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TRIPLE SYMMETRIC IDENTITIES FOR w-CATALAN POLYNOMIALS

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Abstract. In this paper, we introduce w-Catalan polynomials as a generalization of Catalan polynomials and derive fourteen basic identities of symmetry in three variables related to w-Catalan polynomials and analogues of alternating power sums. In addition, specializations of one of the variables as one give us new and interesting identities of symmetry even for two variables. The derivations of identities are based on the p-adic integral expression for the generating function of the w-Catalan polynomials and the quotient of p-adic integrals for that of the analogues of the alternating power sums.

1. Introduction and preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}_p$ will respectively denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of the algebraic closure of $\mathbb{Q}_p$. The p-adic norm $|\cdot|_p$ is normalized as $|p|_p = \frac{1}{p}$. For a continuous function $f : \mathbb{Z}_p \to \mathbb{C}_p$, the p-adic fermionic integral of f is defined by T. Kim and given by

$$\begin{aligned}\int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z) &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)\mu_{-1}(x+p^N\mathbb{Z}_p)\\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x.\end{aligned}$$

Then we can easily show that

$$\int_{\mathbb{Z}_p} f(z+1)d\mu_{-1}(z) + \int_{\mathbb{Z}_p} f(z)d\mu_{-1}(z) = 2f(0). \tag{1.1}$$

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As is well known, the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$ $(n \geq 0)$ are defined by the generating function

$$\frac{1-\sqrt{1-4t}}{2t} = \frac{2}{1+\sqrt{1-4t}} = \sum_{n=0}^{\infty} C_n t^n. \tag{1.2}$$

In addition, the Catalan polynomials are given by the generating function

$$\frac{2}{1+\sqrt{1-4t}}(1-4t)^{\frac{x}{2}} = \sum_{n=0}^{\infty} C_n(x)t^n \quad \text{(cf. [1, 5, 12, 14, 15, 16])}. \tag{1.3}$$

As a generalization of Catalan polynomials $C_n(x)$, for any positive integer w the w-Catalan polynomials $C_{n,w}(x)$ are defined by

$$\frac{2}{1+(1-4t)^{\frac{w}{2}}}(1-4t)^{\frac{w}{2}x} = \sum_{n=0}^{\infty} C_{n,w}(x)t^n. \tag{1.4}$$

For $x = 0$, $C_{n,w} = C_{n,w}(0)$ are called the w-Catalan numbers. We note here that

$$C_{n,1}(x) = C_n(x), \quad C_{n,1} = C_n. \tag{1.5}$$

From (1.1), we note that

$$\begin{aligned} \int_{\mathbb{Z}_p} (1-4t)^{\frac{w}{2}(x+y)} d\mu_{-1}(y) &= \frac{2}{1+(1-4t)^{\frac{w}{2}}}(1-4t)^{\frac{w}{2}x} \\ &= \sum_{n=0}^{\infty} C_{n,w}(x)t^n, \end{aligned} \tag{1.6}$$

where

$$t \in E = \{t \in \mathbb{C}_p \big| |t|_p < p^{-\frac{1}{p-1}}\}. \tag{1.7}$$

Let E be as in (1.7). Then $1+E$ and E are respectively groups under the binary operations $(1+x)(1+y)$ and $x+y$, for $x, y \in E$, and $1+E \to E$ is a group isomorphism under the map $1+x \to \log(1+x)$, with the inverse map given by $e^x \leftarrow x$. Thus, with the restriction $p > 2$ in mind, we see that, for each fixed $t \in E$, $(1-4t)^{\frac{w}{2}y} = e^{\frac{w}{2}y\log(1-4t)}$ is defined for any $y \in \mathbb{Z}_p$. This explains why the restriction in (1.7) is needed in defining the integral in (1.6).

For a nonnegative integer k, and positive integers d, w, we define

$$S_{k,d}(w-1) = \sum_{i=0}^{w-1} (-1)^i \binom{\frac{di}{2}}{k}. \tag{1.8}$$

For $d = 1$, for brevity we will write

$$S_k(w-1) = S_{k,1}(w-1) = \sum_{i=0}^{w-1} (-1)^i \binom{\frac{i}{2}}{k}. \tag{1.9}$$

Here $S_{k,d}(w-1)$ may be viewed as an analogue of the alternating kth power sum

$$T_k(w-1) = \sum_{i=0}^{w-1}(-1)^i i^k. \tag{1.10}$$

For later uses, we observe here that

$$S_{k,d}(0) = \begin{cases} 1, & k = 0, \\ 0, & k > 0. \end{cases} \tag{1.11}$$

From (1.6) and (1.8), we easily derive that, for positive integers w, d with $w \equiv 1 \pmod 2$,

$$\begin{aligned} \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{dx}{2}} d\mu_{-1}(x)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{dwx}{2}} d\mu_{-1}(x)} &= \sum_{i=0}^{w-1}(1-4t)^{\frac{di}{2}}(-1)^i \\ &= \sum_{k=0}^{\infty} S_{k,d}(w-1)(-4)^k t^k. \end{aligned} \tag{1.12}$$

In [13], the fermionic p-adic integrals are used in order to derive symmetric identities in two variables w_1, w_2 involving w-Catalan polynomials and the analogues of alternating power sums in (1.9). We extend this to the case of three variables so that we will have abundant symmetries related to those polynomials and the analogues. Indeed, we will produce 14 basic identities of symmetry in three variables w_1, w_2, w_3 related to w-Catalan polynomials and the analogues of alternating power sums (cf. (4.8), (4.9), (4.12), (4.16), (4.20), (4.23), (4.25), (4.26), (4.29), (4.30), (4.32), (4.35), (4.40), (4.43)). These abundance of symmetries shed new light even on the previously obtained identities for two variables (cf. [13]). For instance, in [13], it was shown that (1.13) and (1.14) are equal and (1.15) and (1.16) are so (cf. [13, Theorems 2 and 5]). In fact, (1.13)-(1.16) are all equal, as they can be derived from one and the same p-adic integral. Also, we have a bunch of new identities in (1.17)-(1.20). Indeed, upon specializing w_3 as 1 in some of the basic identities, we obtain the following identities (cf. Cor. 4.9, 4.12, 4.15). Let w_1, w_2 be any odd positive integers. Then we have:

$$\sum_{k=0}^{n}(-4)^{n-k}C_{k,w_1}(w_2y_1)S_{n-k,w_2}(w_1-1) \tag{1.13}$$

$$= \sum_{k=0}^{n}(-4)^{n-k}C_{k,w_2}(w_1y_1)S_{n-k,w_1}(w_2-1) \tag{1.14}$$

$$= \sum_{i=0}^{w_2-1}(-1)^i C_{n,w_2}(w_1y_1 + \frac{w_1}{w_2}i) \tag{1.15}$$

$$=\sum_{i=0}^{w_1-1}(-1)^i C_{n,w_1}(w_2y_1+\frac{w_2}{w_1}i) \tag{1.16}$$

$$=\sum_{k+l+m=n}(-4)^{l+m}C_{k,w_1w_2}(y_1)S_{l,w_1}(w_2-1)S_{m,w_2}(w_1-1) \tag{1.17}$$

$$=\sum_{k=0}^{n}(-4)^{n-k}S_{n-k,w_2}(w_1-1)\sum_{i=0}^{w_2-1}(-1)^i C_{k,w_1w_2}(y_1+\frac{1}{w_2}i) \tag{1.18}$$

$$=\sum_{k=0}^{n}(-4)^{n-k}S_{n-k,w_1}(w_2-1)\sum_{i=0}^{w_1-1}(-1)^i C_{k,w_1w_2}(y_1+\frac{1}{w_1}i) \tag{1.19}$$

$$=\sum_{j=0}^{w_2-1}\sum_{i=0}^{w_1-1}(-1)^{i+j}C_{n,w_1w_2}(y_1+\frac{1}{w_1}i+\frac{1}{w_2}j). \tag{1.20}$$

In what follows, we will always assume that the p-adic fermionic integrals of the various functions on $\mathbb{Z}_p$ are defined for $t \in E$ (cf. (1.7)), and therefore it will not be mentioned. The reader may refer to [2, 3, 4] and [6, 7, 8, 9, 10, 11] for some of the previous works on identities of symmetry involving various special polynomials.

The derivations of identities are based on the p-adic integral expression for the generating function of the w-Catalan polynomials in (1.6) and the quotient of p-adic integrals in (1.12) for that of the analogues of the alternating power sums.

Before proceeding to the next section, we will state a result about w-Catalan numbers.

Proposition 1.1. *Let w be any positive integer. Then we have the following*:

$$\sum_{m=0}^{\infty}C_{m,w}t^m=\sum_{n=0}^{\infty}C_n\sum_{k_1+\cdots+k_w=n}\binom{n}{k_1,\ldots,k_w}\prod_{l=1}^{w}\binom{w}{l}^{k_l}(-4)^{(l-1)k_l}t^{\sum_{l=1}^{w}lk_l}. \tag{1.21}$$

Proof. The left hand side of (1.21) is

$$\begin{aligned}
&\frac{2}{1+\sqrt{(1-4t)^w}}\\
&=\frac{2}{1+\sqrt{1-4(-\frac{1}{4}((1-4t)^w-1))}}\\
&=\sum_{n=0}^{\infty}C_n\Big(-\tfrac{1}{4}((1-4t)^w-1)\Big)^n\\
&=\sum_{n=0}^{\infty}C_n\left(-\frac{1}{4}\right)^n\left(\sum_{i=1}^{w}\binom{w}{i}(-4t)^i\right)^n
\end{aligned}$$

$$= \sum_{n=0}^{\infty} C_n \left(-\frac{1}{4}\right)^n \sum_{k_1+\cdots+k_w=n} \binom{n}{k_1,\ldots,k_w} \prod_{l=1}^{w} \binom{w}{l}^{k_l} (-4t)^{lk_l}$$
$$= \sum_{n=0}^{\infty} C_n \sum_{k_1+\cdots+k_w=n} \binom{n}{k_1,\ldots,k_w} \prod_{l=1}^{w} \binom{w}{l}^{k_l} (-4)^{(l-1)k_l} t^{\sum_{l=1}^{w} lk_l}. \qquad \square$$

2. Several types of quotients of fermionic integrals

Here we will introduce several types of quotients of p-adic fermionic integrals on $\mathbb{Z}_p$ or $\mathbb{Z}_p^3$ from which some interesting identities follow owing to the built-in symmetries in w_1, w_2, w_3. In the following, w_1, w_2, w_3 are all positive integers and all of the explicit expressions of integrals in (2.2), (2.4), (2.6), and (2.8) are obtained from the identity in (1.6). In below, we will use the following notation:

$$d\mu_{-1}(X) = d\mu_{-1}(x_1)d\mu_{-1}(x_2)d\mu_{-1}(x_3).$$

(a) Type Π_{23}^i (for $i = 0, 1, 2, 3$)

$$\text{(2.1)} \quad I(\Pi_{23}^i) = \frac{\int_{\mathbb{Z}_p^3} (1-4t)^{\frac{1}{2}(w_2w_3x_1+w_1w_3x_2+w_1w_2x_3+w_1w_2w_3(\sum_{j=1}^{3-i} y_j))} d\mu_{-1}(X)}{(\int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_2w_3x_4} d\mu_{-1}(x_4))^i}$$

$$\text{(2.2)} \quad = \frac{2^{3-i}(1-4t)^{\frac{1}{2}w_1w_2w_3(\sum_{j=1}^{3-i} y_j)}((1-4t)^{\frac{1}{2}w_1w_2w_3}+1)^i}{((1-4t)^{\frac{1}{2}w_2w_3}+1)((1-4t)^{\frac{1}{2}w_1w_3}+1)((1-4t)^{\frac{1}{2}w_1w_2}+1)}.$$

(b-0) Type Π_{12}^0

$$\text{(2.3)} \quad I(\Pi_{12}^0) = \int_{\mathbb{Z}_p^3} (1-4t)^{\frac{1}{2}(w_1x_1+w_2x_2+w_3x_3+w_2w_3y+w_1w_3y+w_1w_2y)} d\mu_{-1}(X)$$

$$\text{(2.4)} \quad = \frac{8(1-4t)^{\frac{1}{2}(w_2w_3+w_1w_3+w_1w_2)y}}{((1-4t)^{\frac{1}{2}w_1}+1)((1-4t)^{\frac{1}{2}w_2}+1)((1-4t)^{\frac{1}{2}w_3}+1)}.$$

(b-1) Type Π_{12}^1

$$\text{(2.5)} \quad I(\Pi_{12}^1) = \frac{\int_{\mathbb{Z}_p^3} (1-4t)^{\frac{1}{2}(w_1x_1+w_2x_2+w_3x_3)} d\mu_{-1}(x_1)d\mu_{-1}(x_2)d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p^3} (1-4t)^{\frac{1}{2}(w_2w_3z_1+w_1w_3z_2+w_1w_2z_3)} d\mu_{-1}(z_1)d\mu_{-1}(z_2)d\mu_{-1}(z_3)}$$

$$\text{(2.6)} \quad = \frac{((1-4t)^{\frac{1}{2}w_2w_3}+1)((1-4t)^{\frac{1}{2}w_1w_3}+1)((1-4t)^{\frac{1}{2}w_1w_2}+1)}{((1-4t)^{\frac{1}{2}w_1}+1)((1-4t)^{\frac{1}{2}w_2}+1)((1-4t)^{\frac{1}{2}w_3}+1)}.$$

(c) Type Π_{13}^i (for $i = 0, 1, 2, 3$)

$$\text{(2.7)} \quad I(\Pi_{13}^i) = \frac{\int_{\mathbb{Z}_p^3} (1-4t)^{\frac{1}{2}(w_1x_1+w_2x_2+w_3x_3+w_1w_2w_3(\sum_{j=1}^{3-i} y_j))} d\mu_{-1}(X)}{(\int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_2w_3x_4} d\mu_{-1}(x_4))^i}$$

$$\text{(2.8)} \quad = \frac{2^{3-i}(1-4t)^{\frac{1}{2}w_1w_2w_3(\sum_{j=1}^{3-i} y_j)}((1-4t)^{\frac{1}{2}w_1w_2w_3}+1)^i}{((1-4t)^{\frac{1}{2}w_1}+1)((1-4t)^{\frac{1}{2}w_2}+1)((1-4t)^{\frac{1}{2}w_3}+1)}.$$

All of the above p-adic integrals of various types are invariant under all permutations of w_1, w_2, w_3 as one can see either from p-adic integral representations in (2.1), (2.3), (2.5), and (2.7) or from their explicit evaluations in (2.2), (2.4), (2.6), and (2.8).

3. Identities for Euler polynomials

In the following w_1, w_2, w_3 are all odd positive integers except for (a-0), (b-0), and (c-0), where they are any positive integers.

(a-0) First, let's consider Type Π_{23}^i, for each $i = 0, 1, 2, 3$. The following results can be easily obtained from (1.6) and (1.12).

$$\begin{aligned} I(\Pi_{23}^0) &= \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_2w_3(x_1+w_1y_1)} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_3(x_2+w_2y_2)} d\mu_{-1}(x_2) \\ &\quad \times \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_2(x_3+w_3y_3)} d\mu_{-1}(x_3) \\ &= (\sum_{k=0}^{\infty} C_{k,w_2w_3}(w_1y_1)t^k)(\sum_{l=0}^{\infty} C_{l,w_1w_3}(w_2y_2)t^l)(\sum_{m=0}^{\infty} C_{m,w_1w_2}(w_3y_3)t^m). \end{aligned} \tag{3.1}$$

(a-1) Here we write $I(\Pi_{23}^1)$ in two different ways:

(a-1-1)

$$\begin{aligned} I(\Pi_{23}^1) &= \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_2w_3(x_1+w_1y_1)} d\mu_{-1}(x_1) \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_3(x_2+w_2y_2)} d\mu_{-1}(x_2) \\ &\quad \times \frac{\int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_2x_3} d\mu_{-1}(x_3)}{\int_{Z_p} (1-4t)^{\frac{1}{2}w_1w_2w_3x_4} d\mu_{-1}(x_4)} \qquad (3.2) \\ &= (\sum_{k=0}^{\infty} C_{k,w_2w_3}(w_1y_1)t^k)(\sum_{l=0}^{\infty} C_{l,w_1w_3}(w_2y_2)t^l)(\sum_{m=0}^{\infty} S_{m,w_1w_2}(w_3-1)(-4)^m t^m) \\ &= \sum_{n=0}^{\infty}(\sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)C_{l,w_1w_3}(w_2y_2)S_{m,w_1w_2}(w_3-1)(-4)^m)t^n. \qquad (3.3) \end{aligned}$$

(a-1-2) Using (1.12), (3.2) can also be written as

$$\begin{aligned} I(\Pi_{23}^1) &= \sum_{i=0}^{w_3-1}(-1)^i \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_2w_3(x_1+w_1y_1)} d\mu_{-1}(x_1) \\ &\quad \times \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_1w_3(x_2+w_2y_2+\frac{w_2}{w_3}i)} d\mu_{-1}(x_2) \qquad (3.4) \\ &= \sum_{i=0}^{w_3-1}(-1)^i(\sum_{k=0}^{\infty} C_{k,w_2w_3}(w_1y_1)t^k)(\sum_{l=0}^{\infty} C_{l,w_1w_3}(w_2y_2+\frac{w_2}{w_3}i)t^l) \\ &= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} C_{k,w_2w_3}(w_1y_1)\sum_{i=0}^{w_3-1}(-1)^i C_{n-k,w_1w_3}(w_2y_2+\frac{w_2}{w_3}i))t^n. \end{aligned}$$

(a-2) Here we write $I(\Pi_{23}^2)$ in three different ways:

(a-2-1)

$$I(\Pi_{23}^2) = \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_2w_3(x_1+w_1y_1)} d\mu_{-1}(x_1) \times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_3x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \tag{3.5}$$

$$= (\sum_{k=0}^{\infty} C_{k,w_2w_3}(w_1y_1)t^k)(\sum_{l=0}^{\infty} S_{l,w_1w_3}(w_2-1)(-4)^l t^l)(\sum_{m=0}^{\infty} S_{m,w_1w_2}(w_3-1)(-4)^m t^m)$$

$$= \sum_{n=0}^{\infty}(\sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)S_{l,w_1w_3}(w_2-1)S_{m,w_1w_2}(w_3-1)(-4)^{l+m})t^n. \tag{3.6}$$

(a-2-2) In view of (1.12), (3.5) can also be written as

$$I(\Pi_{23}^2) = \sum_{i=0}^{w_2-1}(-1)^i \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}i)} d\mu_{-1}(x_1)$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \tag{3.7}$$

$$= \sum_{i=0}^{w_2-1}(-1)^i(\sum_{k=0}^{\infty} C_{k,w_2w_3}(w_1y_1+\frac{w_1}{w_2}i)t^k)(\sum_{l=0}^{\infty} S_{l,w_1w_2}(w_3-1)(-4)^l t^l)$$

$$= \sum_{n=0}^{\infty}(\sum_{k=0}^{n} S_{n-k,w_1w_2}(w_3-1)(-4)^{n-k} \sum_{i=0}^{w_2-1}(-1)^i C_{k,w_2w_3}(w_1y_1+\frac{w_1}{w_2}i))t^n. \tag{3.8}$$

(a-2-3) Invoking (1.12) once again, (3.7) can be written as

$$I(\Pi_{23}^2) = \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j} \int_{\mathbb{Z}_p} (1-4t)^{\frac{1}{2}w_2w_3(x_1+w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j)} d\mu_{-1}(x_1)$$

$$= \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j}\sum_{n=0}^{\infty} C_{n,w_2w_3}(w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j)t^n \tag{3.9}$$

$$= \sum_{n=0}^{\infty}(\sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1}(-1)^{i+j} C_{n,w_2w_3}(w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j))t^n.$$

(a-3)

$$I(\Pi_{23}^3) = \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2w_3x_1}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_3x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$

$$= (\sum_{k=0}^{\infty} S_{k,w_2w_3}(w_1-1)(-4)^k t^k)(\sum_{l=0}^{\infty} S_{l,w_1w_3}(w_2-1)(-4)^l t^l)$$
$$\times (\sum_{m=0}^{\infty} S_{m,w_1w_2}(w_3-1)(-4)^m t^m)$$
$$= \sum_{n=0}^{\infty}(-4)^n(\sum_{k+l+m=n} S_{k,w_2w_3}(w_1-1)S_{l,w_1w_3}(w_2-1)S_{m,w_1w_2}(w_3-1))t^n. \tag{3.10}$$

(b-0)

$$I(\Pi_{12}^0) = \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2y)}d\mu_{-1}(x_1)\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2(x_2+w_3y)}d\mu_{-1}(x_2)$$
$$\times \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3(x_3+w_1y)}d\mu_{-1}(x_3) \tag{3.11}$$
$$= (\sum_{n=0}^{\infty} C_{k,w_1}(w_2y)t^k)(\sum_{l=0}^{\infty} C_{l,w_2}(w_3y)t^l)(\sum_{m=0}^{\infty} C_{m,w_3}(w_1y)t^m)$$
$$= \sum_{n=0}^{\infty}(\sum_{k+l+m=n} C_{k,w_1}(w_2y)C_{l,w_2}(w_3y)C_{m,w_3}(w_1y))t^n.$$

(b-1)

$$\frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1x_1}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2z_3}d\mu_{-1}(z_3)} \times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2w_3z_1}d\mu_{-1}(z_1)}$$
$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3w_1z_2}d\mu_{-1}(z_2)} \tag{3.12}$$
$$= (\sum_{k=0}^{\infty} S_{k,w_1}(w_2-1)t^k)(\sum_{l=0}^{\infty} S_{l,w_2}(w_3-1)t^l)(\sum_{m=0}^{\infty} S_{m,w_3}(w_1-1)t^m)$$
$$= \sum_{n=0}^{\infty}(\sum_{k+l+m=n} S_{k,w_1}(w_2-1)S_{l,w_2}(w_3-1)S_{m,w_3}(w_1-1)t^n.$$

(c-0) Let's consider Type Π_{13}^i, for each $i = 0, 1, 2, 3$. The following results can be easily obtained from (1.6) and (1.12).

$$I(\Pi_{13}^0) = \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2w_3y_1)}d\mu_{-1}(x_1)\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2(x_2+w_1w_3y_2)}d\mu_{-1}(x_2)$$
$$\times \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3(x_3+w_1w_2y_3)}d\mu_{-1}(x_3)$$
$$= (\sum_{k=0}^{\infty} C_{k,w_1}(w_2w_3y_1)t^k)(\sum_{l=0}^{\infty} C_{l,w_2}(w_1w_3y_2)t^l)(\sum_{m=0}^{\infty} C_{m,w_3}(w_1w_2y_3)t^m) \tag{3.13}$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)C_{l,w_2}(w_1w_3y_2)C_{m,w_3}(w_1w_2y_3)t^n.$$

(c-1) Here we write $I(\Pi_{13}^1)$ in two different ways:(c-1-1) $I(\Pi_{13}^1)$

$$= \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2w_3y_1)}d\mu_{-1}(x_1)\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2(x_2+w_1w_3y_2)}d\mu_{-1}(x_2)$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \tag{3.14}$$

$$= \Big(\sum_{k=0}^{\infty} C_{k,w_1}(w_2w_3y_1)t^k\Big)\Big(\sum_{l=0}^{\infty} C_{l,w_2}(w_1w_3y_2)t^l\Big)\Big(\sum_{m=0}^{\infty} S_{m,w_3}(w_1w_2-1)(-4)^m t^m\Big)$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)C_{l,w_2}(w_1w_3y_2)S_{m,w_3}(w_1w_2-1)(-4)^m\Big)t^n. \tag{3.15}$$

(c-1-2) Using (1.12), (3.14) can also be written as

$$I(\Pi_{13}^1) = \sum_{i=0}^{w_1w_2-1}(-1)^i\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2w_3y_1)}d\mu_{-1}(x_1)$$

$$\times \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2(x_2+w_1w_3y_2+\frac{w_3}{w_2}i)}d\mu_{-1}(x_2) \tag{3.16}$$

$$= \sum_{i=0}^{w_1w_2-1}(-1)^i\Big(\sum_{k=0}^{\infty} C_{k,w_1}(w_2w_3y_1)t^k\Big)\Big(\sum_{l=0}^{\infty} C_{l,w_2}(w_1w_3y_2+\frac{w_3}{w_2}i)t^l\Big)$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k=0}^{n} C_{k,w_1}(w_2w_3y_1)\sum_{i=0}^{w_1w_2-1}(-1)^i C_{n-k,w_2}(w_1w_3y_2+\frac{w_3}{w_2}i)\Big)t^n.$$

(c-2) Here we write $I(\Pi_{13}^2)$ in three different ways:
(c-2-1)

$$I(\Pi_{13}^2) = \int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2w_3y_1)}d\mu_{-1}(x_1)\times\frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \tag{3.17}$$

$$= \Big(\sum_{k=0}^{\infty} C_{k,w_1}(w_2w_3y_1)t^k\Big)\Big(\sum_{l=0}^{\infty} S_{l,w_2}(w_1w_3-1)(-4)^l t^l\Big)\Big(\sum_{m=0}^{\infty} S_{m,w_3}(w_1w_2-1)(-4)^m t^m\Big)$$

$$= \sum_{n=0}^{\infty}\Big(\sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)S_{l,w_2}(w_1w_3-1)S_{m,w_3}(w_1w_2-1)(-4)^{l+m}\Big)t^n. \tag{3.18}$$

(c-2-2) In view of (1.12), (3.17) can also be written as

$$I(\Pi_{13}^2) = \sum_{i=0}^{w_1w_3-1}(-1)^i\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2w_3y_1+\frac{w_2}{w_1}i)}d\mu_{-1}(x_1)$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \tag{3.19}$$

$$= \sum_{i=0}^{w_1w_3-1}(-1)^i(\sum_{k=0}^{\infty}C_{k,w_1}(w_2w_3y_1+\frac{w_2}{w_1}i)t^k)(\sum_{l=0}^{\infty}S_{l,w_3}(w_1w_2-1)(-4)^lt^l)$$

$$= \sum_{n=0}^{\infty}(\sum_{k=0}^{n}S_{n-k,w_3}(w_1w_2-1)(-4)^{n-k}\sum_{i=0}^{w_1w_3-1}(-1)^iC_{k,w_1}(w_2w_3y_1+\frac{w_2}{w_1}i))t^n. \tag{3.20}$$

(c-2-3) Invoking (1.12) once again, (3.19) can be written as

$$I(\Pi_{13}^2) = \sum_{i=0}^{w_1w_3-1}\sum_{j=0}^{w_1w_2-1}(-1)^{i+j}\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1(x_1+w_2w_3y_1+\frac{w_2}{w_1}i+\frac{w_3}{w_1}j)}d\mu_{-1}(x_1)$$

$$= \sum_{i=0}^{w_1w_3-1}\sum_{j=0}^{w_1w_2-1}(-1)^{i+j}\sum_{n=0}^{\infty}C_{n,w_1}(w_2w_3y_1+\frac{w_2}{w_1}i+\frac{w_3}{w_1}j)t^n \tag{3.21}$$

$$= \sum_{n=0}^{\infty}(\sum_{i=0}^{w_1w_3-1}\sum_{j=0}^{w_1w_2-1}(-1)^{i+j}C_{n,w_1}(w_2w_3y_1+\frac{w_2}{w_1}i+\frac{w_3}{w_1}j))t^n.$$

(c-3)

$$I(\Pi_{13}^3) = \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1x_1}d\mu_{-1}(x_1)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)} \times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_2x_2}d\mu_{-1}(x_2)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$

$$\times \frac{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_3x_3}d\mu_{-1}(x_3)}{\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}w_1w_2w_3x_4}d\mu_{-1}(x_4)}$$

$$= (\sum_{k=0}^{\infty}S_{k,w_1}(w_2w_3-1)(-4)^kt^k)(\sum_{l=0}^{\infty}S_{l,w_2}(w_1w_3-1)(-4)^lt^l)$$

$$\times (\sum_{m=0}^{\infty}S_{m,w_3}(w_1w_2-1)(-4)^mt^m)$$

$$= \sum_{n=0}^{\infty}(-4)^n(\sum_{k+l+m=n}S_{k,w_1}(w_2w_3-1)S_{l,w_2}(w_1w_3-1)S_{m,w_3}(w_1w_2-1))t^n. \tag{3.22}$$

4. Main theorems

As we noted earlier in the last paragraph of Section 2, the various types of quotients of p-adic fermionic integrals are invariant under any permutations of w_1, w_2, w_3. So the corresponding expressions in Section 3 are also invariant under any permutations of w_1, w_2, w_3. Thus our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single one labelled by them, there is a natural transitive action of S_3 on them so that it is in bijective correspondence with a quotient of S_3. In particular, the number of possible distinct expressions are $1, 2, 3$, or 6. (a-0), (a-1-1), (a-1-2), (a-2-2), (c-0), (c-1-1), (c-1-2), and (c-2-2) give the full six identities of symmetry, (a-2-1), (a-2-3), (c-2-1), and (c-2-3) yield three identities of symmetry, and (b-0) and (b-1) give two identities of symmetry, while the expressions in (a-3) and (c-3) yield no identities of symmetry.

Here we will just consider the cases of Theorems 4.8 and 4.17, leaving the others as easy exercises for the reader. As for the case of Theorem 4.8, in addition to (4.15)-(4.17), we get the following three ones:

$$\sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)S_{l,w_1w_2}(w_3-1)S_{m,w_1w_3}(w_2-1)(-4)^{l+m}, \tag{4.1}$$

$$\sum_{k+l+m=n} C_{k,w_1w_3}(w_2y_1)S_{l,w_2w_3}(w_1-1)S_{m,w_1w_2}(w_3-1)(-4)^{l+m}, \tag{4.2}$$

$$\sum_{k+l+m=n} C_{k,w_1w_2}(w_3y_1)S_{l,w_1w_3}(w_2-1)S_{m,w_2w_3}(w_1-1)(-4)^{l+m}. \tag{4.3}$$

But, by interchanging l and m, we see that (4.1), (4.2), and (4.3) are respectively equal to (4.15), (4.16), and (4.17).

As to Theorem 4.17, in addition to (4.26) and (4.27), we have:

$$\sum_{k+l+m=n} S_{k,w_3}(w_2-1)S_{l,w_2}(w_1-1)S_{m,w_1}(w_3-1), \tag{4.4}$$

$$\sum_{k+l+m=n} S_{k,w_1}(w_3-1)S_{l,w_3}(w_2-1)S_{m,w_2}(w_1-1), \tag{4.5}$$

$$\sum_{k+l+m=n} S_{k,w_2}(w_3-1)S_{l,w_3}(w_1-1)S_{m,w_1}(w_2-1), \tag{4.6}$$

$$\sum_{k+l+m=n} S_{k,w_3}(w_1-1)S_{l,w_1}(w_2-1)S_{m,w_2}(w_3-1). \tag{4.7}$$

However, (4.4) and (4.5) are equal to (4.27), as we can see by applying the permutations $k \to m, l \to k, m \to l$ for (4.4) and $k \to l, l \to m, m \to k$ for (4.5). Similarly, we see that (4.6) and (4.7) are equal to (4.26), by applying permutations $k \to l, l \to m, m \to k$ for (4.6) and $k \to m, l \to k, m \to l$ for (4.7).

In below, Theorems 4.1, 4.2, 4.5, 4.8, 4.11, 4.14, 4.16, 4.17, 4.19, 4.20, 4.22, 4.25, 4.28, and 4.31 follow respectively from the equations (3.1), (3.3), (3.4), (3.6), (3.7), (3.9), (3.11), (3.12), (3.13), (3.15), (3.16), (3.18), (3.20), and (3.21).

Theorem 4.1. *Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)C_{l,w_1w_3}(w_2y_2)C_{m,w_1w_2}(w_3y_3) \tag{4.8}$$

$$= \sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)C_{l,w_1w_2}(w_3y_2)C_{m,w_1w_3}(w_2y_3)$$
$$= \sum_{k+l+m=n} C_{k,w_1w_3}(w_2y_1)C_{l,w_2w_3}(w_1y_2)C_{m,w_1w_2}(w_3y_3)$$
$$= \sum_{k+l+m=n} C_{k,w_1w_3}(w_2y_1)C_{l,w_1w_2}(w_3y_2)C_{m,w_2w_3}(w_1y_3)$$
$$= \sum_{k+l+m=n} C_{k,w_1w_2}(w_3y_1)C_{l,w_2w_3}(w_1y_2)C_{m,w_1w_3}(w_2y_3)$$
$$= \sum_{k+l+m=n} C_{k,w_1w_2}(w_3y_1)C_{l,w_1w_3}(w_2y_2)C_{m,w_2w_3}(w_1y_3).$$

Theorem 4.2. *Let w_1, w_2, w_3 be any odd positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)C_{l,w_1w_3}(w_2y_2)S_{m,w_1w_2}(w_3-1)(-4)^m \tag{4.9}$$
$$= \sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)C_{l,w_1w_2}(w_3y_2)S_{m,w_1w_3}(w_2-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_1w_3}(w_2y_1)C_{l,w_2w_3}(w_1y_2)S_{m,w_1w_2}(w_3-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_1w_3}(w_2y_1)C_{l,w_1w_2}(w_3y_2)S_{m,w_2w_3}(w_1-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_1w_2}(w_3y_1)C_{l,w_1w_3}(w_2y_2)S_{m,w_2w_3}(w_1-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_1w_2}(w_3y_1)C_{l,w_2w_3}(w_1y_2)S_{m,w_1w_3}(w_2-1)(-4)^m.$$

Putting $w_3 = 1$ in (4.9), we get the following corollary.

Corollary 4.3. *Let w_1, w_2 be any odd positive integers.*

$$\sum_{k=0}^{n} C_{k,w_2}(w_1y_1)C_{n-k,w_1}(w_2y_2) \tag{4.10}$$
$$= \sum_{k=0}^{n} C_{k,w_1}(w_2y_1)C_{n-k,w_2}(w_1y_2)$$
$$= \sum_{k+l+m=n} C_{k,w_1w_2}(y_1)C_{l,w_1}(w_2y_2)S_{m,w_2}(w_1-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_1}(w_2y_1)C_{l,w_1w_2}(y_2)S_{m,w_2}(w_1-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_1w_2}(y_1)C_{l,w_2}(w_1y_2)S_{m,w_1}(w_2-1)(-4)^m$$

$$= \sum_{k+l+m=n} C_{k,w_2}(w_1 y_1) C_{l,w_1 w_2}(y_2) S_{m,w_1}(w_2 - 1)(-4)^m.$$

Letting further $w_2 = 1$ in (4.10), we have the following corollary.

Corollary 4.4. *Let w_1 be any odd positive integer.*

$$\begin{aligned} &\sum_{k=0}^{n} C_k(w_1 y_1) C_{n-k,w_1}(y_2) \\ &= \sum_{k=0}^{n} C_{k,w_1}(y_1) C_{n-k}(w_1 y_2) \\ &= \sum_{k+l+m=n} C_{k,w_1}(y_1) C_{l,w_1}(y_2) S_m(w_1 - 1)(-4)^m. \end{aligned} \tag{4.11}$$

Theorem 4.5. *Let w_1, w_2, w_3 be any odd positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\begin{aligned} &\sum_{k=0}^{n} C_{k,w_1 w_2}(w_3 y_1) \sum_{i=0}^{w_1 - 1} (-1)^i C_{n-k,w_1 w_3}(w_2 y_2 + \frac{w_2}{w_1} i) \\ &= \sum_{k=0}^{n} C_{k,w_1 w_3}(w_2 y_1) \sum_{i=0}^{w_1 - 1} (-1)^i C_{n-k,w_1 w_2}(w_3 y_2 + \frac{w_3}{w_1} i) \\ &= \sum_{k=0}^{n} C_{k,w_1 w_2}(w_3 y_1) \sum_{i=0}^{w_2 - 1} (-1)^i C_{n-k,w_2 w_3}(w_1 y_2 + \frac{w_1}{w_2} i) \\ &= \sum_{k=0}^{n} C_{k,w_2 w_3}(w_1 y_1) \sum_{i=0}^{w_2 - 1} (-1)^i C_{n-k,w_1 w_2}(w_3 y_2 + \frac{w_3}{w_2} i) \\ &= \sum_{k=0}^{n} C_{k,w_1 w_3}(w_2 y_1) \sum_{i=0}^{w_3 - 1} (-1)^i C_{n-k,w_2 w_3}(w_1 y_2 + \frac{w_1}{w_3} i) \\ &= \sum_{k=0}^{n} C_{k,w_2 w_3}(w_1 y_1) \sum_{i=0}^{w_3 - 1} (-1)^i C_{n-k,w_1 w_3}(w_2 y_2 + \frac{w_2}{w_3} i). \end{aligned} \tag{4.12}$$

Letting $w_3 = 1$ in (4.12), we obtain alternative expressions for the identities in (4.10).

Corollary 4.6. *Let w_1, w_2 be any odd positive integers.*

$$\begin{aligned} &\sum_{k=0}^{n} C_{k,w_2}(w_1 y_1) C_{n-k,w_1}(w_2 y_2) \\ &= \sum_{k=0}^{n} C_{k,w_1}(w_2 y_1) C_{n-k,w_2}(w_1 y_2) \end{aligned} \tag{4.13}$$

$$
\begin{aligned}
&=\sum_{k=0}^{n} C_{k,w_1w_2}(y_1)\sum_{i=0}^{w_1-1}(-1)^i C_{n-k,w_1}(w_2y_2+\frac{w_2}{w_1}i)\\
&=\sum_{k=0}^{n} C_{k,w_1}(w_2y_1)\sum_{i=0}^{w_1-1}(-1)^i C_{n-k,w_1w_2}(y_2+\frac{i}{w_1})\\
&=\sum_{k=0}^{n} C_{k,w_1w_2}(y_1)\sum_{i=0}^{w_2-1}(-1)^i C_{n-k,w_2}(w_1y_2+\frac{w_1}{w_2}i)\\
&=\sum_{k=0}^{n} C_{k,w_2}(w_1y_1)\sum_{i=0}^{w_2-1}(-1)^i C_{n-k,w_1w_2}(y_2+\frac{i}{w_2}).
\end{aligned}
$$

Putting further $w_2 = 1$ in (4.13), we have the alternative expressions for the identities for (4.11).

Corollary 4.7. *Let w_1 be any odd positive integer.*

$$
\begin{aligned}
\sum_{k=0}^{n} C_{k,w_1}(y_1)C_{n-k}(w_1y_2) &= \sum_{k=0}^{n} C_k(w_1y_1)C_{n-k,w_1}(y_2)\\
&= \sum_{k=0}^{n} C_{k,w_1}(y_1)\sum_{i=0}^{w_1-1}(-1)^i C_{n-k,w_1}(y_2+\frac{i}{w_1}). \quad (4.14)
\end{aligned}
$$

Theorem 4.8. *Let w_1, w_2, w_3 be any odd positive integers. Then we have the following three symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n} C_{k,w_2w_3}(w_1y_1)S_{l,w_1w_3}(w_2-1)S_{m,w_1w_2}(w_3-1)(-4)^{l+m} \quad (4.15)$$

$$= \sum_{k+l+m=n} C_{k,w_1w_3}(w_2y_1)S_{l,w_1w_2}(w_3-1)S_{m,w_2w_3}(w_1-1)(-4)^{l+m} \quad (4.16)$$

$$= \sum_{k+l+m=n} C_{k,w_1w_2}(w_3y_1)S_{l,w_2w_3}(w_1-1)S_{m,w_1w_3}(w_2-1)(-4)^{l+m}. \quad (4.17)$$

Putting $w_3 = 1$ in (4.15)-(4.17), we get the following corollary.

Corollary 4.9. *Let w_1, w_2 be any odd positive integers.*

$$
\begin{aligned}
&\sum_{k=0}^{n} C_{k,w_2}(w_1y_1)S_{n-k.w_1}(w_2-1)(-4)^{n-k} \quad (4.18)\\
&= \sum_{k=0}^{n} C_{k,w_1}(w_2y_1)S_{n-k,w_2}(w_1-1)(-4)^{n-k}\\
&= \sum_{k+l+m=n} C_{k,w_1w_2}(y_1)S_{l,w_2}(w_1-1)S_{m,w_1}(w_2-1)(-4)^{l+m}.
\end{aligned}
$$

Letting further $w_2 = 1$ in (4.18), we get the following corollary.

Corollary 4.10. *Let w_1 be any odd positive integer.*

$$C_n(w_1y_1) = \sum_{k=0}^{n} C_{k,w_1}(y_1)S_{n-k}(w_1-1)(-4)^{n-k}. \tag{4.19}$$

Theorem 4.11. *Let w_1, w_2, w_3 be any odd positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\begin{aligned}
&\sum_{k=0}^{n} S_{n-k,w_1w_2}(w_3-1)(-4)^{n-k}\sum_{i=0}^{w_1-1}(-1)^iC_{k,w_1w_3}(w_2y_1+\frac{w_2}{w_1}i) \\
&= \sum_{k=0}^{n} S_{n-k,w_1w_3}(w_2-1)(-4)^{n-k}\sum_{i=0}^{w_1-1}(-1)^iC_{k,w_1w_2}(w_3y_1+\frac{w_3}{w_1}i) \\
&= \sum_{k=0}^{n} S_{n-k,w_1w_2}(w_3-1)(-4)^{n-k}\sum_{i=0}^{w_2-1}(-1)^iC_{k,w_2w_3}(w_1y_1+\frac{w_1}{w_2}i) \\
&= \sum_{k=0}^{n} S_{n-k,w_2w_3}(w_1-1)(-4)^{n-k}\sum_{i=0}^{w_2-1}(-1)^iC_{k,w_1w_2}(w_3y_1+\frac{w_3}{w_2}i) \\
&= \sum_{k=0}^{n} S_{n-k,w_1w_3}(w_2-1)(-4)^{n-k}\sum_{i=0}^{w_3-1}(-1)^iC_{k,w_2w_3}(w_1y_1+\frac{w_1}{w_3}i) \\
&= \sum_{k=0}^{n} S_{n-k,w_2w_3}(w_1-1)(-4)^{n-k}\sum_{i=0}^{w_3-1}(-1)^iC_{k,w_1w_3}(w_2y_1+\frac{w_2}{w_3}i).
\end{aligned} \tag{4.20}$$

Putting $w_3 = 1$ in (4.20), we obtain the following corollary. In Section 1, the identities in (4.18), (4.21), and (4.24) are combined to give those in (1.13)-(1.20).

Corollary 4.12. *Let w_1, w_2 be any odd positive integers.*

$$\begin{aligned}
&\sum_{i=0}^{w_1-1}(-1)^iC_{n,w_1}(w_2y_1+\frac{w_2}{w_1}i) \\
&= \sum_{i=0}^{w_2-1}(-1)^iC_{n,w_2}(w_1y_1+\frac{w_1}{w_2}i) \\
&= \sum_{k=0}^{n} S_{n-k,w_2}(w_1-1)(-4)^{n-k}C_{k,w_1}(w_2y_1) \\
&= \sum_{k=0}^{n} S_{n-k,w_1}(w_2-1)(-4)^{n-k}C_{k,w_2}(w_1y_1) \\
&= \sum_{k=0}^{n} S_{n-k,w_1}(w_2-1)(-4)^{n-k}\sum_{i=0}^{w_1-1}(-1)^iC_{k,w_1w_2}(y_1+\frac{i}{w_1})
\end{aligned} \tag{4.21}$$

$$= \sum_{k=0}^{n} S_{n-k,w_2}(w_1-1)(-4)^{n-k} \sum_{i=0}^{w_2-1} (-1)^i C_{k,w_1w_2}(y_1+\frac{i}{w_2}).$$

Letting further $w_2 = 1$ in (4.21), we get the following corollary.

Corollary 4.13. *Let w_1 be any odd positive integer.*

$$C_n(w_1y_1) = \sum_{i=0}^{w_1-1} (-1)^i C_{n,w_1}(y_1+\frac{i}{w_1}) = \sum_{k=0}^{n} S_{n-k}(w_1-1)(-4)^{n-k} C_{k,w_1}(y_1). \tag{4.22}$$

Theorem 4.14. *Let w_1, w_2, w_3 be any odd positive integers. Then we have the following three symmetries in w_1, w_2, w_3:*

$$\begin{aligned} &\sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} (-1)^{i+j} C_{n,w_1w_2}(w_3y_1+\frac{w_3}{w_1}i+\frac{w_3}{w_2}j) \\ &= \sum_{i=0}^{w_2-1}\sum_{j=0}^{w_3-1} (-1)^{i+j} C_{n,w_2w_3}(w_1y_1+\frac{w_1}{w_2}i+\frac{w_1}{w_3}j) \\ &= \sum_{i=0}^{w_3-1}\sum_{j=0}^{w_1-1} (-1)^{i+j} C_{n,w_1w_3}(w_2y_1+\frac{w_2}{w_3}i+\frac{w_2}{w_1}j). \end{aligned} \tag{4.23}$$

Letting $w_3 = 1$ in (4.23), we have the following corollary.

Corollary 4.15. *Let w_1, w_2 be any odd positive integers.*

$$\begin{aligned} &\sum_{i=0}^{w_1-1} (-1)^i C_{n,w_1}(w_2y_1+\frac{w_2}{w_1}i) \\ &= \sum_{i=0}^{w_2-1} (-1)^i C_{n,w_2}(w_1y_1+\frac{w_1}{w_2}i) \\ &= \sum_{i=0}^{w_1-1}\sum_{j=0}^{w_2-1} (-1)^{i+j} C_{n,w_1w_2}(y_1+\frac{i}{w_1}+\frac{j}{w_2}). \end{aligned} \tag{4.24}$$

Theorem 4.16. *Let w_1, w_2, w_3 be any positive integers. Then we have the following two symmetries in w_1, w_2, w_3:*

$$\begin{aligned} &\sum_{k+l+m=n} C_{k,w_1}(w_2y)C_{l,w_2}(w_3y)C_{m,w_3}(w_1y) \\ &= \sum_{k+l+m=n} C_{k,w_2}(w_1y)C_{l,w_1}(w_3y)C_{m,w_3}(w_2y). \end{aligned} \tag{4.25}$$

Theorem 4.17. *Let w_1, w_2, w_3 be any odd positive integers. Then we have the following two symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n} S_{k,w_1}(w_2-1)S_{l,w_2}(w_3-1)S_{m,w_3}(w_1-1) \tag{4.26}$$

$$= \sum_{k+l+m=n} S_{k,w_2}(w_1-1)S_{l,w_1}(w_3-1)S_{m,w_3}(w_2-1). \tag{4.27}$$

Putting $w_3 = 1$ in (4.26) and (4.27), we get the following corollary.

Corollary 4.18. *Let w_1, w_2 be any odd positive integers.*

$$\sum_{k=0}^{n} S_{k,w_1}(w_2-1)S_{n-k}(w_1-1) = \sum_{k=0}^{n} S_{k,w_2}(w_1-1)S_{n-k}(w_2-1). \tag{4.28}$$

Theorem 4.19. *Let w_1, w_2, w_3 be any positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\begin{aligned}
&\sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)C_{l,w_2}(w_1w_3y_2)C_{m,w_3}(w_1w_2y_3) \qquad (4.29)\\
&= \sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)C_{l,w_3}(w_1w_2y_2)C_{m,w_2}(w_1w_3y_3)\\
&= \sum_{k+l+m=n} C_{k,w_2}(w_1w_3y_1)C_{l,w_1}(w_2w_3y_2)C_{m,w_3}(w_1w_2y_3)\\
&= \sum_{k+l+m=n} C_{k,w_2}(w_1w_3y_1)C_{l,w_3}(w_1w_2y_2)C_{m,w_1}(w_2w_3y_3)\\
&= \sum_{k+l+m=n} C_{k,w_3}(w_1w_2y_1)C_{l,w_1}(w_2w_3y_2)C_{m,w_2}(w_1w_3y_3)\\
&= \sum_{k+l+m=n} C_{k,w_3}(w_1w_2y_1)C_{l,w_2}(w_1w_3y_2)C_{m,w_1}(w_2w_3y_3).
\end{aligned}$$

Theorem 4.20. *Let w_1, w_2, w_3 be any odd positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\begin{aligned}
&\sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)C_{l,w_2}(w_1w_3y_2)S_{m,w_3}(w_1w_2-1)(-4)^m \qquad (4.30)\\
&= \sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)C_{l,w_3}(w_1w_2y_2)S_{m,w_2}(w_1w_3-1)(-4)^m\\
&= \sum_{k+l+m=n} C_{k,w_2}(w_1w_3y_1)C_{l,w_1}(w_2w_3y_2)S_{m,w_3}(w_1w_2-1)(-4)^m\\
&= \sum_{k+l+m=n} C_{k,w_2}(w_1w_3y_1)C_{l,w_3}(w_1w_2y_2)S_{m,w_1}(w_2w_3-1)(-4)^m
\end{aligned}$$

$$= \sum_{k+l+m=n} C_{k,w_3}(w_1w_2y_1)C_{l,w_1}(w_2w_3y_2)S_{m,w_2}(w_1w_3-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_{k,w_3}(w_1w_2y_1)C_{l,w_2}(w_1w_3y_2)S_{m,w_1}(w_2w_3-1)(-4)^m.$$

Putting $w_2 = w_3 = 1$ in (4.9), we get the following corollary.

Corollary 4.21. *Let w_1 be any odd positive integer.*

$$\sum_{k=0}^{n} C_k(w_1y_1)C_{n-k}(w_1y_2) = \sum_{k+l+m=n} C_{k,w_1}(y_1)C_l(w_1y_2)S_m(w_1-1)(-4)^m$$
$$= \sum_{k+l+m=n} C_k(w_1y_1)C_{l,w_1}(y_2)S_m(w_1-1)(-4)^m. \tag{4.31}$$

Theorem 4.22. *Let w_1, w_2, w_3 be any odd positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\sum_{k=0}^{n} C_{k,w_3}(w_1w_2y_1) \sum_{i=0}^{w_2w_3-1} (-1)^i C_{n-k,w_2}(w_1w_3y_2 + \frac{w_1}{w_2}i) \tag{4.32}$$
$$= \sum_{k=0}^{n} C_{k,w_2}(w_1w_3y_1) \sum_{i=0}^{w_2w_3-1} (-1)^i C_{n-k,w_3}(w_1w_2y_2 + \frac{w_1}{w_3}i)$$
$$= \sum_{k=0}^{n} C_{k,w_3}(w_1w_2y_1) \sum_{i=0}^{w_1w_3-1} (-1)^i C_{n-k,w_1}(w_2w_3y_2 + \frac{w_2}{w_1}i)$$
$$= \sum_{k=0}^{n} C_{k,w_1}(w_2w_3y_1) \sum_{i=0}^{w_1w_3-1} (-1)^i C_{n-k,w_3}(w_1w_2y_2 + \frac{w_2}{w_3}i)$$
$$= \sum_{k=0}^{n} C_{k,w_2}(w_1w_3y_1) \sum_{i=0}^{w_1w_2-1} (-1)^i C_{n-k,w_1}(w_2w_3y_2 + \frac{w_3}{w_1}i)$$
$$= \sum_{k=0}^{n} C_{k,w_1}(w_2w_3y_1) \sum_{i=0}^{w_1w_2-1} (-1)^i C_{n-k,w_2}(w_1w_3y_2 + \frac{w_3}{w_2}i).$$

Letting $w_3 = 1$ in (4.12), we obtain the following corollary.

Corollary 4.23. *Let w_1, w_2 be any odd positive integers.*

$$\sum_{k=0}^{n} C_k(w_1w_2y_1) \sum_{i=0}^{w_2-1} (-1)^i C_{n-k,w_2}(w_1y_2 + \frac{w_1}{w_2}i) \tag{4.33}$$
$$= \sum_{k=0}^{n} C_{k,w_2}(w_1y_1) \sum_{i=0}^{w_2-1} (-1)^i C_{n-k}(w_1w_2y_2 + w_1i)$$

$$
\begin{aligned}
&= \sum_{k=0}^{n} C_k(w_1w_2y_1) \sum_{i=0}^{w_1-1} (-1)^i C_{n-k,w_1}(w_2y_2 + \frac{w_2}{w_1} i) \\
&= \sum_{k=0}^{n} C_{k,w_1}(w_2y_1) \sum_{i=0}^{w_1-1} (-1)^i C_{n-k}(w_1w_2y_2 + w_2 i) \\
&= \sum_{k=0}^{n} C_{k,w_2}(w_1y_1) \sum_{i=0}^{w_1w_2-1} (-1)^i C_{n-k,w_1}(w_2y_2 + \frac{1}{w_1} i) \\
&= \sum_{k=0}^{n} C_{k,w_1}(w_2y_1) \sum_{i=0}^{w_1w_2-1} (-1)^i C_{n-k,w_2}(w_1y_2 + \frac{1}{w_2} i).
\end{aligned}
$$

Putting further $w_2 = 1$ in (4.13), we have the alternative expressions for the identities for (4.11).

Corollary 4.24. *Let w_1 be any odd positive integer.*

$$
\begin{aligned}
\sum_{k=0}^{n} C_k(w_1y_1)C_{n-k}(w_1y_2) &= \sum_{k=0}^{n} C_{k,w_1}(y_1) \sum_{i=0}^{w_1-1} (-1)^i C_{n-k}(w_1y_2 + i) \\
&= \sum_{k=0}^{n} C_k(w_1y_1) \sum_{i=0}^{w_1-1} (-1)^i C_{n-k,w_1}(y_2 + \frac{i}{w_1}). \qquad (4.34)
\end{aligned}
$$

Theorem 4.25. *Let w_1, w_2, w_3 be any odd positive integers. Then we have the following three symmetries in w_1, w_2, w_3:*

$$
\sum_{k+l+m=n} C_{k,w_1}(w_2w_3y_1)S_{l,w_2}(w_1w_3-1)S_{m,w_3}(w_1w_2-1)(-4)^{l+m} \qquad (4.35)
$$

$$
= \sum_{k+l+m=n} C_{k,w_2}(w_1w_3y_1)S_{l,w_3}(w_1w_2-1)S_{m,w_1}(w_2w_3-1)(-4)^{l+m} \qquad (4.36)
$$

$$
= \sum_{k+l+m=n} C_{k,w_3}(w_1w_2y_1)S_{l,w_1}(w_2w_3-1)S_{m,w_2}(w_1w_3-1)(-4)^{l+m}. \qquad (4.37)
$$

Putting $w_3 = 1$ in (4.15)-(4.17), we get the following corollary.

Corollary 4.26. *Let w_1, w_2 be any odd positive integers.*

$$
\begin{aligned}
&\sum_{k+l+m=n} C_{k,w_1}(w_2y_1)S_{l,w_2}(w_1-1)S_m(w_1w_2-1)(-4)^{l+m} \qquad (4.38)\\
&= \sum_{k+l+m=n} C_{k,w_2}(w_1y_1)S_l(w_1w_2-1)S_{m,w_1}(w_2-1)(-4)^{l+m} \\
&= \sum_{k+l+m=n} C_k(w_1w_2y_1)S_{l,w_1}(w_2-1)S_{m,w_2}(w_1-1)(-4)^{l+m}.
\end{aligned}
$$

Letting further $w_2 = 1$ in (4.18), we get the following corollary.

Corollary 4.27. *Let w_1 be any odd positive integer.*

$$\sum_{k+l+m=n} C_{k,w_1}(y_1)S_l(w_1-1)S_m(w_1-1)(-4)^{l+m} = \sum_{k=0}^{n} C_k(w_1y_1)S_{n-k}(w_1-1)(-4)^{n-k}. \tag{4.39}$$

Theorem 4.28. *Let w_1, w_2, w_3 be any odd positive integers. Then the following expression is invariant under any permutation of w_1, w_2, w_3, so that it gives us six symmetries.*

$$\begin{aligned}
&\sum_{k=0}^{n} S_{n-k,w_3}(w_1w_2-1)(-4)^{n-k} \sum_{i=0}^{w_2w_3-1}(-1)^i C_{k,w_2}(w_1w_3y_1+\frac{w_1}{w_2}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_2}(w_1w_3-1)(-4)^{n-k} \sum_{i=0}^{w_2w_3-1}(-1)^i C_{k,w_3}(w_1w_2y_1+\frac{w_1}{w_3}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_3}(w_1w_2-1)(-4)^{n-k} \sum_{i=0}^{w_1w_3-1}(-1)^i C_{k,w_1}(w_2w_3y_1+\frac{w_2}{w_1}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_1}(w_2w_3-1)(-4)^{n-k} \sum_{i=0}^{w_1w_3-1}(-1)^i C_{k,w_3}(w_1w_2y_1+\frac{w_2}{w_3}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_2}(w_1w_3-1)(-4)^{n-k} \sum_{i=0}^{w_1w_2-1}(-1)^i C_{k,w_1}(w_2w_3y_1+\frac{w_3}{w_1}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_1}(w_2w_3-1)(-4)^{n-k} \sum_{i=0}^{w_1w_2-1}(-1)^i C_{k,w_2}(w_1w_3y_1+\frac{w_3}{w_2}i).
\end{aligned} \tag{4.40}$$

Putting $w_3 = 1$ in (4.20), we obtain the following corollary.

Corollary 4.29. *Let w_1, w_2 be any odd positive integers.*

$$\begin{aligned}
&\sum_{k=0}^{n} S_{n-k}(w_1w_2-1)(-4)^{n-k} \sum_{i=0}^{w_2-1}(-1)^i C_{k,w_2}(w_1y_1+\frac{w_1}{w_2}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_2}(w_1-1)(-4)^{n-k} \sum_{i=0}^{w_2-1}(-1)^i C_k(w_1w_2y_1+w_1i) \\
=&\sum_{k=0}^{n} S_{n-k}(w_1w_2-1)(-4)^{n-k} \sum_{i=0}^{w_1-1}(-1)^i C_{k,w_1}(w_2y_1+\frac{w_2}{w_1}i) \\
=&\sum_{k=0}^{n} S_{n-k,w_1}(w_2-1)(-4)^{n-k} \sum_{i=0}^{w_1-1}(-1)^i C_k(w_1w_2y_1+w_2i)
\end{aligned} \tag{4.41}$$

$$= \sum_{k=0}^{n} S_{n-k,w_2}(w_1-1)(-4)^{n-k} \sum_{i=0}^{w_1w_2-1} (-1)^i C_{k,w_1}(w_2y_1 + \frac{1}{w_1}i)$$

$$= \sum_{k=0}^{n} S_{n-k,w_1}(w_2-1)(-4)^{n-k} \sum_{i=0}^{w_1w_2-1} (-1)^i C_{k,w_2}(w_1y_1 + \frac{1}{w_2}i).$$

Letting further $w_2 = 1$ in (4.21), we get the following corollary.

Corollary 4.30. *Let w_1 be any odd positive integer.*

$$\sum_{i=0}^{w_1-1} (-1)^i C_n(w_1y_1 + i) = \sum_{k=0}^{n} S_{n-k}(w_1-1)(-4)^{n-k} \sum_{i=0}^{w_1-1} (-1)^i C_{k,w_1}(y_1 + \frac{i}{w_1})$$

$$= \sum_{k=0}^{n} S_{n-k}(w_1-1)(-4)^{n-k} C_k(w_1y_1). \tag{4.42}$$

Theorem 4.31. *Let w_1, w_2, w_3 be any odd positive integers. Then we have the following three symmetries in w_1, w_2, w_3:*

$$\sum_{i=0}^{w_2w_3-1} \sum_{j=0}^{w_1w_3-1} (-1)^{i+j} C_{n,w_3}(w_1w_2y_1 + \frac{w_1}{w_3}i + \frac{w_2}{w_3}j) \tag{4.43}$$

$$= \sum_{i=0}^{w_1w_3-1} \sum_{j=0}^{w_1w_2-1} (-1)^{i+j} C_{n,w_1}(w_2w_3y_1 + \frac{w_2}{w_1}i + \frac{w_3}{w_1}j)$$

$$= \sum_{i=0}^{w_1w_2-1} \sum_{j=0}^{w_2w_3-1} (-1)^{i+j} C_{n,w_2}(w_1w_3y_1 + \frac{w_3}{w_2}i + \frac{w_1}{w_2}j).$$

Letting $w_3 = 1$ in (4.23), we have the following corollary.

Corollary 4.32. *Let w_1, w_2 be any odd positive integers.*

$$\sum_{i=0}^{w_2-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} C_n(w_1w_2y_1 + w_1i + w_2j) \tag{4.44}$$

$$= \sum_{i=0}^{w_1-1} \sum_{j=0}^{w_1w_2-1} (-1)^{i+j} C_{n,w_1}(w_2y_1 + \frac{w_2}{w_1}i + \frac{1}{w_1}j)$$

$$= \sum_{i=0}^{w_1w_2-1} \sum_{j=0}^{w_2-1} (-1)^{i+j} C_{n,w_2}(w_1y_1 + \frac{1}{w_2}i + \frac{w_1}{w_2}j).$$

Letting further $w_2 = 1$ in (4.24), we get the following corollary.

Corollary 4.33. *Let w_1 be any odd positive integer.*

$$\sum_{i=0}^{w_1-1} \sum_{j=0}^{w_1-1} (-1)^{i+j} C_{n,w_1}(y_1 + \frac{1}{w_1}i + \frac{1}{w_1}j) = \sum_{i=0}^{w_1-1} (-1)^i C_n(w_1y_1 + i). \tag{4.45}$$

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