J. Korean Math. Soc. **54** (2017), No. 4, pp. 1231–1242
https://doi.org/10.4134/JKMS.j160434
pISSN: 0304-9914 / eISSN: 2234-3008

FACTORIZATION OF CERTAIN SELF-MAPS OF PRODUCT SPACES

Sangwoo Jun and Kee Young Lee

Abstract. In this paper, we show that, under some conditions, self-maps of product spaces can be represented by the composition of two specific self-maps if their induced homomorphism on the i-th homotopy group is an automorphism for all i in some section of positive integers. As an application, we obtain closeness numbers of several product spaces.

1. Introduction

For a connected pointed topological space X, let $\mathcal{E}(X)$ denote the set of homotopy classes of pointed self-maps of X that are homotopy equivalences. Then, $\mathcal{E}(X)$ is a group with a group operation given by a composition of homotopy classes. Let $[X, X]$ be the set of all based homotopy classes of self-maps of X. When $[X, X]$ is given by a composition of homotopy classes, the set is a monoid. Choi and Lee [5] studied certain submonoid of $[X, X]$ containing $\mathcal{E}(X)$ as a set. If $\mathcal{A}^k_\#(X)$ denotes the set of homotopy classes of self-maps of X that induce an automorphism of $\pi_i(X)$ for $0 \le i \le k$, then $\mathcal{A}^k_\#(X)$ is a submonoid of $[X, X]$ with an operation given by a composition of homotopy classes for any nonnegative integer k. If $k = \infty$, we simply denote $\mathcal{A}^\infty_\#(X)$ as $\mathcal{A}_\#(X)$. By definition, $\mathcal{A}^n_\#(X) \subseteq \mathcal{A}^m_\#(X)$ if $n \ge m$. Therefore, we have the following descending series:

$$\mathcal{E}(X) \subseteq \mathcal{A}_\#(X) \subseteq \cdots \subseteq \mathcal{A}^1_\#(X) \subseteq \mathcal{A}^0_\#(X) = [X, X].$$

For any connected CW-complex X, $\mathcal{A}_\#(X) = \mathcal{E}(X)$ according to the Whitehead theorem.

The group $\mathcal{E}(X \times Y)$ has been studied extensively by several authors, for instance, Booth and Heath [3], Heath [6], Lee [7], Pavešić [8–10] and Sieradski [11]. In particular, Pavešić [9] demonstrated that the group of self-homotopy equivalences $\mathcal{E}(X \times Y)$ can be represented as a product of two subgroups under

Received June 21, 2016; Revised October 21, 2016.
2010 *Mathematics Subject Classification.* Primary 55Q05; Secondary 55P10.
Key words and phrases. reducible, k-reducible, self-homotopy equivalence, self-closeness number.
Supported by a Korea University Grant.

the assumption that the self-equivalences of $X \times Y$ can be diagonalized (or are reducible). In this study, we examine the sufficient conditions under which all elements of the submonoid $\mathcal{A}^k_\#(X \times Y)$ of $[X \times Y, X \times Y]$ can be factorized by two specific self-maps for a non-negative integer k. In Section 2, we introduce the concept of k-reducibility and find several conditions for the factorization of $\mathcal{A}^k_\#(X \times Y)$. In Section 3, we study the split short exact sequences of several monoids. In Section 4, we discuss an alternative idea of the k-reducibility in the category of CW-complexes and their relationships with self-closeness numbers [5] of product spaces.

Let i_X and i_Y denote the inclusions (as slices determined by the base-points) of X and Y in $X \times Y$, respectively, and p_X and p_Y be the projections of $X \times Y$ onto X and Y, respectively. Given a self-map $f : X \times Y \to X \times Y$ and $I, J \in \{X, Y\}$, write $f_I : X \times Y \to I$ for the composition $f_I := p_I \circ f$ so that f is represented componentwise as $f = (f_X, f_Y)$ and $f_{IJ} : J \to I$ for the composition $f_{IJ} := p_I \circ f \circ i_J$. The self-homotopy equivalence f of $X \times Y$ can be *diagonalized* (or *is reducible*) if f_{XX} and f_{YY} are self-homotopy equivalences of X and Y, respectively [8]. Now, we recall that the isomorphism $\Psi : \pi_n(X \times Y) \to \pi_n(X) \times \pi_n(Y)$ is given by $\Psi = (p_{X\#}, p_{Y\#})$ with the inverse Φ, where $\Phi(\alpha, \beta) = i_{X\#}(\alpha) + i_{Y\#}(\beta)$ for $(\alpha, \beta) \in \pi_n(X) \times \pi_n(Y)$. Therefore, for given self-map $f : X \times Y \to X \times Y$, the induced homomorphism $\pi_n(f)$ can be identified with the 2×2 matrix

$$\pi_i(f) = \begin{pmatrix} \pi_i(f_{XX}) & \pi_i(f_{XY}) \\ \pi_i(f_{YX}) & \pi_i(f_{YY}) \end{pmatrix}.$$

We refer to this 2×2-matrix as *the matrix representation* of the homomorphism $\pi_i(f)$ throughout this paper. Given two self-maps $f, g : X \times Y \to X \times Y$, the induced homomorphism $\pi_i(f \circ g)$ of the composition $f \circ g$ can be identified with the multiplication of their matrix representations.

Throughout this paper, all spaces are pointed, connected and have the homotopy type of a CW-complex with an abelian fundamental group. Moreover, all maps and homotopies preserve the base points and we do not distinguish between the notation of a map $f : X \to Y$ and that of its homotopy class in $[X, Y]$.

Acknowledgment. We are very grateful to the referee whose constructive remarks considerably improved the original manuscript.

2. Internal direct product of $\mathcal{A}^k_{X,\#}(X \times Y)$ and $\mathcal{A}^k_{Y,\#}(X \times Y)$

In this section, we discuss the factorization of $\mathcal{A}^k_\#(X \times Y)$ into two submonoids. We begin by introducing the following definition.

Definition 1. The self map $f : X \times Y \to X \times Y$ is said to be k- reducible if $f_{XX} \in \mathcal{A}^k_\#(X)$ and $f_{YY} \in \mathcal{A}^k_\#(Y)$.

According to the definition, if a self-map $f : X \times Y \to X \times Y$ is reducible, then f is k-reducible for each non-negative integer k. However, the converse does not hold. On the other hand, it is easy to show that if a self-homotopy equivalence $f : X \times Y \to X \times Y$ is ∞-reducible, f is reducible on $\mathcal{E}(X \times Y)$ according to the Whitehead theorem.

Example 1. For $2 \leq m < n$, let $f : S^m \to S^m$ and $g : S^n \to S^n$ be maps with deg(f)=2. Because S^m is $(m-1)$-connected, $\pi_k(f) \in \mathrm{Aut}(\pi_k(S^m))$ and $\pi_k(g) \in \mathrm{Aut}(\pi_k(S^n))$ for $0 \leq k \leq m-1$. However, $\pi_m(f)$ is not surjective because deg(f)=2. Therefore, $f \times g$ is $(m-1)$-reducible but not reducible on $\mathcal{E}(X \times Y)$.

Given two abelian groups G and H, an homomorphism $\lambda : H \to G$ is said to be *R-quasi-regular* if for any homomorphism $\mu : G \to H$, the function $id_G - \lambda\mu$ given by $(id_G - \lambda\mu)(g) = g - \lambda((\mu(g))$ is an automorphism of G. Similarly, λ is said to be *L-quasi-regular* if $id_H - \mu\lambda$ is an automorphism of H. Moreover, an homomorphism $\lambda : H \to G$ is said to be *RL-quasi-regular* if it is R-quasi-regular and L-quasi-regular. Clearly, if $\mathrm{Hom}(G, H)$ or $\mathrm{Hom}(H, G)$ is trivial, then each homomorphism in $\mathrm{Hom}(H, G)$ is RL-quasi-regular.

Lemma 1. *If f is an element of $\mathcal{A}^k_\#(X \times Y)$ such that $\pi_i(f_{XY})$ is RL-quasi-regular for $0 \leq i \leq k$, then f is k-reducible.*

Proof. For each $f = (f_X, f_Y) \in \mathcal{A}^k_\#(X \times Y)$, the induced homomorphism $\pi_i(f)$ belongs to $\mathrm{Aut}(\pi_i(X \times Y))$ for $0 \leq i \leq k$. For $0 \leq i \leq k$, the homomorphism

$$\pi_i(f) = \begin{pmatrix} \pi_i(f_{XX}) & \pi_i(f_{XY}) \\ \pi_i(f_{YX}) & \pi_i(f_{YY}) \end{pmatrix}$$

has an inverse homomorphism Φ_i of $\pi_i(f)$. Let

$$\Phi_i = \begin{pmatrix} \varphi_{XX} & \varphi_{XY} \\ \varphi_{YX} & \varphi_{YY} \end{pmatrix}$$

be the matrix representation. Then, $\pi_i(f) \circ \Phi_i = id_{\pi_i(X \times Y)}$ implies that $\pi_i(f_{XX})\varphi_{XX} + \pi_i(f_{XY})\varphi_{YX} = id_{\pi_i(X)}$. Since $\pi_i(f_{XY})$ is R-quasi-regular, $\pi_i(f_{XX})$ is an isomorphism for $0 \leq i \leq k$. Similarly, $\Phi_i \circ \pi_i(f) = id_{\pi_i(X \times Y)}$ implies that $\varphi_{YX}\pi_i(f_{XY}) + \varphi_{YY}\pi_i(f_{YY}) = id_{\pi_i(Y)}$. Since $\pi_i(f_{XY})$ is L-quasi-regular, $\pi_i(f_{YY})$ is an isomorphism for $0 \leq i \leq k$. □

We define the subset $\mathcal{A}^k_{X,\#}(X \times Y)$ as the set of all maps in $\mathcal{A}^k_\#(X \times Y)$ with the form $f = (p_X, f_Y) : X \times Y \to X \times Y$. Similarly, we define the subset $\mathcal{A}^k_{Y,\#}(X \times Y)$ as the set of all maps in $\mathcal{A}^k_\#(X \times Y)$ with the form $f = (f_X, p_Y) : X \times Y \to X \times Y$.

Lemma 2. (a) *$\mathcal{A}^k_{X,\#}(X \times Y)$ and $\mathcal{A}^k_{Y,\#}(X \times Y)$ are submonoids of $\mathcal{A}^k_\#(X \times Y)$ for any nonnegative integer k.*

(b) $(p_X, g) \in \mathcal{A}^k_{X,\#}(X \times Y)$ *if and only if* $g \circ i_Y \in \mathcal{A}^k_\#(Y)$.

(c) $(f, p_Y) \in \mathcal{A}^k_{Y,\#}(X \times Y)$ *if and only if* $f \circ i_X \in \mathcal{A}^k_\#(X)$.

Proof. (a) For the given elements (p_X, f_Y) and (p_X, g_Y) in $\mathcal{A}^k_{X,\#}(X\times Y)$, we have $(p_X, f_Y)\circ(p_X, g_Y) = (p_X, f_Y\circ(p_X, g_Y))$. Moreover, the induced homomorphisms $\pi_i(p_X, f_Y)$ and $\pi_i(p_X, g_Y)$ are in $\mathrm{Aut}(\pi_i(X\times Y))$ for $0\le i\le k$. Therefore, $\pi_i(p_X, f_Y)\circ\pi_i(p_X, g_Y) = \pi_i((p_X, f_Y)\circ(p_X, g_Y))\in\mathrm{Aut}(\pi_i(X\times Y))$ for $0\le i\le k$. It follows that $(p_X, f_Y)\circ(p_X, g_Y)\in\mathcal{A}^k_{X,\#}(X\times Y)$. Clearly, $(p_X, f_Y)\circ((p_X, g_Y)\circ(p_X, h_Y)) = ((p_X, f_Y)\circ(p_X, g_Y))\circ(p_X, h_Y)$ and (p_X, p_Y) is an identity of $\mathcal{A}^k_{X,\#}(X\times Y)$.

(b) Suppose $(p_X, g)\in\mathcal{A}^k_{X,\#}(X\times Y)$. Then, the matrix representation of isomorphism $\pi_i(p_X, g)$ for $0\le i\le k$ is given by

$$\begin{pmatrix} id_{\pi_i(X)} & 0 \\ \pi_i(g\circ i_X) & \pi_i(g\circ i_Y)\end{pmatrix}.$$

Therefore, $\pi_i(g\circ i_Y)$ is an isomorphism on $\pi_i(Y)$ for $0\le i\le k$.

Conversely, suppose that $g\circ i_Y\in\mathcal{A}^k_{\#}(Y)$. Then the inverse of $\pi_i(p_X, g)$ is represented by

$$\begin{pmatrix} id_{\pi_i(X)} & 0 \\ \pi_i(g\circ i_Y)^{-1}\circ(-\pi_i(g\circ i_X)) & \pi_i(g\circ i_Y)^{-1}\end{pmatrix},$$

where $-\pi_i(g\circ i_X):\pi_i(X)\to\pi_i(Y)$ is the homomorphism given by $-\pi_i(g\circ i_X)(\alpha) = -(\pi_i(g\circ i_X)(\alpha))$ in $\pi_i(Y)$ for each $\alpha\in\pi_i(X)$.

(c) This can be proved in a similar method to that of (b). □

Corollary 1. *If $f=(f_X, f_Y)\in\mathcal{A}^k_{\#}(X\times Y)$ is k-reducible, then $(p_X, f_Y)\in\mathcal{A}^k_{X,\#}(X\times Y)$ and $(f_X, p_Y)\in\mathcal{A}^k_{Y,\#}(X\times Y)$.*

Let U be a monoid and S and T be submonoids of U. Then U is called *the internal direct product of S and T* if

(1) U is uniquely factorizable with factors S and T;

(2) for all $s\in S$ and for all $t\in T$, $st=ts$.

On the other hand, the monoid $S\times T=\{(s,t)\mid s\in S, t\in T\}$ is called *the external direct product of the two monoids S and T* if the binary operation is given by $(s,t)(s',t')=(ss',tt')$ on $S\times T$ with the identity $(1_S, 1_T)$.

In [8], Pavešić showed that if X and Y are connected CW-complexes and all self-homotopy equivalences of $X\times Y$ are reducible, then $\mathrm{Aut}(X\times Y) = \mathrm{Aut}_X(X\times Y)\mathrm{Aut}_Y(X\times Y)$. Here, we discuss the factorization of $\mathcal{A}^k_{\#}(X\times Y)$ into $\mathcal{A}^k_{X,\#}(X\times Y)$ and $\mathcal{A}^k_{Y,\#}(X\times Y)$. However, we cannot apply the method in [8] to $\mathcal{A}^k_{\#}(X\times Y)$ directly because not all elements of $\mathcal{A}^k_{\#}(X\times Y)$ are always self-homotopy equivalences.

Theorem 1. *Suppose that each $f=(f_X, f_Y)\in\mathcal{A}^k_{\#}(X\times Y)$ is k-reducible and $f_Y\simeq f_{YY}\circ p_Y$. Then*

$$\mathcal{A}^k_{\#}(X\times Y) = \mathcal{A}^k_{X,\#}(X\times Y)\mathcal{A}^k_{Y,\#}(X\times Y).$$

Furthermore, if $f_X\simeq f_{XX}\circ p_X$, then $(p_X, f_Y)\circ(f_X, p_Y) = (f_X, p_Y)\circ(p_X, f_Y)$.

Proof. According to Corollary 1, $(p_X, f_Y) \in \mathcal{A}^k_{X,\#}(X \times Y)$ and $(f_X, p_Y) \in \mathcal{A}^k_{Y,\#}(X \times Y)$, and therefore, they are contained in $\mathcal{A}^k_{\#}(X \times Y)$. Moreover, because $\mathcal{A}^k_{X,\#}(X \times Y) \cap \mathcal{A}^k_{Y,\#}(X \times Y) = \{(p_X, p_Y)\}$, it is sufficient to show that each element $f = (f_X, f_Y) \in \mathcal{A}^k_{\#}(X \times Y)$ can be factored as $f = g \circ h$, where $g \in \mathcal{A}^k_{X,\#}(X \times Y)$ and $h \in \mathcal{A}^k_{Y,\#}(X \times Y)$. Via a the direct computation, we have

$$\begin{aligned}(p_X, f_Y) \circ (f_X, p_Y) &= (f_X, f_Y \circ (f_X, p_Y)) \\ &\simeq (f_X, f_{YY} \circ p_Y \circ (f_X, p_Y)) \\ &= (f_X, f_{YY} \circ p_Y) \\ &\simeq (f_X, f_Y). \end{aligned}$$

□

From Theorem 1, we arrive at the following corollary.

Corollary 2. *If $f = (f_X, f_Y) \in \mathcal{A}^k_{\#}(X \times Y)$ is k-reducible and $f_X \simeq f_{XX} \circ p_X$ and $f_Y \simeq f_{YY} \circ p_Y$, then $\mathcal{A}^k_{\#}(X \times Y)$ is the internal direct product of $\mathcal{A}^k_{X,\#}(X \times Y)$ and $\mathcal{A}^k_{Y,\#}(X \times Y)$.*

Consider the inclusion map $j : X \vee Y \to X \times Y$, where $X \vee Y$ is the wedge product of X and Y. Then we arrive at the following lemma.

Lemma 3. *$j^\sharp : [X \times Y, X] \to [X \vee Y, X]$ is injective and $[Y, X] = 0$ if and only if for each map $f : X \times Y \to X \times Y$, $f_X \simeq f_{XX} \circ p_X$.*

Proof. Suppose that $j^\sharp$ is injective. It suffices to show that $j^\sharp(f_X) = f_X \circ j = f_{XX} \circ p_X \circ j = j^\sharp(f_{XX} \circ p_X)$. This is true because

$$f_{XX} \circ p_X \circ j \circ i_1 = f_{XX} \circ p_X \circ i_X = f_X \circ i_X = f_X \circ j \circ i_1$$

and

$$f_{XX} \circ p_X \circ j \circ i_2 = f_{XX} \circ p_X \circ i_Y \simeq * \simeq f_{XY} = f_X \circ i_Y = f_X \circ j \circ i_2,$$

where $i_1 : X \to X \vee Y$ and $i_2 : Y \to X \vee Y$ are injective maps defined by $i_1(x) = (x, *)$ and $i_2(y) = (*, y)$, respectively.

Conversely, suppose that for each map $f : X \times Y \to X \times Y$, $f_X \simeq f_{XX} \circ p_X$. For $u, v \in [X \times Y, X]$, define $g : X \times Y \to X \times Y$ and $h : X \times Y \to X \times Y$ by $g = (u, p_Y)$ and $h = (v, p_Y)$, respectively. Then $g_X = u$ and $h_X = v$. If $j^\sharp(u) = j^\sharp(v)$, then

$$u \simeq u \circ i_X \circ p_X = u \circ j \circ i_1 \circ p_X \simeq v \circ j \circ i_1 \circ p_X = v \circ i_X \circ p_X \simeq v$$

according to the hypothesis. Therefore $j^\sharp$ is injective. Moreover, $[Y, X] = 0$. In fact, if we define $f : X \times Y \to X \times Y$ by $f(x, y) = (w(y), y)$ for each map $w : Y \to X$, then $w = f_{XY} = f_X \circ i_Y \simeq f_{XX} \circ p_X \circ i_Y \simeq *$. □

Consider the following cofibre sequence:

$$X \vee Y \xrightarrow{\ j\ } X \times Y \xrightarrow{\ q\ } X \wedge Y\ .$$

This gives rise to the following Barrat-Puppe sequence:

$$\cdots \to [\Sigma(X \vee Y), X] \to [X \wedge Y, X] \xrightarrow{q^\sharp} [X \times Y, X] \xrightarrow{j^\sharp} [X \vee Y, X].$$

From this sequence and Lemma 3, we arrive at the following corollary.

Corollary 3. *If* $[X \wedge Y, X] = 0$ *and* $[Y, X] = 0$*, then* $f_X \simeq f_{XX} \circ p_X$ *for each map* $f : X \times Y \to X \times Y$*.*

According to Lemma 1, Theorem 1 and Corollary 3, we arrive at the following corollary.

Corollary 4. $\mathcal{A}^k_\#(S^1 \times S^n) = \mathcal{A}^k_{S^1,\#}(S^1 \times S^n)\mathcal{A}^k_{S^n,\#}(S^1 \times S^n)$ *for each pair of integers* k *and* n *such that* $1 \leq k < n$*.*

3. Short exact sequences of monoids

In this section, we derive certain short exact sequences related to $\mathcal{A}^k_\#(X \times Y)$. Pavešić [9, Lemma 1.3, Proposition 1.4 and Theorem 1.5] introduced the monoid homomorphism from $\mathrm{Aut}_Y(X \times Y)$ to $\mathrm{Aut}(X)$ and several split short exact sequences. First, we introduce a similar monoid homomorphism.

Lemma 4. *If* $\Phi_X : \mathcal{A}^k_{Y,\#}(X \times Y) \to \mathcal{A}^k_\#(X)$ *is a map defined by* $\Phi_X(f_X, p_Y) = f_{XX}$*, then* Φ_X *is a monoid epimorphism.*

Proof. Clearly, the function Φ_X is surjective according to Lemma 2(b).
Since

$$(f_X, p_Y) \circ i_X = (f_X \circ i_X, p_Y \circ i_X) = (f_{XX}, *) = i_X \circ f_{XX},$$

we have

$$\begin{aligned}\Phi_X((f_X, p_Y) \circ (f'_X, p_Y)) &= p_X \circ (f_X, p_Y) \circ (f'_X, p_Y) \circ i_X \\ &= f_X \circ i_X \circ f'_{XX} = \Phi_X(f_X, p_Y) \circ \Phi_X(f'_X, p_Y)\end{aligned}$$

for $(f_X, p_Y), (f'_X, p_Y) \in \mathcal{A}^k_{Y,\#}(X \times Y)$. Furthermore, because the induced map

$$\pi_i(f_X, p_Y) = \begin{pmatrix} \pi_i(f_{XX}) & \pi_i(f_{XY}) \\ 0 & id_{\pi_i(Y)} \end{pmatrix}$$

is an isomorphism for $0 \leq i \leq k$, $\pi_i(f_{XX})$ is an isomorphism for $0 \leq i \leq k$. □

Let $\mathcal{A}^{X,k}_{Y,\#}(X \times Y)$ denote the submonoid of $\mathcal{A}^k_{Y,\#}(X \times Y)$, which consists of $(f_X, p_Y) \in \mathcal{A}^k_{Y,\#}(X \times Y)$ such that $(f_X, p_Y) \circ i_X = i_X$. Similarly, let $\mathcal{A}^{Y,k}_{X,\#}(X \times Y)$ denote the submonoid of $\mathcal{A}^k_{X,\#}(X \times Y)$ which consists of $(p_X, f_Y) \in \mathcal{A}^k_{X,\#}(X \times Y)$ such that $(p_X, f_Y) \circ i_Y = i_Y$. If $(g_X, p_Y) \in \mathrm{Ker}\Phi_X$, then $g_X \circ i_X = id_X$ (that is, $(g_X, p_Y) \circ i_X = i_X$). Therefore, $\mathrm{Ker}\Phi_X = \mathcal{A}^{X,k}_{Y,\#}(X \times Y)$. Consequently, we have the following lemma.

Lemma 5. *There exists a split short exact sequence of monoids*

$$1 \longrightarrow \mathcal{A}^{X,k}_{Y,\#}(X\times Y) \longrightarrow \mathcal{A}^{k}_{Y,\#}(X\times Y) \xrightarrow{\Phi_X} \mathcal{A}^{k}_{\#}(X) \longrightarrow 1,$$

where $\Phi_X(f_X, p_Y) = f_{XX}$.

Proof. Define $\sigma_X : \mathcal{A}^k_{\#}(X) \to \mathcal{A}^k_{Y,\#}(X\times Y)$ by $\sigma_X(f) = f \times id_Y$. Then σ_X is the section of Φ_X. $\square$

Lemma 6. $\mathcal{A}^{X,k}_{Y,\#}(X\times Y)$ *is trivial if and only if* $f_X \simeq f_{XX}\circ p_X$ *for each* $(f_X, f_Y) \in \mathcal{A}^k_{\#}(X\times Y)$.

Proof. Since $\mathcal{A}^{X,k}_{Y,\#}(X\times Y)$ is trivial, Φ_X is an isomorphism. Moreover,

$$\Phi_X(f_{XX}\circ p_X, p_Y) = f_{XX}\circ p_X \circ i_X = f_{XX} = \Phi_X(f_X, p_Y).$$

Therefore, $f_X \simeq f_{XX}\circ p_X$.

Conversely, suppose that $f_X \simeq f_{XX}\circ p_X$. Then Φ_X is a monoid monomorphism because $\Phi_X(f_X, p_Y) = id_X$ implies $f_X = p_X$. According to Lemma 5, $\mathcal{A}^{X,k}_{Y,\#}(X\times Y)$ is trivial. $\square$

Theorem 2. *Assume that* $\mathcal{A}^{X,k}_{Y,\#}(X\times Y)$ *is trivial and that all elements of* $\mathcal{A}^k_{\#}(X\times Y)$ *are* k*-reducible. Then, there is a split short exact sequence of monoids*

$$1 \longrightarrow \mathcal{A}^{Y,k}_{X,\#}(X\times Y) \longrightarrow \mathcal{A}^{k}_{\#}(X\times Y) \xrightarrow{\Phi} \mathcal{A}^{k}_{\#}(X)\times \mathcal{A}^{k}_{\#}(Y) \longrightarrow 1,$$

where Φ *is given by* $\Phi(f) = (f_{XX}, f_{YY})$ *for each* $f \in \mathcal{A}^k_{\#}(X\times Y)$.

Proof. Because each $f \in \mathcal{A}^k_{\#}(X\times Y)$ is k-reducible, the function Φ is well-defined. Moreover, because $f_{XX}\circ p_X \simeq f_X$ for $(f_X, f_Y) \in \mathcal{A}^k_{\#}(X\times Y)$ according to Lemma 6 and $p_X \circ i_Y = *$, we have $\Phi((f_X, f_Y)\circ(f'_X, f'_Y)) = \Phi((f_{XX}\circ p_X, f_Y)\circ(f'_{XX}\circ p_X, f'_Y)) = \Phi((f_{XX}\circ p_X\circ(f'_{XX}\circ p_X, f'_Y), f_Y\circ(f'_{XX}\circ p_X, f'_Y)) = \Phi((f_{XX}\circ f'_{XX}\circ p_X, f_Y\circ(f'_{XX}\circ p_X, f'_Y)) = ((f_{XX}\circ f'_{XX}\circ p_X)\circ i_X, f_Y\circ(f'_{XX}\circ p_X, f'_Y)\circ i_Y) = (f_{XX}\circ f'_{XX}, f_{YY}\circ f'_{YY}) = \Phi(f_X, f_Y)\circ\Phi(f'_X, f'_Y)$ for $(f_X, f_Y), (f'_X, f'_Y) \in \mathcal{A}^k_{\#}(X\times Y)$. Therefore, Φ is a homomorphism.

Clearly, $\mathrm{Ker}\Phi_Y = \mathcal{A}^{Y,k}_{X,\#}(X\times Y)$. Furthermore, if we define $\sigma : \mathcal{A}^k_{\#}(X)\times \mathcal{A}^k_{\#}(Y) \to \mathcal{A}^k_{\#}(X\times Y)$ by $\sigma(g, g') = g\times g'$, σ is clearly a homomorphism and the section of Φ. $\square$

From Theorem 2 and Lemma 5, we arrive at the following corollary.

Corollary 5. *If both* $\mathcal{A}^{X,k}_{Y,\#}(X\times Y)$ *and* $\mathcal{A}^{Y,k}_{X,\#}(X\times Y)$ *are trivial and all elements of* $\mathcal{A}^k_{\#}(X\times Y)$ *are* k*-reducible, then* $\mathcal{A}^k_{\#}(X\times Y)$ *is isomorphic to the external direct product* $\mathcal{A}^k_{X,\#}(X\times Y)\times\mathcal{A}^k_{Y,\#}(X\times Y)$.

4. Self-closeness number of product spaces

In this section, we discuss the relationship between the k-reducibility and the self-closeness number introduced by Choi and Lee [5].

Lemma 7. *Let* $(f_X, f_Y) \in \mathcal{A}^k_\#(X \times Y)$*. If* $f_X \simeq f_{XX} \circ p_X$ $(f_Y \simeq f_{YY} \circ p_Y)$*, then* f_{XY} (f_{YX}) *is null homotopic.*

Proof. Clearly, $f_{XY} \simeq f_X \circ i_Y \simeq f_{XX} \circ p_X \circ i_Y = *$. □

Theorem 3. *Let* $f = (f_X, f_Y) \in \mathcal{A}^k_\#(X \times Y)$*. If* $f_{XY} \simeq *$ *and* $f_{YX} \simeq *$*, then* f *is* k*-reducible.*

Proof. According to the hypothesis, the induced homomorphisms $\pi_i(f_{XY})$ and $\pi_i(f_{YX})$ are trivial. If

$$\Phi_i = \begin{pmatrix} \varphi_{XX} & \varphi_{XY} \\ \varphi_{YX} & \varphi_{YY} \end{pmatrix}$$

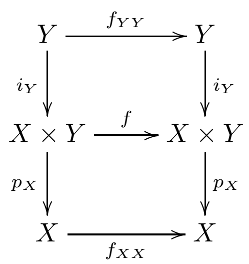
is the inverse homomorphism of $\pi_i(f)$ for $0 \leq i \leq k$, then the homomorphisms φ_{XX} and φ_{YY} are inverse homomorphisms of $\pi_i(f_{XX})$ and $\pi_i(f_{YY})$, respectively. Therefore, f is k-reducible. □

From Theorem 3, we arrive at the following corollary.

Corollary 6. *If for all* $f = (f_X, f_Y) \in \mathcal{A}^k_\#(X \times Y)$*,* $f_X \simeq f_{XX} \circ p_X$ *and* $f_Y \simeq f_{YY} \circ p_Y$*, then* $\mathcal{A}^k_\#(X \times Y) \cong \mathcal{A}^k_{Y,\#}(X \times Y) \times \mathcal{A}^k_{X,\#}(X \times Y) \cong \mathcal{A}^k_\#(X) \times \mathcal{A}^k_\#(Y)$*; moreover,* $\mathcal{A}^k_\#(X \times Y)$ *is the internal direct product of* $\mathcal{A}^k_{X,\#}(X \times Y)$ *and* $\mathcal{A}^k_{Y,\#}(X \times Y)$*.*

Proof. From Corollary 5, $\mathcal{A}^k_\#(X \times Y) \cong \mathcal{A}^k_{Y,\#}(X \times Y) \times \mathcal{A}^k_{X,\#}(X \times Y)$. Moreover, $\mathcal{A}^{X,k}_{Y,\#}(X \times Y)$ and $\mathcal{A}^{Y,k}_{X,\#}(X \times Y)$ are trivial according to Lemma 6. Therefore, $\mathcal{A}^k_\#(X \times Y) \cong \mathcal{A}^k_\#(X) \times \mathcal{A}^k_\#(Y)$ in agreement with Lemma 5. □

For given spaces X and Y, let $f : X \times Y \to X \times Y$ be a map such that $f_X \simeq f_{XX} \circ p_X$. Because the projection map $p_X : X \times Y \to X$ is a fibration, we obtain the following commutative diagram of fibrations:



In fact,

$$f \circ i_Y = (f_{XY}, f_{YY}) \simeq (*, f_{YY}) = i_Y \circ f_{YY}.$$

Conversely, let $g : X \to X$ and $h : Y \to Y$ be maps such that $g \circ p_X \simeq p_X \circ f$ and $i_Y \circ h \simeq f \circ i_Y$. Because $p_X \circ i_X = id_X$, $g \simeq p_X \circ f \circ i_X = f_{XX}$. Similarly, $h \simeq f_{YY}$. Therefore, $f_{XX} : X \to X$ and $f_{YY} : Y \to Y$ are representatives such that the above diagram is homotopy commutative for any $f : X \times Y \to X \times Y$.

Consider the commutative ladder of homotopy groups induced from the above diagram:

$$\begin{array}{ccccccccccc} \cdots \to & \pi_{k+1}(X) & \to & \pi_k(Y) & \to & \pi_k(X \times Y) & \to & \pi_k(X) & \to & \pi_{k-1}(Y) & \to \cdots \\ & \downarrow{\scriptstyle \pi_{k+1}(f_{XX})} & & \downarrow{\scriptstyle \pi_k(f_{YY})} & & \downarrow{\scriptstyle \pi_k(f)} & & \downarrow{\scriptstyle \pi_k(f_{XX})} & & \downarrow{\scriptstyle \pi_{k-1}(f_{YY})} & \\ \cdots \to & \pi_{k+1}(X) & \to & \pi_k(Y) & \to & \pi_k(X \times Y) & \to & \pi_k(X) & \to & \pi_{k-1}(Y) & \to \cdots \end{array}$$

Using this commutative ladder, we will prove Theorem 4.

First, we recall the closeness number introduced by Choi and Lee [5]. The *self-closeness number of* X denoted by $N\mathcal{E}(X)$ is the least nonnegative integer k such that $\mathcal{E}(X) = \mathcal{A}^k_\#(X)$. That is,

$$N\mathcal{E}(X) = \min\{k \mid \mathcal{E}(X) = \mathcal{A}^k_\#(X) \text{ for } k \geq 0\}.$$

Lemma 8 ([5, Theorem 2])**.** *If X is a CW-complex with dimension n, then*

$$N\mathcal{E}(X) \leq n.$$

Lemma 9 ([5, Theorem 3])**.** *Let X and Y be CW-complexes. Then, we have*

$$N\mathcal{E}(X \times Y) \geq \max\{N\mathcal{E}(X), N\mathcal{E}(Y)\}.$$

Theorem 4. *Let X and Y be CW-complexes. If each map $f : X \times Y \to X \times Y$ satisfies the conditions $f_X \simeq f_{XX} \circ p_X$ and $f_{YX} \simeq *$, then*

$$N\mathcal{E}(X \times Y) = \max\{N\mathcal{E}(X), N\mathcal{E}(Y)\}.$$

Proof. Let $N\mathcal{E}(X) = m$ and $N\mathcal{E}(Y) = n$. We assume $m \geq n$. For each $l \geq m$, let $f \in \mathcal{A}^l_\#(X \times Y)$. Then, we have the commutative ladder mentioned above. According to Lemma 7 and Theorem 3, f is l-reducible. Therefore, $f_{XX} \in \mathcal{A}^l_\#(X) \subset \mathcal{A}^m_\#(X)$ and $f_{YY} \in \mathcal{A}^l_\#(Y) \subset \mathcal{A}^n_\#(Y)$. According to the definition of the self-closeness number, $\mathcal{A}^m_\#(X) = \mathcal{E}(X)$ and $\mathcal{A}^n_\#(Y) = \mathcal{E}(Y)$. Therefore, $\pi_k(f_{XX})$ and $\pi_k(f_{YY})$ are automorphisms for all $k \geq 0$. By the Five Lemma, $\pi_k(f)$ is also an automorphism for all $k \geq 0$ in the homotopy commutative ladder. Therefore, f is a homotopy equivalence according to the Whitehead theorem. This implies that $f \in \mathcal{A}^l_\#(X \times Y) = \mathcal{E}(X \times Y)$ for each $l \geq m$. Therefore, $N\mathcal{E}(X \times Y) = m = \max\{N\mathcal{E}(X), N\mathcal{E}(Y)\}$ in accordance with Lemma 9 and the minimality of the self-closeness number. ☐

From Lemma 3, Corollary 3, Theorem 4, and [5, Corollary 2], we obtain the following corollaries.

Corollary 7. *Let X and Y be CW-complexes with $[X \wedge Y, X] = 0$. If $[X, Y] = 0$ and $[Y, X] = 0$, then $N\mathcal{E}(X \times Y) = \max\{N\mathcal{E}(X), N\mathcal{E}(Y)\}$.*

From Corollary 7 and [5, Corollary 2], we obtain the following corollary.

Corollary 8. *Let $m \neq n$. Then,* $N\mathcal{E}(S^m \times S^n) = \max\{m,n\}$ *provided that* $\pi_{m+n}(S^{\min\{m,n\}}) = 0$ *and* $\pi_{\max\{m,n\}}(S^{\min\{m,n\}}) = 0$.

Therefore, if $1 < n$, then $N\mathcal{E}(S^1 \times S^n) = n$. Furthermore, $N\mathcal{E}(S^{12} \times S^7) = 12$ because $\pi_{19}(S^7) = 0$ and $\pi_{12}(S^7) = 0$. Similarly, $N\mathcal{E}(S^8 \times S^{12}) = 12$.

Suppose that X and Y are group-like spaces. Consider the cofibration

$$X \vee Y \xrightarrow{j} X \times Y \xrightarrow{q} X \wedge Y$$

and the short exact sequence of additive groups of homotopy classes obtained from the cofibration:

$$0 \longrightarrow [X \wedge Y, X \times Y] \xrightarrow{q^\sharp} [X \times Y, X \times Y] \xrightarrow{j^\sharp} [X \vee Y, X \times Y] \longrightarrow 0\ .$$

All elements of $[X \vee Y, X \times Y]$ can be identified with the 2×2 matrix

$$(f_{IJ}) = \begin{pmatrix} f_{XX} & f_{XY} \\ f_{YX} & f_{YY} \end{pmatrix}$$

with entries f_{IJ} in the homotopy sets $[I, J]$ for $I, J = X, Y$. In [11, Corollary 7], it was shown that if $[X \wedge Y, X \times Y] = 0$, the group of self-homotopy equivalences of $X \times Y$ is $GL(2, \Lambda_{IJ})$ contained in $[X \vee Y, X \times Y]$, the group of invertible matrices with entries $f_{IJ} \in \Lambda_{IJ} = [I, J]$ for $I, J = X, Y$.

Theorem 5. *Let X and Y be group-like spaces such that $[X \wedge Y, X \times Y] = 0$ and $[Y, X] = 0$. If f is a self-map of $X \times Y$ such that $f_{XX} \in \mathcal{E}(X)$, $f_{YY} \in \mathcal{E}(Y)$ and $(f_{XX})^{-1}$ are H-maps, then f is a self-homotopy equivalence.*

Proof. Let f be a self-map of $X \times Y$ such that $f_{XX} \in \mathcal{E}(X)$, $f_{YY} \in \mathcal{E}(Y)$ and $(f_{XX})^{-1}$ are H-maps. Under the condition $[Y, X] = 0$, each element (f_{IJ}) in $[X \vee Y, X \times Y]$ has a left inverse and a right inverse

$$\begin{pmatrix} (f_{XX})^{-1} & -(f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} \\ 0 & (f_{YY})^{-1} \end{pmatrix}$$

and

$$\begin{pmatrix} (f_{XX})^{-1} & (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1} \\ 0 & (f_{YY})^{-1} \end{pmatrix},$$

respectively. Therefore, if $-(f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} = (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1}$, $[X \vee Y, X \times Y] = GL(2, \Lambda_{IJ})$. Let m and a be the multiplication and the homotopy inverse of X, respectively. Then,

$$\begin{aligned}
* &= (f_{XX})^{-1} \circ * \circ f_{XY} \\
&= (f_{XX})^{-1} \circ m(id \times a) \circ (f_{XY} \times f_{XY})\Delta \\
&= (f_{XX})^{-1} \circ m(f_{XY} \times (a \circ f_{XY}))\Delta \\
&= m((f_{XX})^{-1} \times (f_{XX})^{-1})(f_{XY} \times ((a \circ f_{XY}))\Delta \\
&= m(((f_{XX})^{-1} \circ f_{XY}) \times ((f_{XX})^{-1} \circ ((a \circ f_{XY})))\Delta,
\end{aligned}$$

where $\Delta : Y \to Y \times Y$ is the diagonal map. Therefore, we have

$$\begin{aligned}&((f_{XX})^{-1} \circ f_{XY} + (f_{XX})^{-1} \circ (-f_{XY}))(f_{YY})^{-1} \\ = &((f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} + (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1} = 0,\end{aligned}$$

and further, $-(f_{XX})^{-1} \circ f_{XY} \circ (f_{YY})^{-1} = (f_{XX})^{-1} \circ (-f_{XY}) \circ (f_{YY})^{-1}$. Consequently, there is a unique homotopy inverse for each (f_{IJ}) in $[X \vee Y, X \times Y]$. In accordance with [11, Corollary 7], f is a self-homotopy equivalence. $\square$

From Theorem 5, we obtain the following corollary.

Corollary 9. *For each pair of integers m and n such that $1 \leq m < n$ and the abelian groups G and H,*

$$N\mathcal{E}(K(G, m) \times K(H, n)) = n,$$

where $K(G, m)$ and $K(H, n)$ are Eilenberg-MacLane spaces.

Proof. Let $X = K(G, m)$ and $Y = K(H, n)$. Then, X and Y are group-like spaces and $[X \wedge Y, X \times Y] = 0$. For every map $f_{XX} \in [X, X]$, f_{XX} is an H-map because $X = K(G, m) = \Omega K(G, m + 1)$. Since $m < n$, $[Y, X] = 0$. According to Lemma 1, every element of $\mathcal{A}^k_\#(X \times Y)$ is k-reducible. Moreover, $\mathcal{A}^n_\#(X) = \mathcal{E}(X)$, $\mathcal{A}^n_\#(Y) = \mathcal{E}(Y)$, and $N\mathcal{E}(X \times Y) \geq \max\{N\mathcal{E}(X), N\mathcal{E}(Y)\} = n$ because $N\mathcal{E}(K(G, m)) = m < n = N\mathcal{E}(K(H, n))$. Therefore, $\mathcal{A}^n_\#(X \times Y) = \mathcal{E}(X \times Y)$ in accordance with Theorem 5. $\square$

References

[1] M. Arkowitz, *The group of self-homotopy equivalences - a survey*, Lecture Notes in Math. Vol. 1425, 170–203, Springer, New York, 1990.

[2] ———, *Introduction to Homotopy Theory*, Springer, New York Dordrecht Heidelberg London, 2010.

[3] P. I. Booth and P. R. Heath, *On the groups $\mathcal{E}(X \times Y)$ and $\mathcal{E}^B_B(X \times_B Y)$*, In Group of Self-Equivalences and Related Topics, ed. R. A. Piccinini, 17–31, Springer LNM 1425, 1990.

[4] H. Choi and K. Lee, *Certain self homotopy equivalences on wedge product on Moore spaces*, Pacific J. Math. **272** (2014), no. 1, 35–57.

[5] ———, *Certain numbers on the groups of self-homotopy equivalences*, Topology Appl. **181** (2015), 104–111.

[6] P. R. Heath, *On the group $\mathcal{E}(X \times Y)$ of self homotopy equivalences of a product*, Quaestiones Math. **19** (1996), no. 3-4, 433–451.

[7] K. Lee, *The groups of self pair homotopy equivalences*, J. Korea Math. Soc. **43** (2006), no. 3, 491–506.

[8] P. Pavešić, *Self-homotopy equivalences of product spaces*, Proc. Roy. Soc. Edinburgh Sect. A **129** (1999), no. 1, 181–197.

[9] ———, *Self fibre homotopy equivalences of fibred products*, Topology Appl. **102** (2000), no. 2, 169–180.

[10] ———, *Reducibility of self-homotopy equivalences*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), no. 2, 389–413.

[11] A. J. Sieradski, *Twisted self-homotopy equivalences*, Pacific J. Math. **34** (1970), 789–802.

Sang Woo Jun
Department of Mathematics
Korea University
Seoul 136-701, Korea
E-mail address: junsangwoo81@gmail.com

Kee Young Lee
Department of Mathematics
Korea University
Sejong City 339-700, Korea
E-mail address: keyolee@korea.ac.kr