J. Korean Math. Soc. **54** (2017), No. 4, pp. 1209–1229
https://doi.org/10.4134/JKMS.j160423
pISSN: 0304-9914 / eISSN: 2234-3008

ANALYTICAL TECHNIQUES FOR SYSTEM OF TIME FRACTIONAL NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. We coupled the so-called Sumudu transform with the homotopy perturbation method (HPM) and the homotopy analysis method (HAM), which are called homotopy perturbation Sumudu transform method (HPSTM) and homotopy analysis Sumudu transform method (HASTM), respectively. Then we show how HPSTM and HASTM are more convenient than HPM and HAM by conducting a comparative analytical study for a system of time fractional nonlinear differential equations. A Maple package is also used to enhance the clarity of the involved numerical simulations.

1. Introduction, notations and preliminaries

The systems of fractional (non-integer) order nonlinear differential equations have been found to allow a greater degree of freedom in the mathematical models (see, *e.g.*, [2, 7, 31, 36]). Fractional calculus, the differentiation and integration of arbitrary order, arises naturally in various areas of science and engineering. Fractional calculus is also a tool for modeling phenomena associated with non-locality and genetic effects (see, *e.g.*, [28, 39, 43, 45]). Homotopy perturbation Sumudu transform method (HPSTM) is a method that couples Sumudu transform with homotopy perturbation method and He's polynomials (see [23–25]). Also homotopy analysis Sumudu transform method (HASTM) is a coupling of Sumudu transform with classical homotopy analysis method. Homotopy perturbation method (HPM) was proposed by He (see [23–25]) and has been used to solve a number of problems in various fields of science and engineering (see, *e.g.*, [21, 22, 51]). Watugala [49] introduced Sumudu transform, which has turned out to play a very important role in solving a variety of problems such as ordinary differential equations, partial differential equations, and fractional differential and integral equations (see, *e.g.*, [5, 17–20, 29, 30]). Recently the HPSTM has been applied for solving linear and nonlinear equations

Received June 13, 2016; Revised November 11, 2016.
2010 *Mathematics Subject Classification.* 34A08, 35A20, 35A22.
Key words and phrases. nonlinear differential equations, homotopy perturbation method, homotopy analysis method, Sumudu transform.

including (for example) fractional heat-like equations, fractional heat and wave-like equations, fractional gas dynamics equation (see, *e.g.*, [10,46,47]). Homotopy analysis method (HAM) was first introduced and applied by Liao [32], who has given a systematic and basic idea of the HAM, and carried out a comparative study with other analytical methods, and applied in science and engineering (see [33–35,53]).

The HAM is a recently developed analytical technique, which has been applied to such many problems as nonlinear heat transfer, the American put option, the option pricing under stochastic volatility, linear and nonlinear diffusion wave, space-time advection-dispersion, the electro hydrodynamic flows, and the Poisson–Boltzmann equation for semiconductor devices (see, *e.g.*, [1, 27,38,40–42,52]).

Here, in this paper, we show how the HPSTM and the HASTM are more efficient and convenient than the HPM and the HAM by carrying out a comparative analytical study for system of time fractional nonlinear differential equations. In fact, unlike HPM, HPSTM is uniformly valid for either small or large parameters and variables; also HASTM is more convenient than HAM in that the differentiation property of Sumudu transform is used in HASTM without the assumption of auxiliary linear operator in HAM. A Maple package is used to enhance the clarity of the involved numerical simulations.

For our purpose, we recall some definitions with some of their properties. In the following, let $\mathbb{C}$ and $\mathbb{N}$ be the sets of complex numbers and positive integers, respectively, and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\alpha \in \mathbb{C}$ with $\Re(\alpha) \geq 0$ and $[a,\, b]$ be a finite interval of the real line $\mathbb{R} := (-\infty,\, \infty)$. Also let

$$n = [\Re(\alpha)] + 1 \quad (\alpha \notin \mathbb{N}_0)\,; \quad n = \alpha \quad (\alpha \in \mathbb{N}_0)\,. \tag{1.1}$$

- Among various fractional derivatives, the fractional derivative of $f(t)$ in the Caputo sense is defined as follows (see, *e.g.*, [28, Section 2.4]; see also [15]):

$$\left(D^{\alpha}_{a+} f\right)(t) := \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{1.2}$$

where $f^{(n)}(\tau)$ is the usual derivative of integer order n.

Very recently, by modifying the Caputo fractional derivative (1.2) when $n = 1$, Caputo and Fabrizio [16] have proposed a new non-local fractional derivative able to describe material heterogeneities and structures with different scales, which cannot be well described by classical local theories:

$$\mathcal{D}^{(\alpha)}_t f(t) = \frac{M(\alpha)}{1-\alpha} \int_a^t f'(\tau) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{1.3}$$

$$\left(\alpha \in [0,1],\ a \in [-\infty, t),\ f \in H^1(a,b),\ b > a\right),$$

where $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$. It is noted that, contrary to the definition (1.2), the kernel in (1.3) does not have singularity at $t = \tau$.

They [16] also showed that the definition (1.3) can be applied to functions that do not belong to $H^1(a,b)$ as follows:

$$\mathcal{D}_t^{(\alpha)} f(t) = \frac{\alpha\, M(\alpha)}{1-\alpha} \int_{-\infty}^{t} (f(t) - f(\tau)) \exp\left[-\frac{\alpha(t-\tau)}{1-\alpha}\right] d\tau \tag{1.4}$$

$$\left(\alpha \in [0,1],\ f \in L^1(-\infty, b)\right).$$

After the introduction of the new fractional derivative in (1.3), Losada and Nieto [37] proposed the following associative fractional integral (anti-derivative) of order α of a function f (see also [6]):

$$I_\alpha^t\,(f(t)) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)}\, f(t) + \frac{2\alpha}{(2-\alpha)\, M(\alpha)} \int_0^t f(\tau)\, d\tau \tag{1.5}$$

$$(0 < \alpha < 1,\ t \geq 0).$$

Consider the following generalized Mittag-Leffler function:

$$E_\alpha\left(t^\alpha\right) = \sum_{k=0}^{\infty} \frac{t^{\alpha k}}{\Gamma(\alpha k + 1)}. \tag{1.6}$$

Atangana and Baleanu [9] have proposed the following new fractional derivative:

$${}^{ABC}_{\ \ \ b}D_t^\alpha\,(f(t)) = \frac{B(\alpha)}{1-\alpha} \int_b^t f'(\tau)\, E_\alpha\left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha}\right] d\tau \tag{1.7}$$

$$\left(\alpha \in [0,1],\ f \in H^1(a,b),\ b > a\right),$$

where $B(\alpha)$ has the same properties as those of $M(\alpha)$ in (1.3).

It is noted that the case $\alpha = 0$ in (1.7) does not recover the original function except when $f(b) = 0$. To revise this issue, they [9] have also proposed the following definition:

$${}^{ABR}_{\ \ \ b}D_t^\alpha\,(f(t)) = \frac{B(\alpha)}{1-\alpha}\, \frac{d}{dt} \int_b^t f(\tau)\, E_\alpha\left[-\alpha \frac{(t-\tau)^\alpha}{1-\alpha}\right] d\tau. \tag{1.8}$$

In connection with the new fractional derivative (1.7), Atangana and Koca [11] proposed the following associate fractional integral (anti-derivative) with nonlocal kernel of a function f (see also [4]):

$${}^{AB}_{\ \ a}I_t^\alpha\{f(t)\} = \frac{1-\alpha}{B(\alpha)}\, f(t) + \frac{\alpha}{B(\alpha)\,\Gamma(\alpha)} \int_a^t f(\tau)\,(t-\tau)^{\alpha-1}\, d\tau. \tag{1.9}$$

For more detailed properties and various applications of the newly-proposed fractional derivatives in (1.3), (1.4), (1.7), and (1.8), one may refer to the works [3, 4, 6, 8, 9, 11, 16, 37] and the references therein.

- Suppose that $f(t)$ is a real- or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter. The Laplace transform of the function $f(t)$ is defined by

$$F(s) = (\mathcal{L}f)(s) = \mathcal{L}\{f(t) : s\} = \int_0^\infty e^{-st} f(t)\, dt = \lim_{\tau \to \infty} \int_0^\tau e^{-st} f(t)\, dt, \tag{1.10}$$

whenever the limit exits (as a finite number). Then the Laplace transform of the Caputo derivative is given as follows (see, *e.g.*, [28, Eq. (2.4.62)]; see also [15]):

$$\mathcal{L}\left(D_{0+}^{\alpha} f\right)(s) = s^{\alpha} (\mathcal{L}f)(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \left(D^k f\right)(0), \tag{1.11}$$

where $\alpha > 0$ and $n - 1 < \alpha \leq n$ $(n \in \mathbb{N})$.

- The so-called Sumudu transform is an integral transform which was defined and studied by Watugala [50] to facilitate the process of solving differential and integral equations in the time domain. The Sumudu transform has been applied to various problems in engineering and applied physics. For some fundamental properties of the Sumudu transform, one may refer to the works including (for example) [5, 13, 14, 50]. It turns out that the Sumudu transform has very special properties which are useful in solving problems in science and engineering.

  Let $\mathfrak{A}$ be the class of exponentially bounded functions $f : \mathbb{R} \to \mathbb{R}$, that is,

$$|f(t)| < \begin{cases} M \exp\left(-\dfrac{t}{\tau_1}\right) & (t \leqq 0) \\ M \exp\left(\dfrac{t}{\tau_2}\right) & (t \geqq 0), \end{cases} \tag{1.12}$$

where M, τ_1 and τ_2 are some positive real constants. The Sumudu transform defined on the set $\mathfrak{A}$ is given by the following formula (see [50]; see also [17])

$$G(u) = \mathcal{S}[f(t); u] = \int_0^\infty e^{-t} f(ut)\, dt \qquad (-\tau_1 < u < \tau_2). \tag{1.13}$$

The Sumudu transform given in (1.13) can also be derived directly from the Fourier integral. Moreover, it can be easily verified that the Sumudu transform is a linear operator and the function $G(u)$ in (1.13) keeps the same units as $f(t)$; that is, for any real or complex number λ, we have

$$\mathcal{S}[f(\lambda t); u] = G(\lambda u).$$

The Sumudu transform $G(u)$ and the Laplace transform $F(s)$ exhibit a duality relation that may be expressed as follows:

$$G\left(\frac{1}{s}\right) = s\,F\,(s) \qquad \text{or} \qquad G\,(u) = \frac{1}{u}\,F\left(\frac{1}{u}\right). \tag{1.14}$$

The Sumudu transform has been shown to be the theoretical dual of the Laplace transform. The Sumudu transform of the Caputo fractional derivative of a function $f(t)$ can be given as follows (see, *e.g.*, [14, Theorem 2.2]):

$$\mathcal{S}\left[D_{0+}^{\alpha} f(t); u\right] = u^{-\alpha}\,\mathcal{S}\left[f(t); u\right] - \sum_{k=0}^{n-1} u^{-\alpha+k}\,\left(D^k f\right)(0), \tag{1.15}$$

where $\alpha > 0$ and $n - 1 < \alpha \leq n$ $(n \in \mathbb{N})$.

- Let f be a function of l variables t_i $(i = 1, \ldots, l)$ defined on a domain $D \subset \mathbb{R}^l$. Then define a partial Caputo fractional derivative of order α of the function f with respect to t_i by

$$\partial_{t_i}^{\alpha} f := \frac{1}{\Gamma(n-\alpha)} \int_a^{t_i} (t_i - \tau)^{n-\alpha-1} \partial_{t_i}^{n} f(x)\, d\tau, \tag{1.16}$$

where $x := (t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)$ and α, n are given as in (1.1). Here $\partial_{t_i}^{n}$ is the usual partial derivative of integer order n.

It may be remarked in passing that, throughout this paper, the fractional derivative is the Caputo fractional derivative.

2. Basic idea of HPSTM

We illustrate the basic idea of HPSTM by considering a general time-fractional nonlinear non-homogeneous partial differential equation with the initial condition of the general form:

$$\left(D^{\alpha} y\right)(t) = \mathcal{R}\,y + \mathcal{N}\,y + g(x,t) \quad (\alpha > 0,\ n-1 < \alpha \leq n\ (n \in \mathbb{N})), \tag{2.1}$$

with the following initial conditions:

$$\left(D^{m} y\right)(0) = f_m(x)\ (m = 0, \ldots, n-1),\ \left(D^{n} y\right)(0) = 0,\ n = [\alpha],$$

where $y := y(x,t)$, $\left(D^{\alpha} y\right)(t)$ is the Caputo fractional derivative of the function $y(x,t)$ without the specified starting point $a+$ as in (1.2), $\mathcal{R}$ is the linear differential operator, $\mathcal{N}$ represents the general nonlinear differential operator. Here, by applying the Sumudu transform on both sides of (2.1), we get

$$\mathcal{S}\left[\left(D^{\alpha} y\right)(t); u\right] = \mathcal{S}[\mathcal{R}\,y; u] + \mathcal{S}[\mathcal{N}\,y; u] + \mathcal{S}[g(x,t); u], \tag{2.2}$$

which, upon using a property of the Sumudu transform (see (1.15)), yields

$$\mathcal{S}[y; u] = f(x,u) + u^{\alpha}\,\mathcal{S}[\mathcal{R}y; u] + u^{\alpha}\,\mathcal{S}[\mathcal{N}y; u] + u^{\alpha}\mathcal{S}[g(x,t); u], \tag{2.3}$$

where $f(x,u)$ is a function of x and u. Taking the inverse of Sumudu transform on both sides of (2.3), we get

$$y = F(x,t) + \mathcal{S}^{-1}\left(u^{\alpha}\,\mathcal{S}[\mathcal{R}y;u]\right) + \mathcal{S}^{-1}\left(u^{\alpha}\,\mathcal{S}[\mathcal{N}y;u]\right), \tag{2.4}$$

where $F(x,t)$ is a function of x and t given by

$$F(x,t) := \mathcal{S}^{-1}\left(f(x,u) + u^{\alpha}\mathcal{S}[g(x,t);u]\right),$$

which may be concretely determined, since $g(x,t)$ is a known function.

Now we apply the HPM to the equation (2.1) to get the solution (see, *e.g.*, [23–25]):

$$y(x,t) \;=\; \sum_{n=0}^{\infty} p^n y_n(x,t), \tag{2.5}$$

where $p \in [0, 1]$ is an embedding (or homotopy) parameter, y_0 is an initial approximation which satisfies the boundary conditions, and y_n $(n \in \mathbb{N})$ are the nth order approximations which are functions yet to be determined. It is noted that setting $p = 1$ in (2.5) gives an approximate solution of the given nonlinear differential equation (2.1).

The nonlinear term in (2.1) can be decomposed as follows (see, *e.g.*, [26,44]):

$$\mathcal{N}y(x,t) \;=\; \sum_{n=0}^{\infty} p^n\, H_n\left(y_0,\ldots,y_n\right), \tag{2.6}$$

where H_n are the He's polynomials given by

$$H_n(y_0,\ldots,y_n) \;=\; \frac{1}{n!}\frac{\partial^n}{\partial p^n}\mathcal{N}\left(\sum_{r=0}^{n} p^r y_r(x,t)\right) \quad (n \in \mathbb{N}_0). \tag{2.7}$$

Substituting (2.5) and (2.6) in (2.4) gives

$$\sum_{n=0}^{\infty} p^n y_n = F(x,t) + p\,\mathcal{S}^{-1}\left(u^{\alpha}\mathcal{S}\left[\mathcal{R}\left(\sum_{n=0}^{\infty} p^n y_n\right)\right] + u^{\alpha}\mathcal{S}\left[\sum_{n=0}^{\infty} p^n\, H_n\right]\right). \tag{2.8}$$

We have completed the coupling of the Sumudu transform and the HPM using He's polynomials. Comparing the coefficients of same power of p in (2.8), we obtain the following approximations for y_n $(n \in \mathbb{N}_0)$:

$$p^0 : \; y_0 \;=\; F(x,t),$$

$$p^1 : \; y_1 \;=\; \mathcal{S}^{-1}\left(u^{\alpha}\,\mathcal{S}\left[\mathcal{R}(y_0) + H_0(y_0)\right]\right),$$

$$p^2 : \; y_2 \;=\; \mathcal{S}^{-1}\left(u^{\alpha}\,\mathcal{S}\left[\mathcal{R}(y_1) + H_1(y_0,y_1)\right]\right),$$

$$\vdots$$

$$p^n : \; y_n \;=\; \mathcal{S}^{-1}\left(u^{\alpha}\,\mathcal{S}\left[\mathcal{R}(y_{n-1}) + H_{n-1}(y_0,\ldots,y_{n-1})\right]\right) \quad (n \in \mathbb{N}). \tag{2.9}$$

Finally the approximate and analytical solution of (2.1) is given by truncating the following series:

$$y(x,t) = y_0(x,t) + \sum_{k=1}^{\infty} y_k(x,t). \tag{2.10}$$

3. Basic idea of HASTM

In order to illustrate the basic idea of HASTM, we consider a general fractional nonlinear partial differential equation of the form:

$$\mathcal{N}_\alpha\left[y(x,t)\right] \;=\; g(x,t), \tag{3.1}$$

where $\mathcal{N}_\alpha$ represents the general linear and nonlinear partial fractional differential operator, x denote an independent variable, $y(x,t)$ is an unknown function. The linear terms of $\mathcal{N}_\alpha$ are decomposed into $D^\alpha + \mathcal{R}$, where D^α is the highest order linear operator and $\left(D^\alpha y\right)(t)$ is the Caputo fractional derivative of the function $y(x,t)$ without the specified starting point $a+$ as in (1.2), and $\mathcal{R}$ is the remaining of the linear operator. The equation (3.1), therefore, can be written as follows:

$$\left(D^\alpha y\right)(t) + \mathcal{R}y + \mathcal{N}y = g(x,t) \quad (\alpha > 0,\ n-1 < \alpha \leq n\ (n \in \mathbb{N})), \tag{3.2}$$

where $\mathcal{N}y$ indicates the nonlinear terms.

Taking the Sumudu transform (1.13) on both sides of the equation (3.2), we get

$$S\left[\left(D^\alpha y\right)(t)\right] + \mathcal{S}\left[\mathcal{R}y\right] + \mathcal{S}\left[\mathcal{N}y\right] = \mathcal{S}\left[g(x,t)\right]. \tag{3.3}$$

Using the property of the Sumudu transform (1.15), we have

$$\frac{\mathcal{S}\left[y\right]}{u^\alpha} - \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{u^{\alpha-k}} + \mathcal{S}\left[\mathcal{R}y\right] + \mathcal{S}\left[\mathcal{N}y\right] = \mathcal{S}\left[g(x,t)\right]. \tag{3.4}$$

Or, equivalently,

$$\mathcal{S}\left[y\right] - u^\alpha \sum_{k=0}^{n-1} \frac{y^{(k)}(0)}{u^{\alpha-k}} + u^\alpha \left(S\left[Ry\right] + S\left[\mathcal{N}y\right] - S\left[g(x,t)\right]\right) = 0. \tag{3.5}$$

We define the nonlinear operator

$$\begin{aligned} \mathrm{N}\left[\delta(x,t;q)\right] := \mathcal{S}\left[\delta(x,t;q)\right] - u^\alpha \sum_{k=0}^{n-1} \frac{\delta^{(r)}(x,t;q)(0)}{u^{(\alpha-r)}} \\ + u^\alpha \left(\mathcal{S}\left[\mathcal{R}\delta(x,t;q)\right] + \mathcal{S}\left[\mathcal{N}\delta(x,t;q)\right] - \mathcal{S}\left[g(x,t)\right]\right), \end{aligned} \tag{3.6}$$

where $q \in [0,1]$ is a parameter, and $\delta(x,t;q)$ is a real-valued function of x, t and q. We construct a homotopy as follows:

$$(1-q)\, S\left[\delta(x,t\,;q) - y_0(x,t)\right] = \hbar\, q H(x,t)\, \mathrm{N}_\alpha\left[y(x,t\,)\right], \tag{3.7}$$

where $\mathcal{S}$ denotes the Sumudu transform, $q \in [0,\,1]$ is the embedding parameter, $H(x,t)$ depicts a nonzero auxiliary function, $\hbar \neq 0$ is an auxiliary parameter,

$y_0(x,t)$ is an initial guess of $y(x,t)$ and $\delta(x,t;q)$ is a unknown function. Obviously, choosing the embedding parameter $q=0$ and $q=1$ in (3.7) gives

$$\delta(x,t\,;0)=y_0(x,t) \quad \text{and} \quad \delta(x,t\,;1)=y(x,t), \tag{3.8}$$

respectively.

Thus, as q increases from 0 to 1, the solution $\delta(x,t;q)$ varies from the initial guess $y_0(x,t)$ to the solution $y(x,t)$. Expanding $\delta(x,t;q)$ as Taylor series with respect to q, we obtain

$$\delta(x,t;q)=y_0(x,t)+\sum_{m=1}^{\infty} y_m(x,t)\,q^m, \tag{3.9}$$

where

$$y_m(x,t)=\frac{1}{m\,!}\,\frac{\partial^m \delta(x,t\,;q)}{\partial q^m}\bigg|_{q=0}. \tag{3.10}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (3.9) may converges at $q=1$. Then we have

$$y(x,t)=y_0(x,t)+\sum_{m=1}^{\infty} y_m(x,t), \tag{3.11}$$

which should be one of the solutions of the original nonlinear equation (3.1).

By (3.11), the governing equation can be deduced from the zero-order deformation (3.7). Define vectors

$$\overrightarrow{y}_m := \{y_0(x,t),\, y_1(x,t),\, \ldots,\, y_m(x,t)\}. \tag{3.12}$$

Differentiating the zeroth-order deformation Eq. (3.7) m-times with respect to q and then dividing them by $m!$ and finally setting $q=0$, we have the following mth-order deformation equation:

$$S\left[y_m(x,t)-\chi_m y_{m-1}(x,t)\right]=\hbar\, H(x,t)\,\mathfrak{R}_m(\overrightarrow{y}_{m-1}), \tag{3.13}$$

where

$$\mathfrak{R}_m\left(\overrightarrow{y}_{m-1}\right)=\frac{1}{(m-1)!}\,\frac{\partial^{m-1}\,\mathrm{N}\left[\delta(x,t\,;q)\right]}{\partial q^{m-1}}\bigg|_{q=0} \tag{3.14}$$

and

$$\chi_m=\begin{cases} 0, & m\le 1,\\ 1, & m>1. \end{cases} \tag{3.15}$$

Taking the inverse Sumudu transform on both sides of (3.13), we obtain the following equation:

$$y_m(x,t)=\chi_m y_{m-1}(x,t)]+\hbar\,\mathcal{S}^{-1}\left[H(x,t)\,\mathfrak{R}_m(\overrightarrow{y}_{m-1})\right]. \tag{3.16}$$

It is noted that the success of the HASTM is based on a proper selection of the initial guess.

4. System of fractional nonlinear partial differential equations

In order to apply the HPSTM and the HASTM in Sections 2 and 3, respectively, consider the following simple system of nonlinear fractional differential equations:

$$\left\{\begin{array}{l}(D^\alpha a)(t)+b_x c_y-b_y c_x=-a,\\ (D^\alpha b)(t)+c_x a_y+c_y a_x=b,\\ (D^\alpha c)(t)+a_x b_y+a_y b_x=c,\end{array}\right. \tag{4.1}$$

with the initial conditions:

$$a(x,y,0)=e^{x+y},\quad b(x,y,0)=e^{x-y},\quad \text{and}\quad c(x,y,0)=e^{-x+y}, \tag{4.2}$$

where $\alpha \in (0,1]$ is a parameter describing the order of the time fractional derivative, D^α is the Caputo fractional derivative in (1.2). It is noted that

$$a(x,y,t)=e^{x+y-t},\quad b(x,y,t)=e^{x-y+t},\quad \text{and}\quad c(x,y,t)=e^{-x+y+t}$$

is an exact solution of the system of nonlinear differential equations (4.1) for $\alpha = 1$.

5. Use of the HPSTM

Taking Sumudu transform on the system (4.1) with the initial condition (4.2), we get

$$\left\{\begin{array}{l}\mathcal{S}[a]=e^{x+y}+u^\alpha\mathcal{S}[-b_x c_y+b_y c_x-a],\\ \mathcal{S}[b]=e^{x-y}+u^\alpha\mathcal{S}[-c_x a_y-c_y a_x+b],\\ \mathcal{S}[c]=e^{-x+y}+u^\alpha\mathcal{S}[-a_x b_y-a_y b_x+c].\end{array}\right. \tag{5.1}$$

Taking inverse of Sumudu transform on the system (5.1), we find

$$\left\{\begin{array}{l}a=e^{x+y}+\mathcal{S}^{-1}\left(u^\alpha\mathcal{S}\left[-b_x c_y+b_y c_x-a\right]\right),\\ b=e^{x-y}+\mathcal{S}^{-1}\left(u^\alpha\mathcal{S}\left[-c_x a_y-c_y a_x+b\right]\right),\\ c=e^{-x+y}+\mathcal{S}^{-1}\left(u^\alpha\mathcal{S}\left[-a_x b_y-a_y b_x+c\right]\right).\end{array}\right. \tag{5.2}$$

We then apply the HPM to get the following system:

$$\left\{\begin{array}{l}\sum_{m=0}^{\infty}p^m a_m=e^{x+y}+pS^{-1}\Big\{u^\alpha\Big(S\Big(-\sum_{m=0}^{\infty}p^m H_m(b_x c_y)\\ \qquad\qquad +\sum_{m=0}^{\infty}p^m H_m(b_y c_x)-\sum_{m=0}^{\infty}p^m a_m\Big)\Big)\Big\},\\ \sum_{m=0}^{\infty}p^m b_m=e^{x-y}+p\,S^{-1}\Big\{u^\alpha\Big(S\Big((-\sum_{m=0}^{\infty}p^m H_m(c_x a_y)\\ \qquad\qquad -\sum_{m=0}^{\infty}p^m H_m(c_y a_x)+\sum_{m=0}^{\infty}p^m b_m\Big)\Big)\Big\},\\ \sum_{m=0}^{\infty}p^m c_m=e^{-x+y}+pS^{-1}\Big\{u^\alpha\Big(S\Big(-\sum_{m=0}^{\infty}p^m H_m(a_x b_y)\\ \qquad\qquad -\sum_{m=0}^{\infty}p^m H_m(a_y b_x)+\sum_{m=0}^{\infty}p^m c_m\Big)\Big)\Big\}.\end{array}\right. \tag{5.3}$$

Comparing the coefficients of same powers of p, we have

$$p^0:\ a_0 = e^{x+y}, \quad p^0:\ b_0 = e^{x-y}, \quad p^0:\ c_0 = e^{-x+y};$$

$$\begin{cases} p^1: a_1 = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_0(b_x c_y) + H_0(b_y c_x) - a_0])\right\} \\ \qquad = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-b_{0,x}\, c_{0,y} + b_{0,y}\, c_{0,x} - a_0])\right\} \\ \qquad = -e^{x+y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ p^1: b_1 = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_0(c_x a_y) - H_0(c_y a_x) + b_0])\right\} \\ \qquad = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-c_{0,x}\, a_{0,y} - c_{0,y}\, a_{0,x} + b_0])\right\} \\ \qquad = e^{x-y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ p^1: c_1 = \mathcal{S}^{-1}\mathcal{S}\left\{u^\alpha([-H_0(a_x b_y) - H_0(a_y b_x) + c_0])\right\} \\ \qquad = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-a_{0,x}\, b_{0,y} - a_{0,y}\, b_{0,x} + c_0])\right\} \\ \qquad = e^{-x+y}\,\frac{t^\alpha}{\Gamma(\alpha+1)}; \end{cases}$$

$$\begin{cases} p^2: a_2 = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_1(b_x c_y) + H_1(b_y c_x) - a_1])\right\} \\ \qquad = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-b_{0,x}\, c_{1,y} - b_{1,x}\, c_{0,y} + b_{0,y}\, c_{1,x} + b_{1,y}\, c_{0,x} - a_1])\right\} \\ \qquad = e^{x+y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ p^2: b_2 = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_1(c_x a_y) - H_1(c_y a_x) + b_1])\right\} \\ \qquad = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-c_{0,x}\, a_{1,y} - c_{1,x}\, a_{0,y} - c_{0,y}\, a_{1,x} - c_{1,y}\, a_{0,x} + b_1])\right\} \\ \qquad = e^{x-y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ p^2: c_2 = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_1(a_x b_y) - H_1(a_y b_x) + c_1])\right\} \\ \qquad = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-a_{0,x}\, b_{1,y} - a_{1,x}\, b_{0,y} - a_{0,y}\, b_{1,x} - a_{1,y}\, b_{0,x} + c_1])\right\} \\ \qquad = e^{-x+y}\,\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}; \end{cases}$$

$$\vdots$$

$$\begin{cases} p^n: a_n = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_{n-1}(b_x c_y) + H_{n-1}(b_y c_x) - a_{n-1}])\right\} \\ \qquad = (-1)^n e^{x+y}\,\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ p^n: b_n = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_{n-1}(c_x a_y) - H_{n-1}(c_y a_x) + b_{n-1}])\right\} \\ \qquad = e^{x-y}\,\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \\ p^n: c_n = \mathcal{S}^{-1}\left\{u^\alpha(\mathcal{S}[-H_{n-1}(a_x b_y) - H_{n-1}(a_y b_x) + c_{n-1}])\right\} \\ \qquad = e^{-x+y}\,\frac{t^{n\alpha}}{\Gamma(n\alpha+1)}. \end{cases} \tag{5.4}$$

The series solution of (4.1) with the initial conditions (4.2) is, therefore, found to be given as follows:

$$\begin{cases} a(x,t) = a_0(x,t) + \sum_{m=1}^{\infty} a_m(x,t), \\ b(x,t) = b_0(x,t) + \sum_{m=1}^{\infty} b_m(x,t), \\ c(x,t) = c_0(x,t) + \sum_{m=1}^{\infty} c_m(x,t). \end{cases} \tag{5.5}$$

In particular, setting $\alpha = 1$ in (5.5), the solution is seen to converge to the exact solution:

$$a(x,t) = e^{x+y-t}, \quad b(x,t) = e^{x-y+t}, \quad \text{and} \quad c(x,t) = e^{-x+y+t}.$$

6. Use of the HASTM

Here we apply the HASTM to the system (4.1) subject to the initial condition (4.2).

In view of the result in Section 5, we may take the initial guess as follows:

$$a_0 = e^{x+y}, \quad b_0 = e^{x-y}, \quad \text{and} \quad c_0 = e^{-x+y}. \tag{6.1}$$

As in Section 5, we then obtain the same system as in (5.1):

$$\begin{cases} \mathcal{S}[a] = e^{x+y} + u^\alpha \mathcal{S}[-b_x c_y + b_y c_x - a], \\ \mathcal{S}[b] = e^{x-y} + u^\alpha \mathcal{S}[-c_x a_y - c_y a_x + b], \\ \mathcal{S}[c] = e^{-x+y} + u^\alpha \mathcal{S}[-a_x b_y - a_y b_x + c]. \end{cases} \tag{6.2}$$

We define the involved nonlinear operator as follows:

$$\begin{cases} \mathcal{N}^1 = \mathcal{S}[\delta_1(x,y,t;q)] - e^{x+y} + u^\alpha \big(\mathcal{S}[\delta_2(x,y,t;q)_x \delta_3(x,y,t;q)_y \\ \qquad\qquad -\delta_2(x,y,t;q)_y \delta_3(x,y,t;q)_x + \delta_1(x,y,t;q)]\big), \\ \mathcal{N}^2 = \mathcal{S}[\delta_2(x,y,t;q)] - e^{x-y} + u^\alpha \big(\mathcal{S}[\delta_3(x,y,t;q)_x \delta_1(x,y,t;q)_y \\ \qquad\qquad +\delta_3(x,y,t;q)_y \delta_1(x,y,t;q)_x - \delta_2(x,y,t;q)]\big), \\ \mathcal{N}^3 = \mathcal{S}[\delta_3(x,y,t;q)] - e^{-x+y} + u^\alpha \big(\mathcal{S}[\delta_1(x,y,t;q)_x \delta_2(x,y,t;q)_y \\ \qquad\qquad +\delta_1(x,y,t;q)_y \delta_2(x,y,t;q)_x - \delta_3(x,y,t;q)]\big), \end{cases} \tag{6.3}$$

where

$$\mathcal{N}^j := \mathcal{N}^j[\delta_1(x,y,t;q), \delta_2(x,y,t;q), \delta_3(x,y,t;q)] \quad (j = 1,\, 2,\, 3).$$

Then the m^{th}-order deformation equation is given by

$$\begin{cases} \mathcal{S}\left[a_m(x,y,t) - \chi_m a_{m-1}(x,y,t)\right] = \hbar \mathcal{R}_{1,m}\big[\underset{m-1}{\vec{a}}, \underset{m-1}{\vec{b}}, \underset{m-1}{\vec{c}}\big], \\ \mathcal{S}\left[b_m(x,y,t) - \chi_m b_{m-1}(x,y,t)\right] = \hbar \mathcal{R}_{2,m}\big[\underset{m-1}{\vec{a}}, \underset{m-1}{\vec{b}}, \underset{m-1}{\vec{c}}\big], \\ \mathcal{S}\big[c_m(x,y,t) - \chi_m c_{m-1}(x,y,t)\big] = \hbar \mathcal{R}_{3,m}\big[\underset{m-1}{\vec{a}}, \underset{m-1}{\vec{b}}, \underset{m-1}{\vec{c}}\big], \end{cases} \tag{6.4}$$

where

$$\begin{cases} \mathcal{R}_{1,m}\big[\underset{m-1}{\vec{a}}, \underset{m-1}{\vec{b}}, \underset{m-1}{\vec{c}}\big] = \mathcal{S}[a_{m-1}] - (1-\chi_m)\, e^{x+y} \\ \qquad + u^\alpha \left(\mathcal{S}\left[\sum_{i=0}^{m-1} b_{i,x} c_{m-1-i,y} - \sum_{i=0}^{m-1} b_{i,y}\, c_{m-1-i,x} + a_{m-1}\right]\right), \\ \mathcal{R}_{2,m}\big[\underset{m-1}{\vec{a}}, \underset{m-1}{\vec{b}}, \underset{m-1}{\vec{c}}\big] = \mathcal{S}[b_{m-1}] - (1-\chi_m)\, e^{x-y} \\ \qquad + u^\alpha \left(\mathcal{S}\left[\sum_{i=0}^{m-1} c_{i,x}\, a_{m-1-i,y} + \sum_{i=0}^{m-1} c_{i,y}\, a_{m-1-i,x} - b_{m-1}\right]\right), \\ \mathcal{R}_{3,m}\big[\underset{m-1}{\vec{a}}, \underset{m-1}{\vec{b}}, \underset{m-1}{\vec{c}}\big] = \mathcal{S}[c_{m-1}] - (1-\chi_m)\, e^{-x+y} \\ \qquad + u^\alpha \left(\mathcal{S}\left[\sum_{i=0}^{m-1} a_{i,x}\, b_{m-1-i,y} + \sum_{i=0}^{m-1} a_{i,y}\, b_{m-1-i,x} - c_{m-1}\right]\right). \end{cases} \tag{6.5}$$

Taking the inverse Sumudu transform on each equation in (6.4), we have

$$\text{(6.6)}\qquad \begin{cases} a_m\,(x,y,t) = \chi_m a_{m-1}(t) + \hbar S^{-1}\left\{\mathcal{R}_{1,m}[\vec{a}_{m-1}, \vec{b}_{m-1}, \vec{c}_{m-1}]\right\}, \\ b_m\,(x,y,t) = \chi_m b_{m-1}(t) + \hbar \mathcal{S}^{-1}\left\{\mathcal{R}_{2,m}[\vec{a}_{m-1}, \vec{b}_{m-1}, \vec{c}_{m-1}]\right\}, \\ c_m\,(x,y,t) = \chi_m c_{m-1}(t) + \hbar \mathcal{S}^{-1}\left\{\mathcal{R}_{3,m}[\vec{a}_{m-1}, \vec{b}_{m-1}, \vec{c}_{m-1}]\right\}. \end{cases}$$

Solving the above system (6.6), we get the following results:

$$a_0 = e^{x+y}, \quad b_0 = e^{x-y}, \quad c_0 = e^{-x+y};$$

$$a_1 = \hbar\, e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad b_1 = -\hbar\, e^{x-y}\frac{t^\alpha}{\Gamma(\alpha+1)}, \quad c_1 = -\hbar\, e^{-x+y}\frac{t^\alpha}{\Gamma(\alpha+1)};$$

$$\begin{cases} a_2 = (1+\hbar)\hbar e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ b_2 = -(1+\hbar)\hbar e^{x-y}\frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 e^{x-y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ c_2 = -(1+\hbar)\hbar e^{-x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 e^{-x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}; \end{cases}$$

(6.7)

$$\begin{cases} a_3 = (1+\hbar)^2\hbar e^{x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + 2(1+\hbar)\hbar^2 e^{x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \hbar^3 e^{x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ b_3 = -(1+\hbar)^2\hbar e^{x-y}\frac{t^\alpha}{\Gamma(\alpha+1)} + 2(1+\hbar)\hbar^2 e^{x-y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3 e^{x-y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}, \\ c_3 = -(1+\hbar)^2\hbar e^{-x+y}\frac{t^\alpha}{\Gamma(\alpha+1)} + 2(1+\hbar)\hbar^2 e^{-x+y}\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3 e^{-x+y}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}; \end{cases}$$

and so on. Likewise the rest of the components can be obtained. Thus the approximate series solutions can be given by

$$\text{(6.8)}\qquad \begin{cases} a(x,y,t) = a_0(x,y,t) + \sum_{m=1}^{\infty} a_m(x,y,t), \\ b(x,y,t) = b_0(x,y,t) + \sum_{m=1}^{\infty} b_m(x,y,t), \\ c(x,y,t) = c_0(x,y,t) + \sum_{m=1}^{\infty} c_m(x,y,t). \end{cases}$$

Setting $\hbar = -1$ in the solution system (6.8), it converts to the series solution in the HPSTM. Setting $\hbar = -1$ and $\alpha = 1$ in the solution system (6.8) is seen to converge to the exact solution

$$a(x,y,t) = e^{x+y-t}, \quad b(x,y,t) = e^{x-y+t}, \quad c(x,y,t) = e^{-x+y+t}.$$

It is easy to see that $b(x,y,t)$ and $c(x,y,t)$ for $x = y$ give the same series solution for any value t. For this reason, we present two-dimensional graphs for $b = c$. The graphical representations show that low-order approximate HPSTM and HASTM solutions lead to high accuracy. Here we use only 4th order approximations for solution simulation.

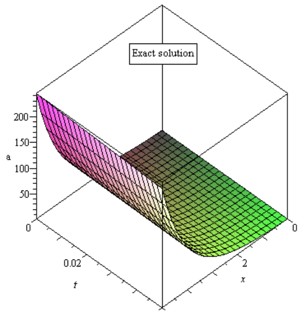

FIGURE 1. The behaviour of the exact solution $a(x, y, t)$ at $y = 0.5$ w.r.t. x and t.

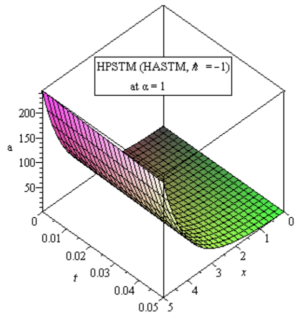

FIGURE 2. The behaviour of the HPSTM and HASTM ($\hbar = -1$) solutions $a(x, y, t)$ at $\alpha = 1$ and $y = 0.5$ w.r.t. x and t.

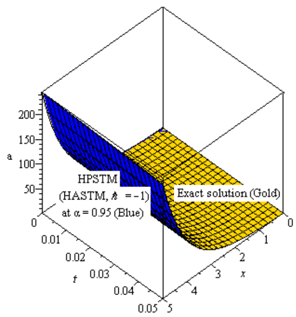

FIGURE 3. The comparison between of the exact, HPSTM and HASTM ($\hbar = -1$) solutions $a(x, y, t)$ at $\alpha = 0.95$ and $y = 0.5$ w.r.t. x and t.

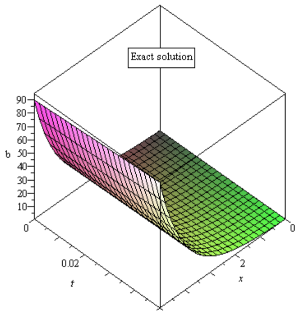

FIGURE 4. The behaviour of the exact solution $b(x, y, t)$ at $y = 0.5$ w.r.t. x and t.

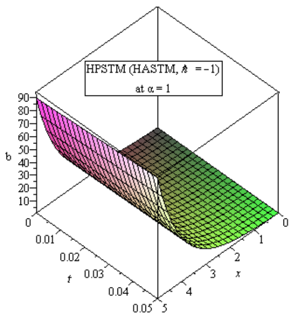

FIGURE 5. The behaviour of the HPSTM and HASTM ($\hbar = -1$) solutions $b(x, y, t)$ at $\alpha = 1$ and $y = 0.5$ w.r.t. x and t.

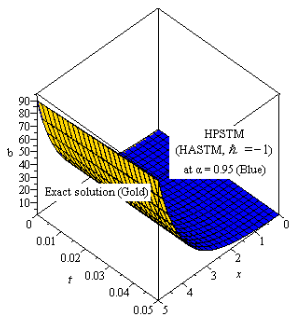

FIGURE 6. The comparison between of the exact, HPSTM and HASTM ($\hbar = -1$) solutions $b(x, y, t)$ at $\alpha = 0.95$ and $y = 0.5$ w.r.t. x and t.

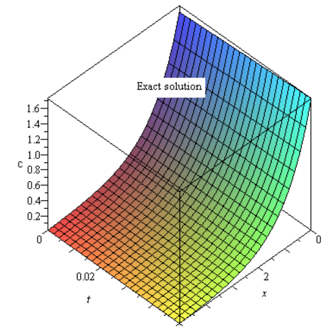

FIGURE 7. The behaviour of the exact solution $c(x, y, t)$ at $y = 0.5$ w.r.t. x and t.

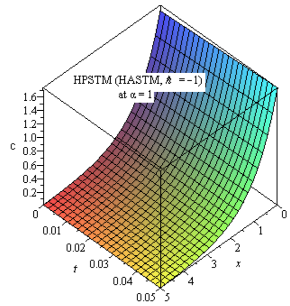

FIGURE 8. The behaviour of the HPSTM and HASTM ($\hbar = -1$) solutions $c(x, y, t)$ at $\alpha = 1$ and $y = 0.5$ w.r.t. x and t.

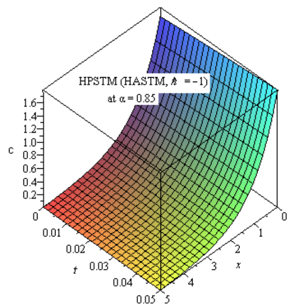

FIGURE 9. The comparison between of the HPSTM and HASTM ($\hbar = -1$) solutions $c(x, y, t)$ at $\alpha = 0.85$ and $y = 0.5$ w.r.t. x and t.

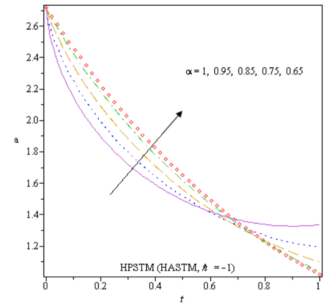

FIGURE 10. Plots of HPSTM and HASTM ($\hbar = -1$) solutions $a(x, y, t)$ vs. t at $x = y = 0.5$ for different values of α.

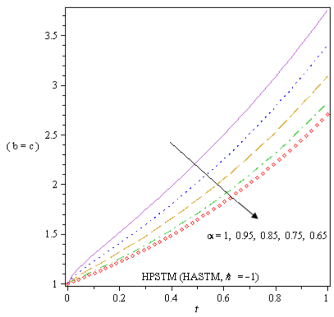

FIGURE 11. Plots of HPSTM and HASTM ($\hbar = -1$) solutions $b(x, y, t) = c(x, y, t)$ vs. t at $x = y = 0.5$ for different values of α.

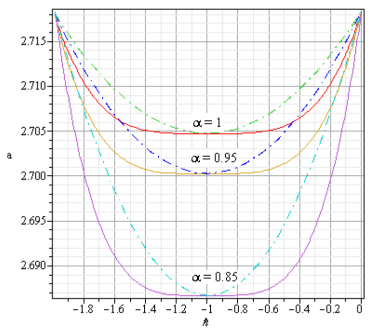

FIGURE 12. Plots of HASTM solution $a(x, y, t)$ vs. $\hbar$ at $x = y = 0.5$ and $t = 0.005$ for different values of α.

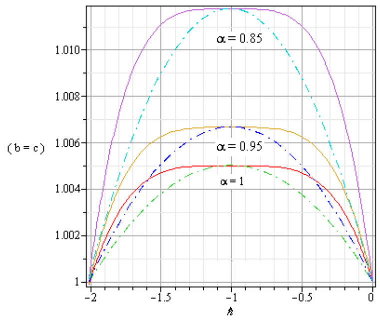

FIGURE 13. Plots of HASTM solution $b(x, y, t) = c(x, y, t)$ vs. $\hbar$ at $x = y = 0.5$ and $t = 0.005$ for different values of α.

7. Numerical results and discussions

Figs. 1-3 show the comparison between exact, HPSTM and HASTM ($\hbar = -1$) solutions $a(x, y, t)$ of the system (4.1) with different values α at $y = 0.5$. Figs. 4-6 present the comparison between exact, HPSTM and HASTM ($\hbar = -1$) solutions $b(x, y, t)$ of the system (4.1) with different values α at $y = 0.5$. Figs. 7-9 depict the comparison between exact, HPSTM ($\hbar = -1$) and HASTM solutions $c(x, y, t)$ of the system (4.1) with different values α at $y = 0.5$.

From Figs. 1-9, we can see that results obtained by the proposed methods are in a very good agreement with the exact solution.

Fig. 10 shows the comparative behavior of the 4th order HPSTM and HASTM ($\hbar = -1$) approximate solutions $a(x, y, t)$ versus t of the system (4.1) at $x = y = 0.5$ with different order α. Fig. 11 shows the comparative behavior of the 4th order HPSTM and HASTM ($\hbar = -1$) approximate solutions $b(x, y, t) = c(x, y, t)$ of the system (4.1) at $x = y = 0.5$ for a time t with different order α. Fig. 12 depicts the validity of absolute convergence range of $\hbar$ curve for solution $a(x, y, t)$ and Fig. 13 represents the validity of absolute convergence range of $\hbar$ curve for solution $b(x, y, t) = c(x, y, t)$.

In Figs. 12 and 13, solid line represents 4th order and dash dot line represents 2nd order HASTM approximation at $x = y = 0.5$, $t = 0.005$ for different order α of the system (4.1) and show that the valid range of convergence and the horizontal line segments show the absolute convergence range. Fig. 12 reveals that the valid range of $\hbar$ is $-1.986 \leq \hbar < 0$ for $a(x, y, t)$. Fig. 13 shows that the valid range of $\hbar$ is $-2.01 \leq \hbar < 0$ for $b(x, y, t) = c(x, y, t)$. It is obvious that the middle point of $\hbar$-curves interval, i.e., $\hbar = -1$ is an appropriate selection, at the point of which the numerical solution converges to series solution of HPSTM for standard motion as well as Brownian motions.

Concluding remarks

In our investigation here, our choice to make use of the Sumudu transform instead of the classical Laplace transform is prompted by the various computations considered here which happen to be much simpler (see also [48]). It may be carefully asserted that both the methods HPSTM and HASTM give series solutions, which converge rapidly, and require less computational work, and provide high accurate results for systems of nonlinear equations.

Here only fourth order approximation in solution series is used for numerical simulation. A diagrammatical illustration of convergence control parameter $\hbar$, arising in HASTM iteration, can provide multiple options in series solution with very small absolute error compared to HPSTM and other existing traditional perturbation methods in same term iterations. The proposed techniques may be more reliable than analytical techniques used in their respective coupling.

Acknowledgments. The authors would like to express their deep-felt thanks for the reviewer's useful comments with an introduction of (very recently presented) new fractional derivatives.

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