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HAUSDORFF DIMENSION OF THE SET CONCERNING WITH BOREL-BERNSTEIN THEORY IN LÜROTH EXPANSIONS

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Abstract. It is well known that every $x \in (0,1]$ can be expanded to an infinite Lüroth series with the form of

$$x = \frac{1}{d_1(x)} + \cdots + \frac{1}{d_1(x)(d_1(x)-1)\cdots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)} + \cdots,$$

where $d_n(x) \geq 2$ for all $n \geq 1$. In this paper, the set of points with some restrictions on the digits in Lüroth series expansions are considered. Namely, the Hausdorff dimension of following the set

$$F_\phi = \{x \in (0,1] : d_n(x) \geq \phi(n),\ \text{i. o. } n\}$$

is determined, where ϕ is an integer-valued function defined on $\mathbb{N}$, and $\phi(n) \to \infty$ as $n \to \infty$.

1. Introduction

For each $x \in (0,1]$, let $d_1 = d_1(x) \in \mathbb{N}$ be the unique integer such that

$$\frac{1}{d_1(x)} < x \leq \frac{1}{d_1(x)-1}, \tag{1}$$

and the transformation $T : (0,1] \to (0,1]$ be defined as

$$T(x) := d_1(x)(d_1(x)-1)\Big(x - \frac{1}{d_1(x)}\Big). \tag{2}$$

Then the algorithm of (2) leads to an infinite series expansion for every $x \in (0,1]$ with the form

$$x = \frac{1}{d_1(x)} + \sum_{n\geq 2} \frac{1}{d_1(x)(d_1(x)-1)\cdots d_{n-1}(x)(d_{n-1}(x)-1)d_n(x)}, \tag{3}$$

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where $d_n(x) = d_1(T^{n-1}(x)) \geq 2 (\forall n \geq 1)$ are called the digits of x. The infinite series (3) is named the Lüroth expansion of x, which was first introduced by J. Lüroth in 1883 ([11]). Lüroth series expansion played an important role in the representation theory of numbers, probability theory, and dynamical system. For metrical properties, the digits $\{d_n, n \geq 1\}$ are stochastically independent but with infinite mean ([7], p. 66). For dynamical properties, the transformation T is invariant and ergodic with respect to Lebesgue measure ([2,7,9,13]). And for more researches related to Lüroth expansion, we can refer to [3], [7], [12] and [15]. For the exceptional sets in Lüroth expansions, the earliest research was conducted by T. Šalát in [12], the author obtained the Hausdorff dimension of the sets $M_k = \{x \in (0,1] : d_n(x) = k, n = 1, 2, \dots\}$ for any $k \in \mathbb{N}$, and by the conformal system theory, K. J. Falconer ([5]) obtained the Huasdorff dimension for general case of the above sets, i.e., the set $J_A = \{x \in (0,1] : d_n(x) \in A \text{ for all } n \geq 1\}$, where $A \subset \mathbb{N}\backslash\{1\}$. In recent years, great importance was attached with the Lüroth expansions. For given probability sequence $\vec{p} = (p_1, p_2, \dots)$, i.e., $p_j \geq 0$ for all $j \in \mathbb{N}$ and $\sum_{j=1}^{\infty} p_j = 1$, A. H. Fan et al. ([6]) obtained the dimension of the Besicovitch-Eggleston set

$$E(\vec{p}) := \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} 1_{\{d_1(x)=j+1\}}(T^k(x)) = p_j \text{ for all } \ j \geq 1 \right\}.$$

L. Barreira and G. Iommi ([1]) computed the Hausdorff dimension of the set of the frequency of digits $F_\alpha = \{x \in (0,1] : \tau_k(x) = \alpha_k \text{ for each k} \in \mathbb{N}\}$, where $\tau_k(x,n) = card\{i \in \{1, \dots, n\} : d_i(x) = k\}$, $\tau_k(x) = \lim\limits_{n\to\infty} \frac{\tau_k(x,n)}{n}$, and $\alpha = (\alpha_1, \dots, \alpha_2, \dots), \Sigma_{i=1}^{\infty} \alpha_i = 1$. L. M. Shen and K. Fang [14] considered the set

$$F_\varphi = \{x \in (0,1] : d_n(x) \geq \varphi(n) \text{ for all } n \geq 1\},$$

where $\varphi : \mathbb{N} \to \mathbb{N}$ is a function satisfying $\varphi \to \infty$ as $n \to \infty$, in which not only the set whose digits assume arbitrary is computed, but also the set $F(a,b) = \{x \in (0,1] : d_n(x) \geq e^{a^b} \text{ i. o. } n\}$ is determined.

Theorem 1.1 (Borel-Bernstein). *Let ϕ be an arbitrary positive function on natural numbers $\mathbb{N}$ and $F(\phi) = \{x \in (0,1] : d_n(x) \geq \phi(n) \text{ i. o. } n\}$, then $\mathcal{L}(F(\phi))$ is null or full according to the series $\sum_{n=1}^{\infty} \frac{1}{\phi(n)}$ converges or not, where $\mathcal{L}$ denotes the Lebesgue measure.*

In light of Borel-Bernstein Theorem, the sets whose digits obey some restrictions would be of null Lebesgue measure. Let ϕ be a positive function defined on natural numbers. In this paper, the Hausdorff dimension of the set

$$F_\phi = \{x \in (0,1] : d_n(x) \geq \phi(n), \text{ i. o. } n\}$$

is determined, which strengthens the result of [14].

The paper is organized as follows. In Section 2, we present some elementary properties of Lüroth expansion which will be used later. Section 3 is devoted to determining the Hausdorff dimension of the set $F(B) = \{x \in [0,1) : d_n(x) \geq$

$B^n + 1,$ i. o. $n\}$ $(B > 1)$, and the Hausdorff dimension of the set $F(\phi)$ is determined in Section 4.

Throughout this paper, we denote I the interval $(0, 1]$, $|\cdot|$ the diameter of a set, H^t the Hausdorff measure, $\dim_{\mathrm{H}}$ the Hausdorff dimension and 'cl' the closure of a subset of I respectively.

It should be mentioned that the proof idea is derived from [17].

2. Preliminaries

In this section, we will present some elementary properties which are enjoyed by Lüroth expansion and present some lemmas that will be used later.

Proposition 2.1 ([7], p. 18)**.** *The series on the right hand side of* (3) *is the expansion of its sum by the algorithm* (1) *and* (2) *if and only if*

$$d_n \geq 2 \text{ for all } n \geq 1.$$

For any $d_1, \ldots, d_n \in \mathbb{N}$ with $d_k \geq 2$ for all $1 \leq k \leq n$, and we call

$$I(d_1, \ldots, d_n) = \{x \in I : d_k(x) = d_k, 1 \leq k \leq n\}$$

an n-th order basic cylinder.

Proposition 2.2 ([7], p. 67)**.** *For any $d_1, \ldots, d_n \in \mathbb{N}$ with $d_k \geq 2$ $(1 \leq k \leq n)$, the n-th order basic interval $I(d_1, \ldots, d_n)$ is the interval with the endpoints*

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n},$$

and

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n} + \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}.$$

As a consequence,

$$|I(d_1, \ldots, d_n)| = \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}. \tag{4}$$

The following lemma is essential to the determination of the Hausdorff dimension of the set $\{x \in I : d_n(x) \geq B^n + 1,$ i. o. $n\}$.

Lemma 2.3. *Let $B > 1$ and $s(B)$ be the unique solution of the equation*

$$\sum_{k=2}^{\infty} \Big(\frac{1}{Bk(k-1)}\Big)^s = 1. \tag{5}$$

Then 1) *$s(B)$ is continuous with respect to B;* 2) $\lim_{B\to+\infty} s(B) = \frac{1}{2}$; 3) $\lim_{B\to 1} s(B) = 1$.

Proof. 1) By the definition of $s(B)$, for any $0<\epsilon<1$ and $1<B^{'}<B<B^{'}+\epsilon$, we have

$$\sum_{k=2}^{\infty}\big(\frac{1}{B'k(k-1)}\big)^{s(B)+\epsilon}\le\frac{1}{B'^{\epsilon}}\sum_{k=2}^{\infty}\big(\frac{1}{B'k(k-1)}\big)^{s(B)}=\frac{1}{B'^{\epsilon}}\big(\frac{B}{B'}\big)^{s(B)}$$
$$\le\frac{B}{B'^{(1+\epsilon)}}<1.$$

By the monotonicity of $s(\cdot)$, one has $s(B)<s(B^{'})<s(B)+\epsilon$.

2) For any $n\ge 1$ and $B>1$,

$$\sum_{k=2}^{\infty}\big(\frac{1}{Bk(k-1)}\big)^{\frac{1}{2}}\ge\frac{1}{B^{\frac{1}{2}}}\sum_{k=2}^{\infty}\frac{1}{k}=\infty.$$

Thus $s(B)\ge\frac{1}{2}$.

For any $0<\epsilon<1$, take $B_0=\big(\sum_{k=2}^{\infty}(\frac{1}{k-1})^{2\epsilon+1}\big)^{\frac{1}{\frac{1}{2}+\epsilon}}$, then for any $B>B_0$, one has

$$\sum_{k=2}^{\infty}\big(\frac{1}{k(k-1)}\big)^{\frac{1}{2}+\epsilon}\le\sum_{k=2}^{\infty}\big(\frac{1}{k-1}\big)^{2\epsilon+1}=B_0^{\frac{1}{2}+\epsilon}<B^{\frac{1}{2}+\epsilon}.$$

Thus $s(B)<\frac{1}{2}+\epsilon$.

3) For any $0<\epsilon<1$ and any $B>1$ with $1<B<B_0=2^{\epsilon}$, one has

$$\sum_{k=2}^{\infty}\big(\frac{1}{Bk(k-1)}\big)^{1-\epsilon}\ge 2^{\epsilon}\sum_{k=2}^{\infty}\frac{1}{Bk(k-1)}>1,$$

which implies $s(B)>1-\epsilon$. On the other hand, it is obviously that $s(B)\le 1$. □

3. The Hausdorff dimension of $\{x\in I: d_n(x)\ge B^n+1,\ \text{i. o. } n\}$.

In this section, we will determine the Hausdorff dimension of

$$F(B)=\{x\in I: d_n(x)\ge B^n+1,\text{i. o. } n\},$$

which is the main part of this paper.

Before giving the lower bound estimation of $F(B)$, we state the mass distribution principle, see ([7], Proposition 4.2).

Lemma 3.1. *Let $E\subset I$ be a Borel set and μ be a measure with $\mu(E)>0$. If for any $x\in E$,*

$$\liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}\ge s,$$

where $B(x,r)$ denotes the open ball of center x and radius r, then $\dim_H E\ge s$.

Theorem 3.2. *For any $B>1$, we have $\dim_H F(B)=s(B)$.*

Firstly, we give the upper bound of $F(B)$.

$$\begin{aligned}
F(B) &= \limsup_{n\to\infty}\{x \in [0,1) : d_{n+1}(x) \geq B^{n+1}+1\} \\
&= \bigcap_{N\geq 1}\bigcup_{n\geq N}\{x \in [0,1) : d_{n+1}(x) \geq B^{n+1}+1\} \\
&= \bigcap_{N\geq 1}\bigcup_{n\geq N}\bigcup_{d_1,\ldots,d_n}\{x \in [0,1) : d_k(x)=d_k, 1\leq k\leq n, d_{n+1}(x) \geq B^{n+1}+1\} \\
&:= \bigcap_{N\geq 1}\bigcup_{n\geq N}\bigcup_{d_1,\ldots,d_n}|J(d_1,\ldots,d_n)|.
\end{aligned}$$

From (4), we have

$$|J(d_1,\ldots,d_n)| = \sum_{d_{n+1}\geq B^{n+1}+1}|I(d_1,\ldots,d_n,d_{n+1})| = \frac{1}{Q_nB^{n+1}},$$

where $Q_n = d_1(d_1-1)\cdots d_n(d_n-1)$.

For any $\epsilon > 0$,

$$\begin{aligned}
H^{s(B)+\epsilon}(F(B)) &\leq \liminf_{N\to\infty}\sum_{n\geq N}\sum_{d_1,\ldots,d_n}|J(d_1,\ldots,d_n)|^{s(B)+\epsilon} \\
&= \liminf_{N\to\infty}\sum_{n\geq N}\sum_{d_1,\ldots,d_n}(\frac{1}{Q_nB^{n+1}})^{s(B)+\epsilon} \\
&= \liminf_{N\to\infty}\sum_{n\geq N}\sum_{d_1,\ldots,d_n}(\frac{1}{Q_nB^{n+1}})^{s(B)}(\frac{1}{Q_nB^{n+1}})^{\epsilon} \\
&\leq \liminf_{N\to\infty}\sum_{n\geq N}\sum_{d_1,\ldots,d_n}(\frac{1}{Q_nB^{n}})^{s(B)}(\frac{1}{2^nB^{n+1}})^{\epsilon} \\
&= \liminf_{N\to\infty}\sum_{n\geq N}(\frac{1}{2^nB^{n+1}})^{\epsilon} = 0.
\end{aligned}$$

This implies that $\dim_H F(B) \leq s(B)+\epsilon$. By the arbitrary of ϵ, we have $\dim_H F(B) \leq s(B)$.

In order to get the lower bound of $\dim_H F(B)$, we will construct a subset $F_\alpha(B) \subset F(B)$ and use the Hausdorff dimension of $F_\alpha(B)$ to approximate that of $F(B)$.

3.1. Structure of $F_\alpha(B)$

Let $\alpha \in \mathbb{N}$ and a sequence $\Lambda = \{n_1, n_2, \ldots\}$ satisfying $n_k \in \mathbb{N}$

$$n_1 = 1 \text{ and } n_1 + n_2 + \cdots + n_k \leq \frac{1}{k+1}n_{k+1} \tag{6}$$

for all $k \geq 1$. For $\forall\alpha \geq 2$, we define

$$\begin{aligned}
F_\alpha(B) = \{x \in I : [B^{n_k}]+2 \leq d_{n_k}(x) \leq 2[B^{n_k}]+1, \\
\forall k\geq 1, \text{and } 2\leq d_j(x)\leq\alpha, \forall j\neq n_k\},
\end{aligned}$$

and define $s_\alpha(B)$ be the unique solution of the equation

$$\sum_{k=2}^{\alpha}\Big(\frac{1}{Bk(k-1)}\Big)^s = 1.$$

Remark 3.3. By the definition of $s(B)$ and $s_\alpha(B)$, we have

$$\lim_{\alpha\to\infty} s_\alpha(B) = s(B). \tag{7}$$

In order to illustrate the structure of $F_\alpha(B)$, we will use the following symbolic space. Let

$$D_n = \{(\sigma_1,\dots,\sigma_n)\in\mathbb{N}^n : [B^{n_k}]+2 \le \sigma_{n_k} \le 2[B^{n_k}]+1$$
$$\forall k\ge 1, \text{and } 2\le\sigma_j\le\alpha, \forall j\neq n_k\}.$$

$$D = \bigcup_{n=0}^{\infty} D_n \ (D_0 := \phi).$$

For any $n \ge 1$ and $(\sigma_1,\dots,\sigma_n) \in D_n$, we call $I_n(\sigma_1,\dots,\sigma_n)$ a *n-th basic interval* and

$$J((\sigma_1,\dots,\sigma_n)) = \bigcup_{\sigma_{n+1}} cl((\sigma_1,\dots,\sigma_{n+1})) \tag{8}$$

an *n-th fundamental interval*, where the union in (8) is taken over all σ_{n+1} such that $(\sigma_1,\dots,\sigma_n,\sigma_{n+1}) \in D_{n+1}$. Then, if $n+1 \neq n_k$ for some $k \ge 1$,

$$|J((\sigma_1,\dots,\sigma_n))| = \frac{1}{Q_n}(1-\frac{1}{\alpha}), \tag{9}$$

if $n+1 = n_k$ for some $k \ge 1$,

$$|J(\sigma_1,\dots,\sigma_n)| = \frac{1}{Q_n}\frac{[B^n]}{([B^n]+1)(2[B^n]+1)}. \tag{10}$$

It is clear that

$$F_\alpha(B) = \bigcap_{n\ge 1}\ \bigcup_{(\sigma_1,\dots,\sigma_n)\in D_n} J(\sigma_1,\dots,\sigma_n). \tag{11}$$

3.2. Gaps in $F_\alpha(B)$

In this subsection, we estimate the gaps between the adjoint fundamental of the same order. Given a fundamental interval $J(d_1,\dots,d_n)$, the distance between $J(d_1,\dots,d_n)$ and fundamental intervals of the same order which lies on the right of $J(d_1,\dots,d_n)$ is denoted by $g^r(d_1,\dots,d_n)$, the distance between $J(d_1,\dots,d_n)$ and fundamental intervals of the same order which lies on the left of $J(d_1,\dots,d_n)$ is denoted by $g^l(d_1,\dots,d_n)$.

Also, let $J_n^{(l)}(d_1^{(l)},\dots,d_n^{(l)})$ and $J_n^{(r)}(d_1^{(r)},\dots,d_n^{(r)})$ be two fundamental intervals (if exist) lying the left most and right most to $J_n(d_1,\dots,d_n)$. Without loss of generality, we can assume that these three fundamental intervals are lying in the same interval $I_{n-1}(d_1,\dots,d_{n-1})$, otherwise, we consider the gaps between their mother fundamental intervals. Then, $J_n^{(l)}$ and $J_n^{(r)}$ are just

$$J_n^{(l)} = J(d_1,\dots,d_{n-1},d_n+1), J_n^{(r)} = J(d_1,\dots,d_{n-1},d_n-1).$$

Let $G(d_1,\dots,d_n) = \min\{g^r(d_1,\dots,d_n), g^l(d_1,\dots,d_n)\}$.

Since we care only about the gaps between fundamental intervals, all the digits blocks $(d_1,\dots,d_n)$ appearing below will be assumed in the symbolic space D without special announcement.

Case I. $n+1 \neq n_k$ for any $k \geq 1$.

(i) $I(d_1,\dots,d_n)$ is the rightmost subinterval of $I(d_1,\dots,d_{n-1})$. Then $d_n = [B^n]+2$ if $n = n_k$ for some $k \geq 1$ and $d_n = 2$ if $n \neq n_k$ for any $k \geq 1$. So $g^l(d_1,\dots,d_n)$ is just the distance between the left endpoint of $J(d_1,\dots,d_n)$ and the left endpoint of $I(d_1,\dots,d_n)$. Hence,

$$\begin{aligned} &g^l(d_1,\dots,d_n)\\ &= \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1}\frac{1}{d_k(d_k-1)}\frac{1}{d_n} + \prod_{k=1}^{n}\frac{1}{d_k(d_k-1)}\frac{1}{\alpha}\\ &\quad - \Big(\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1}\frac{1}{d_k(d_k-1)}\frac{1}{d_n}\Big)\\ &= \prod_{k=1}^{n}\frac{1}{d_k(d_k-1)}\frac{1}{\alpha}. \end{aligned}$$

$g^r(d_1,\dots,d_n)$ is larger than the distance between the right endpoint of $I(d_1,\dots,d_n)$ and the right endpoint of $I(d_1,\dots,d_{n-1})$. Hence

$$\begin{aligned} &g^r(d_1,\dots,d_n)\\ &\geq \frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-2}\frac{1}{d_k(d_k-1)}\frac{1}{d_{n-1}-1}\\ &\quad - \Big(\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-2}\frac{1}{d_k(d_k-1)}\frac{1}{d_{n-1}} + \prod_{k=1}^{n}\frac{1}{d_k(d_k-1)}\frac{1}{d_n-1}\Big)\\ &= \prod_{k=1}^{n-1}\frac{1}{d_k(d_k-1)}\Big(1-\frac{1}{d_n-1}\Big). \end{aligned}$$

Thus

$$G(d_1,\dots,d_n) = \prod_{k=1}^{n}\frac{1}{d_k(d_k-1)}\frac{1}{\alpha}. \tag{12}$$

(ii) $I(d_1,\dots,d_n)$ lies on the leftmost of all basic interval of order n contained in $I(d_1,\dots,d_{n-1})$. In this case, $d_n = 2[B^{n_k}]+1$ if for some $n = n_k$ and $d_n = \alpha$

if $n \neq n_k$ for any $k \geq 1$. $g^r(d_1, \dots, d_n)$ is just the distance between the right endpoint of $J(d_1, \dots, d_n)$ and the left endpoint of $J(d_1, \dots, d_n - 1)$. Hence

$$g^r(d_1, \dots, d_n) = \frac{1}{d_1(d_1-1)\cdots d_{n-1}(d_{n-1}-1)(d_n-1)(d_n-2)\alpha}.$$

$g^l(d_1, \dots, d_n)$ is just the distance between the left endpoint of $I(d_1, \dots, d_n)$ and the left endpoint of $J(d_1, \dots, d_n)$, thus

$$g^l(d_1, \dots, d_n) = \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} \frac{1}{\alpha}.$$

Hence

$$G(d_1, \dots, d_n) = \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} \frac{1}{\alpha}. \tag{13}$$

(iii) If $I(d_1, \dots, d_n - 1), I(d_1, \dots, d_n), I(d_1, \dots, d_n + 1)$ are contained in $I(d_1, \dots, d_{n-1})$. In this case, $g^l(d_1, \dots, d_n)$ is just the distance between the right endpoint of $J(d_1, \dots, d_n + 1)$ and the left endpoint of $J(d_1, \dots, d_n)$. $g^r(d_1, \dots, d_n)$ is just the distance between the left endpoint of $J(d_1, \dots, d_n)$ and the right endpoint of $J(d_1, \dots, d_n)$.

Thus

$$g^l(d_1, \dots, d_n) = \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} (\frac{1}{\alpha} + \frac{d_n - 1}{2(d_n+1)}),$$
$$g^r(d_1, \dots, d_n) = \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} (\frac{1}{2} + \frac{d_n}{(d_n-2)\alpha}).$$

Thus

$$G(d_1, \dots, d_n) = \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} (\frac{1}{\alpha} + \frac{d_n - 1}{2(d_n+1)}). \tag{14}$$

Case II. $n + 1 = n_k$ for some $k \geq 1$.

In this case, $g^l(d_1, \dots, d_n)$ is larger than the distance between the left endpoint of $I(d_1, \dots, d_n)$ and the left endpoint of $J(d_1, \dots, d_n)$. $g^r(d_1, \dots, d_n)$ is larger than the distance between the right endpoint of $I(d_1, \dots, d_n - 1)$ and the right endpoint of $J(d_1, \dots, d_n)$. Thus

$$g^l(d_1, \dots, d_n) \geq \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} \frac{1}{2[B^n]+1},$$
$$g^r(d_1, \dots, d_n) \geq \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} (1 - \frac{1}{[B^n]+1}).$$

Hence

$$G(d_1, \dots, d_n) \geq \frac{1}{d_1(d_1-1)\cdots d_n(d_n-1)} \frac{1}{2[B^n]+1}. \tag{15}$$

By (9) and (12)-(14), we have in case I,

$$G(d_1, \dots, d_n) \geq \frac{1}{\alpha - 1} |J(d_1, \dots, d_n)|. \tag{16}$$

And for case II, by (10) and (15), we have

$$G(d_1,\ldots,d_n) \geq \frac{1}{2}|J(d_1,\ldots,d_n)|. \tag{17}$$

3.3. A measure supported on $F_\alpha(B)$

In order to estimate the Hausdorff dimension of the lower bound of $F_\alpha(B)$, we define a probability on this set.

Let $m_1 = n_1$ and $m_k = n_k - n_{k-1} - 1$ for any $k \geq 2$. Then a set function $\mu : \{J(\sigma), \sigma \in D \setminus D_0\} \to \mathbb{R}^+$ is defined as follows.

Write $Q_k(\sigma_1,\ldots,\sigma_k) = \prod_{j=1}^k \sigma_j(\sigma_j - 1)$.

For any $\sigma_1 \in D_1$, let

$$\mu(J(\sigma_1)) = \frac{1}{[B]}.$$

For any $(\sigma_1,\ldots,\sigma_{n_2-1}) \in D_{n_2-1}$, let

$$\mu(J(\sigma_1,\ldots,\sigma_{n_2-1})) = \mu(J(\sigma_1))\Big(\frac{1}{B^{m_2}Q_{m_2}(\sigma_2,\ldots,\sigma_{n_2-1})}\Big)^{s_\alpha(B)}$$

and for any $(\sigma_1,\ldots,\sigma_{n_2}) \in D_{n_2}$, let

$$\mu(J(\sigma_1,\ldots,\sigma_{n_2})) = \frac{1}{[B^{n_2}]}\mu(J(\sigma_1,\ldots,\sigma_{n_2-1})).$$

For any $1 < n < n_2 - 1$ and $(\sigma_1,\ldots,\sigma_n) \in D_n$, let

$$\mu(J(\sigma_1,\ldots,\sigma_n)) = \sum_{2\leq\sigma_{n+2},\ldots,\sigma_{n_2-1}\leq\alpha} \mu(J(\sigma_1,\ldots,\sigma_n,\sigma_{n+1},\sigma_{n_2-1})).$$

Suppose for some $k \geq 2$, $\mu(J(\sigma_1,\ldots,\sigma_{n_k-1}))$ has been defined for any $(\sigma_1,\ldots,\sigma_{n_k-1}) \in D_{n_k-1}$, let

$$\mu(J(\sigma_1,\ldots,\sigma_{n_k})) = \frac{1}{[B^{n_k}]}\mu(J(\sigma_1,\ldots,\sigma_{n_k-1})).$$

For any $n_{k-1} < n < n_k - 1$ and $(\sigma_1,\ldots,\sigma_n) \in D_n$, let

$$\mu(J(\sigma_1,\ldots,\sigma_n)) = \sum_{2\leq\sigma_{n+2},\ldots,\sigma_{n_k-1}\leq\alpha} \mu(J(\sigma_1,\ldots,\sigma_n,\sigma_{n+1},\ldots,\sigma_{n_k-1})),$$

and let

$$\begin{aligned}&\mu(J(\sigma_1,\ldots,\sigma_{n_{k+1}-1}))\\ &= \mu(J(\sigma_1,\ldots,\sigma_{n_{k+1}-1}))\Big(\frac{1}{B^{m_{k+1}}Q_{m_{k+1}}(\sigma_{n_k+1},\ldots,\sigma_{n_{k+1}-1})}\Big)^{s_\alpha(B)}.\end{aligned}$$

It is easy to check that for any $n \geq 1$ and $(\sigma_1,\ldots,\sigma_n) \in D_n$, we have

$$\mu(J(\sigma_1,\ldots,\sigma_n)) = \sum_{\sigma_{n+1}} \mu(J(\sigma_1,\ldots,\sigma_n,\sigma_{n+1})),$$

where the summation is taken over all σ_{n+1} such that $(\sigma_1, \ldots, \sigma_n, \sigma_{n+1}) \in D_{n+1}$. Notice that

$$\sum_{\sigma_1 \in D_1} \mu(J(\sigma_1)) = 1,$$

by Kolmogorov extension theorem, the set function μ can be extended to a probability measure supported on $F_\alpha(B)$, which is still denoted by μ. From the definition of μ, we have for any $k \geq 1$ and $(\sigma_1, \ldots, \sigma_{n_k}) \in D_{\sigma_{n_k}}$

$$\begin{aligned}\mu(J(\sigma_1, \ldots, \sigma_{n_k})) &= \frac{1}{[B^{n_k}]}\mu(J(\sigma_1, \ldots, \sigma_{n_k-1})) \\ &= \mu(J(\sigma_1, \ldots, \sigma_{n_{k-1}}))\frac{1}{[B^{n_k}]}\Big(\frac{1}{B^{m_k}Q_{m_k}}\Big)^{s_\alpha(B)} \\ &= \prod_{j=1}^{k}\frac{1}{[B^{n_j}]}\Big(\frac{1}{B^{m_j}Q_{m_j}}\Big)^{s_\alpha(B)}. \qquad (18)\end{aligned}$$

3.4. Estimation of $\mu(J(\sigma_1, \ldots, \sigma_n))$

Fix $0 < t < s_\alpha(B)$, take $\tau = \frac{s_B(\alpha)-t}{2}, \tau' = \frac{\log 2}{\log \alpha(\alpha-1)B}\tau$. Choose N_0 and k_0 large enough such that

$$m_k \geq N_0,\ \frac{m_k}{n_k} \geq \frac{\log 2}{(t+\tau)n_k \log B} + \frac{t-\tau}{(t+\tau)\log B}, \forall k > k_0, \qquad (19)$$

$$m_k \geq N_0,\ \frac{m_k}{n_k} \geq \frac{t\log B}{(t+\tau)\log B + \log 2\tau}, \forall k > k_0, \qquad (20)$$

$$2^{3k}(1-\frac{1}{\alpha})^\tau \leq 2^{n_{k-1}\tau},\ \forall k > k_0. \qquad (21)$$

Take $c_0 = [\alpha(\alpha-1)]^{n_{k_0}-k_0}B^{2(n_1+\cdots+n_{k_0})}$.

For any $n > n_{k_0}$ and $(\sigma_1, \ldots, \sigma_n) \in D_n$, we estimate $\mu(J(\sigma_1, \ldots, \sigma_n))$.

Case I. $n = n_k$ for some $k \geq k_0$.

$$\begin{aligned}&\mu(J(\sigma_1, \ldots, \sigma_{n_k})) \\ &= \prod_{j=1}^{k}\frac{1}{[B^{n_j}]}\Big(\frac{1}{Q_{m_j}B^{m_j}}\Big)^{s_\alpha(B)} \text{ (by (18))} \\ &= \prod_{j=1}^{k_0}\frac{1}{[B^{n_j}]}\Big(\frac{1}{Q_{m_j}B^{m_j}}\Big)^{s_\alpha(B)} \prod_{j=k_0+1}^{k}\frac{1}{[B^{n_j}]}\Big(\frac{1}{Q_{m_j}B^{m_j}}\Big)^{s_\alpha(B)} \\ &\leq c_0\prod_{j=1}^{k_0}\Big(\frac{1}{Q_{m_j}B^{2n_j}}\Big)^{t} \prod_{j=k_0+1}^{k}\frac{1}{[B^{n_j}]}\Big(\frac{1}{Q_{m_j}B^{m_j}}\Big)^{t+\tau} \text{ (by (18))} \\ &\leq c_0\prod_{j=1}^{k_0}\Big(\frac{1}{Q_{m_j}B^{2n_j}}\Big)^{t} \prod_{j=k_0+1}^{k}2\Big(\frac{1}{Q_{m_j}B^{m_j+n_j}}\Big)^{t+\tau}\end{aligned}$$

$$\le c_0 \prod_{j=1}^{k_0} \big(\frac{1}{Q_{m_j} B^{2n_j}}\big)^t \prod_{j=k_0+1}^{k} \big(\frac{1}{Q_{m_j}}\big)^t \prod_{j=k_0+1}^{k} 2\big(\frac{1}{B^{m_j+n_j}}\big)^{t+\tau}$$

$$\le c_0 \prod_{j=1}^{k_0} \big(\frac{1}{Q_{m_j} B^{2n_j}}\big)^t \prod_{j=k_0+1}^{k} \big(\frac{1}{Q_{m_j}}\big)^t \prod_{j=k_0+1}^{k} \big(\frac{1}{B^{2n_j}}\big)^t \text{ (by (19)-(21))}$$

$$= c_0 \prod_{j=1}^{k} \big(\frac{1}{Q_{m_j} B^{2n_j}}\big)^t \le c_0 \frac{2^{3k}}{Q_{n_k}^t} \le c_0 \frac{2^{3k}}{Q_{n_k}^t}(1-\frac{1}{\alpha})^t$$

$$\le c_0 |J(\sigma_1, \ldots, \sigma_{n_k})|^{t-\tau} \text{ (by (9) and (21))} \tag{22}$$

Case II. $n = n_k - 1$ for some $k > k_0$.

Choose $\sigma_{n_k} \in \mathbb{N}$ such that $(\sigma_1, \ldots, \sigma_{n_k-1}, \sigma_{n_k}) \in D_{n_k}$. Similar to the proof of (22), we have

$$\mu(J(\sigma_1, \ldots, \sigma_{n_k-1})) = [B^{n_k}]\mu(J(\sigma_1, \ldots, \sigma_{n_k}))$$

$$= \prod_{j=1}^{k-1} \frac{1}{[B^{n_j}]} \big(\frac{1}{Q_{m_j} B^{m_j}}\big)^{s_\alpha(B)} \big(\frac{1}{Q_{m_k} B^{m_k}}\big)^{s_\alpha(B)}$$

$$\le c_0 \frac{2^{3(k-1)}}{Q_{n_{k-1}}^t} \big(\frac{1}{Q_{m_k} B^{m_k}}\big)^{t+\tau} \le c_0 \frac{2^{3(k-1)}}{Q_{n_k-1}^t} \big(\frac{1}{B^{n_k}}\big)^t \text{ (by (19))}$$

$$\le c_0 |J(\sigma_1, \ldots, \sigma_{n_k-1})|^{t-\tau}. \tag{23}$$

Case III. $n_{k-1} < n < n_k - 1$ for some $k > k_0$.

Let $l = n - n_{k-1}$, $l' = n_k - 1 - n$. Similar to the proof of (22), we have

$$\mu(J(\sigma_1, \ldots, \sigma_n)) = \sum_{\sigma_{n+1}, \ldots, \sigma_{n_k-1} \le \alpha} \mu(J(\sigma_1, \ldots, \sigma_n \sigma_{n+1}, \ldots, \sigma_{n_k-1}))$$

$$= \prod_{j=1}^{k-1} \frac{1}{[B^{n_j}]} \big(\frac{1}{Q_{m_j} B^{m_j}}\big)^{s_\alpha(B)} \sum_{\sigma_{n+1}, \ldots, \sigma_{n_k-1} \le \alpha} \big(\frac{1}{Q_{m_k} B^{m_k}}\big)^{s_\alpha(B)}$$

$$\le c_0 2^{3k} \frac{1}{Q_n^t} \sum_{2 \le d_1, \ldots, d_{l'} \le \alpha} \big(\frac{1}{Q_{l'} B^{m_k}}\big)^{s_\alpha(B)}.$$

Notice that

$$1 = \sum_{2 \le d_1, \ldots, d_{l'} \le \alpha} \big(\frac{1}{Q_{m_k} B^{m_k}}\big)^{s_\alpha(B)}$$

$$= \sum_{2 \le a_1, \ldots, a_{l'} \le \alpha} \big(\frac{1}{Q_{l'} B^{l'}}\big)^{s_\alpha(B)} \sum_{2 \le b_1, \ldots, b_l \le \alpha} \big(\frac{1}{Q_l B^l}\big)^{s_\alpha(B)}.$$

(i) If $l \le N_0$, then

$$1 = \sum_{2 \le d_1, \ldots, d_{l'} \le \alpha} \big(\frac{1}{Q_{m_k} B^{m_k}}\big)^{s_\alpha(B)}$$

$$= \sum_{2\le a_1,\dots,a_{l'}\le\alpha} \Big(\frac{1}{Q_{l'}B^{l'}}\Big)^{s_\alpha(B)} \sum_{2\le b_1,\dots,b_l\le\alpha} \Big(\frac{1}{Q_l B^l}\Big)^{s_\alpha(B)}$$
$$\ge \sum_{2\le a_1,\dots,a_{l'}\le\alpha} \Big(\frac{1}{Q_{l'}B^{l'}}\Big)^{s_\alpha(B)} \frac{1}{B^{N_0}(\alpha(\alpha-1))^{N_0}},$$

thus, by (9) and (22), we have

$$\begin{aligned}\mu(J(\sigma_1,\dots,\sigma_n)) &\le c_0 2^{3k} B^{N_0}(\alpha(\alpha-1))^{N_0}\frac{1}{Q_n^t}\\ &\le c_0|J(\sigma_1,\dots,\sigma_n)|^{t-\tau}. \qquad (24)\end{aligned}$$

(ii) If $l > N_0$. In this case, we have

$$\begin{aligned}\sum_{2\le b_1,\dots,b_l\le\alpha} \Big(\frac{1}{Q_l B^l}\Big)^{s_\alpha(B)} &\ge \sum_{2\le b_1,\dots,b_l\le\alpha} \Big(\frac{1}{Q_l B^l}\Big)^{s_\alpha(B)+\tau'}\\ &\ge \sum_{2\le b_1,\dots,b_l\le\alpha} \Big(\frac{1}{B^l\alpha(\alpha-1)}\Big)^{l\tau'} \frac{1}{B^l Q_l}^{s_\alpha(B)}\\ &= \Big(\frac{1}{B^l\alpha(\alpha-1)}\Big)^{l\tau'} \ge \frac{1}{2^{l\tau}}.\end{aligned}$$

Thus, by (9) and (22), we have

$$\begin{aligned}\mu(J(\sigma_1,\dots,\sigma_n)) &\le c_0 2^{3k} 2^{n\tau}\frac{1}{Q_n^t} \le c_0|J(\sigma_1,\dots,\sigma_n)|^{t-\tau}\\ &\le c_0|J(\sigma_1,\dots,\sigma_n)|^{t-2\tau}. \qquad (25)\end{aligned}$$

3.5. The estimation of $\mu(B(x,r))$

For any $n\ge 1$ and $(\sigma_1,\dots,\sigma_n)\in D_n$, if $n\ne n_k-1 (k\ge 1)$, define

$$G_1(\sigma_1,\dots,\sigma_n) := \frac{1}{Q_n\alpha} \ge \frac{1}{\alpha-1}|J(\sigma_1,\dots,\sigma_n)|, \qquad (26)$$

if $n = n_k - 1$ for some $k\ge 1$,

$$G_1(\sigma_1,\dots,\sigma_n) := \frac{1}{Q_n}\frac{1}{2[B^n]+1} \ge \frac{1}{2}|J(\sigma_1,\dots,\sigma_n)|. \qquad (27)$$

Take $r_0 = \min_{1\le j\le n_{k_0}} \min_{(\sigma_1,\dots,\sigma_j)\in D_j} G_1(\sigma_1,\dots,\sigma_j)$. Fix $x\in F_\alpha(B)$ and $0 < r < r_0$. There exists a unique sequence $\sigma_1,\sigma_2,\dots$ such that $x\in J(\sigma_1,\dots,\sigma_k)$ for all $k\ge 1$ and for some $n\ge n_{k_0}$,

$$G_1(\sigma_1,\dots,\sigma_n,\sigma_{n+1}) \le r < G_1(\sigma_1,\dots,\sigma_n).$$

From the definition of $G_1(\sigma_1,\dots,\sigma_n)$, we know that the ball $B(x,r)$ can intersect only one fundamental interval of order n, which is just $J(\sigma_1,\dots,\sigma_n)$.

Case I. $n = n_k - 1$ for some $k > k_0$.

(i) $r \le |I(\sigma_1, \dots, \sigma_{n_k})|$. In this case, the ball $B(x,r)$ can intersect at most four basic intervals of order n_k, which is $I(\sigma_1, \dots, \sigma_{n_k} - 1)$, $I(\sigma_1, \dots, \sigma_{n_k})$, $I(\sigma_1, \dots, \sigma_{n_k} + 1)$ and $I(\sigma_1, \dots, \sigma_{n_k} + 2)$. Thus, by (22) and (26),

$$\mu(B(x,r)) \le 4\mu(J(\sigma_1, \dots, \sigma_{n_k})) \le c_0 |J(\sigma_1, \dots, \sigma_{n_k})|^{t-\tau} \tag{28}$$
$$\le 4(\alpha-1)c_0|G_1(\sigma_1, \dots, \sigma_{n_k})|^{t-\tau} \le 4(\alpha-1)c_0 r^{t-\tau}.$$

(ii) $r > |I(\sigma_1, \dots, \sigma_{n_k})|$. In this case, since

$$|I(\sigma_1, \dots, \sigma_{n_k}| = \frac{1}{Q_{n_k-1}} \frac{[B^{n_k}]}{([B^{n_k}]+1)([2B^{n_k}]+1)} \ge \frac{1}{Q_{n_k-1}} \frac{1}{[B^{n_k}]^2},$$

we have that the number of fundamental intervals of order n_k contained in $J(\sigma_1, \dots, \sigma_{n_k} - 1)$ that the ball $B(x,r)$ intersects is at most $2rQ_{n_k-1}[B^{n_k}]^2 + 2 \le 4rQ_{n_k-1}[B^{n_k}]^2$. Therefore, by (10) and (23),

$$\begin{aligned} \mu(B(x,r)) &\le \mu(J(\sigma_1, \dots, \sigma_{n_k-1})) \min\{1, 4rQ_{n_k-1}[B^{n_k}]^2 \frac{1}{[B^{n_k}]}\} \\ &\le c_0 |J(\sigma_1, \dots, \sigma_{n_k})|^{t-\tau} \\ &\le c_0 |J(\sigma_1, \dots, \sigma_{n_k-1})|^{t-\tau} \min\{1, 4rQ_{n_k-1}[B^{n_k}]\} \\ &\le c_0 \big(\frac{1}{[B^{n_k}]Q_n}\big)^{t-\tau} 1^{1-t-\tau} (4rQ_{n_k-1}[B^{n_k}])^{t-\tau} \\ &\le 4c_0 r^{t-\tau}. \end{aligned} \tag{29}$$

Case II. $n = n_k - 2$ for some $k \ge 1$. For any $2 \le \xi \le \zeta \le \alpha$, we have

$$\frac{\mu(\sigma_1, \dots, \sigma_{n_k-2}, \xi)}{\mu(\sigma_1, \dots, \sigma_{n_k-2}, \tau)} = \frac{Q_{m_k}(\sigma_{n_{k-1}+1}, \dots, \sigma_{n_k-2}, \zeta)}{Q_{m_k}(\sigma_{n_{k-1}+1}, \dots, \sigma_{n_k-2}, \xi)} \le \frac{\alpha(\alpha-1)}{2}.$$

Thus by (24) and (25), we have

$$\begin{aligned} \mu(B(x,r)) &\le \frac{\alpha^2(\alpha-1)}{2} \mu(J(\sigma_1, \dots, \sigma_{n_k-1})) \\ &\le \frac{c_0\alpha^2(\alpha-1)}{2} |J(\sigma_1, \dots, \sigma_{n_k-1})|^{t-\tau} \\ &\le c_0\alpha^2(\alpha-1)|G_1(\sigma_1, \dots, \sigma_{n_k-1})|^{t-\tau} \le c_0\alpha^2(\alpha-1)r^{t-\tau}. \end{aligned} \tag{30}$$

Case III. $n \ne n_k - 1$ and $n \ne n_k - 2$ for any $k \ge 1$. Similar to case II, we have

$$\mu(B(x,r)) \le c_0\alpha^3(\alpha-1)r^{t-2\tau}. \tag{31}$$

From (28)-(31), and the distribution principle, we have

$$\dim_H F_\alpha(B) \ge t - 2\tau = 2t - s_\alpha(B).$$

Since $t < s_\alpha(B)$ is arbitrary, we have

$$\dim_H F(B) \ge \dim_H F_\alpha(B) \ge s_\alpha(B).$$

Corollary 3.4. *Let $\Lambda \subset \mathbb{N}$ be an infinite set. Then we have*

$$\dim_H\{x \in I : d_n(x) \geq B^n, \text{ i. o. } n \in \Lambda\} = s(B).$$

4. The Hausdorff dimension of $\{x \in I : d_n(x) \geq \phi(n), \text{ i. o. } n\}$

In this section, we generalize Theorem 3.2 to the general case in order to give a complete characterization on the Hausdorff dimension of the set $\{x \in I : d_n(x) \geq \phi(n), \text{ i. o. } n\}$.

Lemma 4.1 ([14])**.** *For any $a, b > 1$, set $E(a,b) = \{x \in I : d_n(x) \geq a^{b^n}\}$ and $F(a,b) = \{x \in (0,1] : d_n(x) \geq e^{a^b}, \text{ i. o. } n\}$, then $\dim_H(E(a,b)) = \dim_H(F(a,b)) = \frac{1}{b+1}$.*

Theorem 4.2. *Let ϕ be an arbitrary positive function defined on natural numbers and*

$$\{x \in I : d_n(x) \geq \phi(n), \text{ i. o. } n\}.$$

Suppose that $\liminf_{n\to\infty} \frac{\log \phi(n)}{n} = \log B$.

(1) *If $B = 1$, then $\dim_H E(\phi(n)) = 1$.*

(2) *If $1 < B < \infty$, then $\dim_H E(\phi(n)) = s(B)$.*

(3) *If $B = \infty$, and let $\liminf_{n\to} \frac{\log\log \phi(n)}{n} = \log b$.*

(3a) *If $b = 1$, then $\dim_H E(\phi(n)) = \frac{1}{2}$.*

(3b) *If $1 < b < \infty$, then $\dim_H E(\phi(n)) = \frac{1}{1+b}$.*

(3c) *If $b = \infty$, then $\dim_H E(\phi(n)) = 0$.*

Proof. (1) $B = 1$. In this case, for any $\epsilon > 0$, $\frac{\log \phi(n)}{n} \leq \log(1+\epsilon)$, i.e., $\phi(n) \leq (1+\epsilon)^n$ holds for infinitely many times. Let

$$\Lambda = \{n : \phi(n) \leq (1+\epsilon)^n\}.$$

Then,

$$\{x \in I : d_n(x) \geq (1+\epsilon)^n \text{ i. o. } n \in \Lambda\} \subset E(\phi(n)).$$

By Lemma 2.3, Theorem 3.2 and Corollary 3.3, we have

$$\dim_H E(\phi(n)) \geq \sup_{\epsilon>0} s(1+\epsilon).$$

(2) $1 < B < \infty$. In this case, for any $\epsilon > 0$, $\frac{\log \phi(n)}{n} \leq \log(B+\epsilon)$, i.e., $\phi(n) \leq (1+\epsilon)^n$ holds for infinitely many times. At the same time, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, we have $(B-\epsilon)^n \leq \phi(n)$. Let

$$\Lambda = \{n : \phi(n) \leq (B+\epsilon)^n\}.$$

Then

$$\begin{aligned}&\{x \in I : d_n(x) \geq (B+\epsilon)^n, \text{i. o. } n \in \Lambda\}\\ &\subset E(\phi(n)) \subset \{x \in I : d_n(x) \geq (B-\epsilon)^n, \text{i. o. } n \in \Lambda\}.\end{aligned}$$

By Lemma 2.3, Theorem 3.2 and Corollary 3.3, we have

$$s_B = \lim_{\epsilon\to 0} s(B+\epsilon) \le \dim_H E(\phi(n)) \le \lim_{\epsilon\to 0} s(B-\epsilon) = s(B).$$

(3) $B = \infty$.

(3a) $b = 1$. In this case, for any $\epsilon > 0$, $\frac{\log\log\phi(n)}{n} \le \log(1+\epsilon)$, i.e., $\phi(n) \le e^{(1+\epsilon)^n}$ holds for infinitely many times. Let

$$\Lambda = \{n : \phi(n) \le e^{(1+\epsilon)^n}\}.$$

Then

$$\{x \in I : d_n(x) \ge e^{(1+\epsilon)^n}\ \forall n \ge 1\} \subset \{x \in I : d_n(x) \ge e^{(1+\epsilon)^n} \text{ i. o. } n \in \Lambda\} \subset E(\phi(n)).$$

By Lemma 4.1, we have

$$\dim_H E(\phi(n)) \ge \lim_{\epsilon\to 0} \frac{1}{2+\epsilon} = \frac{1}{2}.$$

On the other hand, since $B = \infty$, we have for any $B_1 > 0$, $\frac{\log\phi(n)}{n} \ge \log B_1$, i.e., $\phi(n) \ge B_1^n$ holds for infinitely many times. Then

$$E(\phi(n)) \subset \{x \in I : d_n(x) \ge B_1^n, \text{i. o.}\}.$$

By Lemma 2.3, Theorem 3.2 and Corollary 3.3, we have

$$\dim E(\phi(n)) \le \lim_{B_1\to\infty} s(B_1) = \frac{1}{2}.$$

(3b) $1 < b < \infty$. In this case, for any $\epsilon > 0$, $\frac{\log\log\phi(n)}{n} \le \log(b+\epsilon)$, i.e., $\phi(n) \le e^{(b+\epsilon)^n}$ holds for infinitely many times. At the same time, $\phi(n) \ge e^{(b-\epsilon)^n}$ holds for n ultimately. Let

$$\Lambda = \{n : \phi(n) \le e^{(b+\epsilon)^n}\}.$$

Then

$$\{x \in I : d_n(x) \ge e^{(b+\epsilon)^n},\ \forall n \ge 1\} \subset \{x \in I : d_n(x) \ge e^{(b+\epsilon)^n}, \text{i. o. } n \in \Lambda\}$$
$$\subset E(\phi(n)) \subset \{x \in I : d_n(x) \ge e^{(b-\epsilon)^n}, \text{i. o. } n\}.$$

By Lemma 4.1, we have

$$\frac{1}{b+1} \le \lim_{\epsilon\to 0} \frac{1}{b+1+\epsilon} \le \dim_H E(\phi(n)) \le \lim_{\epsilon\to 0} \frac{1}{b+1-\epsilon} = \frac{1}{b+1}.$$

(3c) $b = \infty$. In this case, for any $B_1 > 0$, $\frac{\log\log\phi(n)}{n} \ge \log B_1$, i.e., $\phi(n) \le e^{B_1^n}$ holds for n ultimately. Then

$$E(\phi(n)) \subset \{x \in I : d_n(x) \ge e^{B_1^n}, \text{i. o. } n\}.$$

By Lemma 4.1, we have

$$\dim_H E(\phi(n)) \le \lim_{B_1\to\infty} \frac{1}{B_1+1} = 0.$$

□

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