{k-1} \oplus O_{n-k})$, and that $D_{p,q} = L(E_{1,1}) \not\sqsubseteq L(O_1 \oplus J_{k-1} \oplus O_{n-k}) = O_1 \oplus J_{k-1} \oplus O_{n-k}$ since L is bijective on the base elements. This is a contradiction since $\rho_{\mathbf{sa}}(E_{1,1}+(O_1 \oplus J_{k-1} \oplus O_{n-k})) = k$ and $\rho_{\mathbf{sa}}(D_{p,q}+(O_1 \oplus J_{k-1} \oplus O_{n-k})) = \infty$.

Thus L, and hence T, preserves SA_1. $\square$

Corollary 3.5. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then T is bijective and preserves SA_k for some $4 \le k \le n-1$ if and only if T is a (P, P^t)-operator.*

Proof. Suppose T is bijective and preserves SA_k for some $4 \le k \le n-1$. By Lemma 3.4 T preserves SA_1. Now, by Theorem 2.3 there exist scalars $\alpha_{i,j}, \beta_{i,j} \in \mathbb{B}$ for $1 \le i < j \le n$ and a permutation matrix P such that

$$T(X) = P\left(X + \sum_{1\le i<j\le n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$$

for all $X \in \mathcal{S}_n(\mathbb{B})$. If $\alpha_{i,j} = 1$, then $T(D_{i,j}) = T(D_{i,j} + E_{i,i})$, contradicting that T is bijective. Thus, $\alpha_{i,j} = 0$ for all $i \ne j$. Similarly $\beta_{i,j} = 0$ for all $i \ne j$. That is $T(X) = PXP^t$ for all $X \in \mathcal{S}_n(\mathbb{B})$.

The converse is routinely established. $\square$

Lemma 3.6. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_2, then T preserves SA_1.*

Proof. Suppose T is bijective. Then T is bijective on the set of base elements since $\mathbb{B}$ is antinegative and each member of $\mathcal{S}_n(\mathbb{B})$ is a unique combination of base elements.

Suppose that $T(E_{i,i}) = D_{p,q}$ for some i and $p \ne q$. Let $T(E_{j,j}) = F$, a base element. Then $T(E_{i,i} + E_{j,j}) = D_{p,q} + F$. However $\rho_{\mathbf{sa}}(E_{i,i} + E_{j,j}) = 2$ while the sum of $D_{p,q}$ plus any base element is never in SA_2. Thus we have a contradiction and hence, T preserves SA_1. $\square$

Theorem 3.7. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_2, then T is a (P, P^t)-operator.*

Proof. Suppose that T is bijective and preserves SA_2. By Lemma 3.6, T preserves SA_1 and hence T is bijective on SA_1. Let P be the permutation matrix such that $T(E_{i,i}) = PE_{i,i}P^t$ for all $i = 1, \ldots, n$, and let $T(D_{i,j}) = F_{i,j}$, a base element. Then $T(D^+_{i,j}) = T(E_{i,i} + E_{j,j} + D_{i,j}) = T(E_{i,i}) + T(E_{j,j}) + T(D_{i,j}) = PE_{i,i}P^t + PE_{j,j}P^t + F_{i,j}$. Since the only base element that can be added to $PE_{i,i}P^t + PE_{j,j}P^t$ to get an element of SA_2 is $PD_{i,j}P^t$, we must have that $F = PD_{i,j}P^t$. That is, $T(X) = PXP^t$ for all $X \in \mathcal{S}_n(\mathbb{B})$. $\square$

Lemma 3.8. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_3, then T preserves SA_2.*

Proof. If T is bijective and preserves SA_3, then by Lemma 3.4, T preserves SA_1. It now follows that T is bijective on SA_1 and T is bijective on the set of digons.

Let P be a permutation matrix such that $L(X) = P^t T(X) P$ for all $X \in \mathcal{S}_n(\mathbb{B})$ and such that $L(E_{i,i}) = E_{i,i}$ for all i. Further, L is bijective on the set of digons and preserves sets SA_1 and SA_3. Suppose that L does not preserve SA_2. Then since L is the identity on the SA_1, we must have that $L(D_{i,j}) \neq D_{i,j}$ for some $i \neq j$. Say $L(D_{i,j}) = D_{r,s}$ with $s \neq i, j$. Let k be any integer, $k \neq i, j, s$. Then $\rho_{\mathbf{sa}}(E_{i,i} + E_{j,j} + E_{k,k} + D_{i,j} + D_{i,k} + D_{j,k}) = 3$ and $L(E_{i,i} + E_{j,j} + E_{k,k} + D_{i,j} + D_{i,k} + D_{j,k}) = E_{i,j} + E_{j,j} + E_{k,k} + D_{r,s} + L(D_{i,k} + D_{j,k})$. Since $E_{s,s} \not\sqsubseteq E_{i,i} + E_{j,j} + E_{k,k} + D_{r,s} + L(D_{i,k} + D_{j,k})$, $\rho_{\mathbf{sa}}(L(E_{i,i} + E_{j,j} + E_{k,k} + D_{i,j} + D_{i,k} + D_{j,k})) = \infty$, a contradiction. Thus T preserves SA_2. $\square$

Theorem 3.9. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_3, then T is a (P, P^t)-operator.*

Proof. By Lemma 3.8, T preserves SA_2, and then by Theorem 3.7 the theorem follows. $\square$

Lemma 3.10. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_n, then T preserves SA_2.*

Proof. If T is bijective and preserves SA_n, then T preserves SA_1 by Lemma 3.4. Thus, T is bijective on the set of diagonal cells. Let P be a permutation matrix such that $L(X) = P^t T(X) P$ for all $X \in \mathcal{S}_n(\mathbb{B})$ and such that $L(E_{i,i}) = E_{i,i}$ for all i. Further, L is bijective on the set of digons and preserves sets SA_1 and SA_n. By Lemma 3.8 we may assume that $n \geq 4$.

Suppose that $L(D_{i,j}) \neq D_{i,j}$ for some $i \neq j$. Without loss of generality we may assume that $(i, j) = (1, 2)$. Then $L(D_{1,2}) = D_{p.q}$ for some $q \geq 4$. Then, $L(E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{1,2} + D_{1,3}) = E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{p,q} + D_{r,s}$ for some $r < s$. But, $\rho_{\mathbf{sa}}(E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{1,2} + D_{1,3}) = n$ by Example 2.1, while $\rho_{\mathbf{sa}}(E_{1,1} + E_{2,2} + \cdots + E_{q-1,q-1} + E_{q+1,q+1} + \cdots + E_{n,n} + D_{p,q} + D_{r,s}) = \infty$ for any choice of $r \neq s$ since the (q, q) entry is zero while the (p, q) entry is nonzero, which is not possible in any member of $\mathcal{F}_{sym}$. This contradicts that L preserves SA_n. Thus, L preserves SA_2. $\square$

Theorem 3.11. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T is bijective and preserves SA_n, then T is a (P, P^t)-operator.*

Proof. By Lemma 3.10, T preserves SA_2, and then by Theorem 3.7 the theorem follows. $\square$

4. Strong preservers of symmetric arctic rank

In this section we shall classify all linear operators on $\mathcal{S}_n(\mathbb{B})$ that strongly preserve the set of matrices of symmetric arctic rank k.

Theorem 4.1. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T strongly preserves SA_k for some $3 \leq k \leq n$, then T is bijective on the set of base elements.*

Proof. Suppose that $T(X) = O$ for some $X \in \mathcal{S}_n(\mathbb{B})$. Then, for some $i \leq j$ $T(D_{i,j}) = O$. Without loss of generality, we may assume that $1 \leq i, j \leq 2$. Then, $T((J_k \oplus O) \setminus D_{i,j}) = T(J_k \oplus O)$, contradicting that T strongly preserves SA_k since $\rho_{\mathbf{sa}}(J_k \oplus O) = k$ while $\rho_{\mathbf{sa}}((J_k \oplus O) \setminus D_{i,j})$ is ∞ if $i = j$ or is $2k-2 \neq k$ if $i \neq j$ since $k \geq 3$. Thus T is nonsingular. Note that this does not mean that T is bijective, just that $T(X) = O$ only for $X = O$.

Now, suppose that $\#(T(D_{i,j})) \geq 2$ for some $i \leq j$. Again, without loss of generality we may assume that $1 \leq i \leq j \leq 2$. But then, there is a base element F such that $T((J_k \oplus O) \setminus F) = T(J_k \oplus O)$, again a contradiction. Thus, T maps base elements to base elements. Suppose that E and F are base elements, $E \neq F$, and $T(E) = T(F)$. If for some permutation matrix P, $(E + F) \sqsubseteq P^t(J_k \oplus O)P$, then

$$\begin{aligned} T(P^t(J_k \oplus O)P) &= T((P^t(J_k \oplus O)P \setminus E) + E) \\ &= T(P^t(J_k \oplus O)P \setminus E) + T(E) \\ &= T(P^t(J_k \oplus O)P \setminus E) + T(F) \\ &= T((P^t(J_k \oplus O)P \setminus E) + F) \\ &= T(P^t(J_k \oplus O)P \setminus E), \end{aligned}$$

a contradiction since $\rho_{\mathbf{sa}}(P^t(J_k \oplus O)P \setminus E) \neq k$ while $\rho_{\mathbf{sa}}(P^t(J_k \oplus O)P) = k$. Note that if $k \geq 4$, then it is always possible to find such a permutation matrix P.

Thus, if there is no permutation matrix P such that $P(E+F)P^t \sqsubseteq J_k \oplus O$, then $k \leq 3$. However, there is a permutation P such that $E \sqsubseteq P^t(J_k \oplus O)P$. In this case, as above we have that

$$\begin{aligned} T(P^t(J_k \oplus O)P) &= T((P^t(J_k \oplus O)P \setminus E) + E) \\ &= T(P^t(J_k \oplus O)P \setminus E) + T(E) \\ &= T(P^t(J_k \oplus O)P \setminus E) + T(F) \\ &= T((P^t(J_k \oplus O)P \setminus E) + F), \end{aligned}$$

which is a contradiction since $\rho_{\mathbf{sa}}(P^t(J_k \oplus O)P) = k$ while $(P^t(J_k \oplus O)P \setminus E) + F \notin \mathcal{F}_{sym}$ if F is a digon or E is a diagonal cell, and is in SA_{2k+1} if E is a digon and F is a diagonal cell, so that $\rho_{\mathbf{sa}}((P^t(J_k \oplus O)P \setminus E) + F) \neq k$.

We now have that T is injective on the set of base elements. Since $\mathcal{S}_n(\mathbb{B})$ is finite, T is bijective on the set of base elements. $\square$

Corollary 4.2. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. If T strongly preserves SA_k for some $3 \leq k \leq n$, then T is bijective on $\mathcal{S}_n(\mathbb{B})$.*

Proof. By the above theorem, T is bijective on the set of base elements. Every member of $\mathcal{S}_n(\mathbb{B})$ is a unique sum of base elements, so that T is bijective on $\mathcal{S}_n(\mathbb{B})$. □

Theorem 4.3. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then T strongly preserves SA_k for some $4 \leq k \leq n-1$ if and only if T is a (P, P^t)-operator.*

Proof. Suppose that T strongly preserves SA_k. By Corollary 4.2 we have that T is bijective and by Lemma 3.4 that T preserves SA_1. Now by Theorem 2.3, there exist scalars $\alpha_{i,j}, \beta_{i,j} \in \mathbb{B}$ for $1 \leq i < j \leq n$ and a permutation matrix P such that

$$T(X) = P\left(X + \sum_{1\leq i<j\leq n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$$

for all $X \in \mathcal{S}_n(\mathbb{B})$. Suppose that $\alpha_{i,j} \neq 0$ for some i, j. Without loss of generality we may assume that $(i, j) = (1, 2)$. Then $\rho_{\mathbf{sa}}(D_{1,2} + (O_1 \oplus I_{k-1} \oplus O_{n-k})) = \infty$ but $T(D_{1,2} + (O_1 \oplus I_{k-1} \oplus O_{n-k})) = P(D_{1,2} + (I_k \oplus O_{n-k}))P^t$ which has symmetric arctic rank k, a contradiction. Thus, $\alpha_{i,j} = \beta_{i,j} = 0$ for all i, j. That is T is a (P, P^t)-operator. Since every (P, P^t)-operator preserves all symmetric arctic ranks, the converse follows. □

5. Summary

We summarize the above results below.

Theorem 5.1. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then the following are equivalent*:

(1) *T preserves SA_1 and SA_k for some $4 \leq k \leq n-1$;*
(2) *T preserves SA_2 and SA_k for some $4 \leq k \leq n-1$;*
(3) *there exist scalars $\alpha_{i,j}, \beta_{i,j} \in \mathbb{B}$ for $1 \leq i < j \leq n$ and a permutation matrix P such that*

$$T(X) = P\left(X + \sum_{1\leq i<j\leq n} x_{i,j}(\alpha_{i,j}E_{i,i} + \beta_{i,j}E_{j,j})\right)P^t$$

for all $X \in \mathcal{S}_n(\mathbb{B})$.

Proof. (1) implies (3) by Theorem 2.3, and (2) implies (3) by Theorem 2.4, To show (3) implies (1) and (2), one observes that such a T defined in (3) preserves all symmetric arctic ranks except ∞. □

Theorem 5.2. *Let $T : \mathcal{S}_n(\mathbb{B}) \to \mathcal{S}_n(\mathbb{B})$ be a linear operator. Then the following are equivalent*:

(1) *T is bijective and preserves SA_k for some $2 \leq k \leq n$;*

(2) T *strongly preserves* SA_k *for some* $4 \leq k \leq n-1$;
(3) T *is a* (P, P^t)*-operator.*

Proof. (1) implies (3) by Theorem 3.7, Theorem 3.9, Corollary 3.5, and Theorem 3.11; (2) implies (3) by Theorem 4.3; and to show (3) implies (1) and (2) one observes that any (P, P^t)-operator is bijective and strongly preserves all symmetric arctic ranks. □

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LeRoy B. Beasley
Department of Mathematics and Statistics
Utah State University
Logan, Utah 84322-3900, USA
E-mail address: leroy.b.beasley@aggiemail.usu.edu

Seok-Zun Song
Department of Mathematics
Jeju National University
Jeju 63243, Korea
E-mail address: szsong@jejunu.ac.kr