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HYPERSURFACES OF INFINITE TYPE WITH NULL TANGENTIAL HOLOMORPHIC VECTOR FIELDS

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Abstract. In this paper, we introduce the condition (I) (cf. Section 2) and prove that there is no nontrivial tangential holomorphic vector field of a certain hypersurface of infinite type in $\mathbb{C}^2$.

1. Introduction

Let (M, p) be a real $\mathcal{C}^1$-smooth hypersurface germ at $p \in \mathbb{C}^n$. A smooth vector field germ (X, p) on M is called *a real-analytic infinitesimal CR automorphism germ at p of M* if there exists a holomorphic vector field germ (H, p) in $\mathbb{C}^n$ such that H is tangent to M, i.e., Re H is tangent to M, and $X = \text{Re } H\mid_M$. We denote by $\text{hol}_0(M, p)$ the real vector space of holomorphic vector field germs (H, p) vanishing at p which are tangent to M.

In several complex variables, such tangential holomorphic vector fields arise naturally from the action by the automorphism group of a domain. If Ω is a smoothly bounded domain in $\mathbb{C}^n$ and if its automorphism group $\text{Aut}(\Omega)\cap\mathcal{C}^1(\overline{\Omega})$ contains a one-parameter subgroup, say $\{\varphi_t\}$, i.e., $\varphi_{t+s} = \varphi_t \circ \varphi_s$ for all $t, s \in \mathbb{R}$ and $\varphi_0 = \text{id}_\Omega$, then the t-derivative generates a holomorphic vector field tangent to $\partial\Omega$.

In [1], J. Byun et al. proved that $\text{hol}_0(M, p) = \{i\beta z_2 \frac{\partial}{\partial z_2} \colon \beta \in \mathbb{R}\}$ for any $\mathcal{C}^\infty$-smooth radially symmetric real hypersurface $M \subset \mathbb{C}^2$ of infinite type at the origin. Recently, A. Hayashimoto and the author [3] showed that $\text{hol}_0(M_P, 0)$ is trivial for any non-radially symmetric infinite type model

$$M_P := \{(z_1, z_2) \in \mathbb{C}^2 \colon \text{Re } z_1 + P(z_2) = 0\},$$

where P is non-radially symmetric real-valued $\mathcal{C}^\infty$-smooth function satisfying that P vanishes to infinite order at $z_2 = 0$ and that the connected component

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of 0 in the zero set of P is $\{0\}$. However, many functions, such as

$$P(z_2) = \exp\Big(-\frac{1}{|\mathrm{Re}(z_2)|^2}\Big),$$

do not satisfy this condition.

In this paper, we shall introduce the condition (I) (cf. Section 2) and prove that $\mathrm{hol}_0(M, p)$ of a certain hypersurface of infinite type M in $\mathbb{C}^2$ is trivial. To state the result explicitly, we need some notations and a definition. Taking the risk of confusion we employ the notations

$$P'(z) = P_z(z) = \frac{\partial P}{\partial z}(z)$$

throughout the article. Also denote by $\Delta_r = \{z \in \mathbb{C}\colon |z| < r\}$ for $r > 0$ and by $\Delta = \Delta_1$. A function f defined on Δ_r $(r > 0)$ is called to be *flat* at the origin if $f(z) = o(|z|^n)$ for each $n \in \mathbb{N}$ (cf. Definition 1). In what follows, $\lesssim$ and $\gtrsim$ denote inequalities up to a positive constant multiple. In addition, we use $\approx$ for the combination of $\lesssim$ and $\gtrsim$.

The aim of this paper is to prove the following theorem.

Theorem 1. *If a $\mathcal{C}^1$-smooth hypersurface germ $(M, 0)$ is defined by the equation $\rho(z) := \rho(z_1, z_2) = \mathrm{Re}\ z_1 + P(z_2) + (\mathrm{Im}\ z_1)Q(z_2, \mathrm{Im}\ z_1) = 0$, satisfying the conditions*:

(i) $P \not\equiv 0$, $P(0) = Q(0, 0) = 0$;
(ii) *P satisfies the condition* (I) (*cf. Definition 2 in Section 2*);
(iii) *P is flat at $z_2 = 0$,*

then any holomorphic vector field vanishing at the origin tangent to $(M, 0)$ is identically zero.

Remark 1. If P and Q are $\mathcal{C}^\infty$-smooth, then Theorem 1 gives a partial answer to the Greene-Krantz conjecture, which states that for a smoothly bounded pseudoconvex domain admitting a non-compact automorphism group, the point orbits can accumulate only at a point of finite type [2].

This paper is organized as follows. In Section 2, the condition (I) and several examples are introduced. In Section 3, several technical lemmas are proved and the proof of Theorem 1 is finally given.

2. Functions vanishing to infinite order

First of all, we recall the following definition.

Definition 1. A function $f : \Delta_{\epsilon_0} \to \mathbb{C}$ $(\epsilon_0 > 0)$ is called to be *flat* at $z = 0$ if for each $n \in \mathbb{N}$ there exist positive constants $C, \epsilon > 0$, depending only on n, with $0 < \epsilon < \epsilon_0$ such that

$$|f(z)| \le C|z|^n$$

for all $z \in \Delta_\epsilon$.

We note that in the above definition we do not need the smoothness of the function f. For example, the following function

$$f(z) = \begin{cases} \frac{1}{n} e^{-\frac{1}{|z|^2}} & \text{if } \frac{1}{n+1} < |z| \leq \frac{1}{n} \ , n = 1, 2, \dots \\ 0 & \text{if } z = 0 \end{cases}$$

is flat at $z = 0$ but not continuous on Δ. However, if $f \in \mathcal{C}^\infty(\Delta_{\epsilon_0})$, then it follows from the Taylor's theorem that f is flat at $z = 0$ if and only if

$$\frac{\partial^{m+n}}{\partial z^m \partial \bar{z}^n} f(0) = 0$$

for every $m, n \in \mathbb{N}$, i.e., f vanishes to infinite order at 0. Consequently, if $f \in \mathcal{C}^\infty(\Delta_{\epsilon_0})$ is flat at 0, then $\frac{\partial^{m+n} f}{\partial z^m \partial \bar{z}^n}$ is also flat at 0 for each $m, n \in \mathbb{N}$.

We now introduce the condition (I) and give several examples of functions defined on the open unit disc in the complex plane with infinite order of vanishing at the origin.

Definition 2. We say that a real $\mathcal{C}^1$-smooth function f defined on a neighborhood U of the origin in $\mathbb{C}$ satisfies the *condition (I)* if

(I.1) $\limsup_{\tilde{U} \ni z \to 0} |\mathrm{Re}(b z^k \frac{f'(z)}{f(z)})| = +\infty$;

(I.2) $\limsup_{\tilde{U} \ni z \to 0} |\frac{f'(z)}{f(z)}| = +\infty$

for all $k = 1, 2, \dots$ and for all $b \in \mathbb{C}^*$, where $\tilde{U} := \{z \in U : f(z) \neq 0\}$.

Example 1. The function $P(z) = e^{-C/|\mathrm{Re}(z)|^\alpha}$ if $\mathrm{Re}(z) \neq 0$ and $P(z) = 0$ if otherwise, where $C, \alpha > 0$, satisfies the condition (I). Indeed, a direct computation shows that

$$P'(z) = P(z) \frac{C\alpha}{2|\mathrm{Re}(z)|^{\alpha+1}}$$

for all $z \in \mathbb{C}$ with $\mathrm{Re}(z) \neq 0$. Therefore, it is easy to see that $|P'(z)/P(z)| \to +\infty$ as $z \to 0$ in the domain $\{z \in \mathbb{C} \colon \mathrm{Re}(z) \neq 0\}$.

Now we shall prove that the condition (I.1) holds. Let k be an arbitrary positive integer. Let $z_l := 1/l + i/l^\beta$, where $0 < \beta < \min\{1, \alpha/(k-1)\}$ if $k > 1$ and $\beta = 1/2$ if $k = 1$, for all $l \in \mathbb{N}^*$. Then $z_l \to 0$ as $l \to \infty$ and $\mathrm{Re}(z_l) = 1/l \neq 0$ for all $l \in \mathbb{N}^*$. Moreover, for each $b \in \mathbb{C}^*$ we have that

$$|\mathrm{Re}\Big(b z_l^k \frac{P'(z_l)}{P(z_l)}\Big)| \gtrsim \frac{l^{\alpha+1}}{l^{\beta(k-1)+1}} = l^{\alpha - \beta(k-1)}.$$

This implies that

$$\lim_{l \to \infty} |\mathrm{Re}\Big(b z_l^k \frac{P'(z_l)}{P(z_l)}\Big)| = +\infty.$$

Hence, the function P satisfies the condition (I).

Remark 2. i) Any rotational function P does not satisfy the condition (I.1) because $\mathrm{Re}(izP'(z)) = 0$ (see [1] or [4]).

ii) It follows from [4, Lemma 2] that if P is a non-zero $\mathcal{C}^1$-smooth function defined on a neighborhood U of the origin in $\mathbb{C}$, $P(0) = 0$, and $\tilde{U} := \{z \in U\colon P(z) \neq 0\}$ contains a $\mathcal{C}^1$-smooth curve $\gamma : (0,1] \to \tilde{U}$ such that γ' stays bounded on $(0,1]$ and $\lim_{t\to 0^-} \gamma(t) = 0$, then P satisfies the condition (I.2).

Lemma 1. *Suppose that $g : (0,1] \to \mathbb{R}$ is a $\mathcal{C}^1$-smooth unbounded function. Then we have $\limsup_{t\to 0^+} t^\alpha |g'(t)| = +\infty$ for any real number $\alpha < 1$.*

Proof. Fix an arbitrary $\alpha < 1$. Suppose that, on the contrary,

$$\limsup_{t\to 0^+} t^\alpha |g'(t)| < +\infty.$$

Then there is a constant $C > 0$ such that

$$|g'(t)| \leq \frac{C}{t^\alpha},\ \forall\, 0 < t < 1.$$

We now have the following estimate

$$\begin{aligned} |g(t)| &\leq |g(1)| + \int_t^1 |g'(\tau)| d\tau \leq |g(1)| + C \int_t^1 \frac{d\tau}{\tau^\alpha} \\ &\leq |g(1)| + \frac{C}{1-\alpha}(1 - t^{1-\alpha}) \lesssim 1. \end{aligned}$$

However, this is impossible since g is unbounded on $(0,1]$, and thus the lemma is proved. □

In general, the above lemma does not hold for $\alpha \geq 1$. This follows from that $|t^{1+\beta} \frac{d}{dt} \frac{1}{t^\beta}| = \beta$ and $|t \frac{d}{dt} \log(t)| = 1$ for all $0 < t < 1$, where $\beta > 0$. However, the following lemmas show that there exists such a function g such that $\liminf_{t\to 0^+} \sqrt{t}|g'(t)| < +\infty$ and $\limsup_{t\to 0^+} t^\beta |g'(t)| = +\infty$ for all $\beta < 2$. Furthermore, several examples of smooth functions vanishing to infinite order at the origin in $\mathbb{C}$ and satisfying the condition (I) are constructed.

Lemma 2. *There exists a $\mathcal{C}^\infty$-smooth real-valued function $g : (0,1) \to \mathbb{R}$ satisfying*

(i) *$g(t) \equiv -2n$ on the closed interval $[\frac{1}{n+1}(1 + \frac{1}{3n}), \frac{1}{n+1}(1 + \frac{2}{3n})]$ for $n = 4, 5, \ldots$;*

(ii) *$g(t) \approx -\frac{1}{t}$, $\forall\, t \in (0,1)$;*

(iii) *for each $k \in \mathbb{N}$ there exists $C(k) > 0$, depending only on k, such that $|g^{(k)}(t)| \leq \frac{C(k)}{t^{3k+1}}$, $\forall\, t \in (0,1)$.*

Remark 3. Let

$$P(z) := \begin{cases} \exp(g(|z|^2)) & \text{if } 0 < |z| < 1 \\ 0 & \text{if } z = 0. \end{cases}$$

Then this function is a $\mathcal{C}^\infty$-smooth function on the open unit disc Δ that vanishes to infinite order at the origin. Moreover, we see that $P'(\frac{2n+1}{2n(n+1)}) = 0$ for any $n \geq 4$, and hence $\liminf_{z\to 0} |P'(z)|/P(z) = 0$.

Lemma 2 was stated in [4] without proof. For the convenience of the reader, we now introduce a detailed proof of this lemma as follows.

Proof of Lemma 2. Let $G : (0, +\infty) \to \mathbb{R}$ be the piecewise linear function such that $G(a_n - \epsilon_n) = G(b_n + \epsilon_n) = -2n$ and $G(x) = -8$ if $x \geq \frac{9}{40}$, where $a_n = \frac{1}{n+1}(1 + \frac{1}{3n})$, $b_n = \frac{1}{n+1}(1 + \frac{2}{3n})$, and $\epsilon_n = \frac{1}{n^3}$ for every $n \geq 4$.

Let ψ be a $\mathcal{C}^\infty$-smooth function on $\mathbb{R}$ given by

$$\psi(x) = C \begin{cases} e^{-\frac{1}{1-|x|^2}} & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 1, \end{cases}$$

where $C > 0$ is chosen so that $\int_{\mathbb{R}} \psi(x)dx = 1$. For $\epsilon > 0$, set $\psi_\epsilon := \frac{1}{\epsilon}\psi(\frac{x}{\epsilon})$. For $n \geq 4$, let g_n be the $\mathcal{C}^\infty$-smooth on $\mathbb{R}$ defined by the following convolution

$$g_n(x) := G * \psi_{\epsilon_{n+1}}(x) = \int_{-\infty}^{+\infty} G(y)\psi_{\epsilon_{n+1}}(y - x)dy.$$

Now we show the following.

(a) $g_n(x) = G(x) = -2n$ if $a_n \leq x \leq b_n$;
(b) $g_n(x) = G(x) = -2(n + 1)$ if $a_{n+1} \leq x \leq b_{n+1}$;
(c) $|g_n^{(k)}(x)| \leq \frac{2(n+1)\|\psi^{(k)}\|_1}{\epsilon_{n+1}^k}$ if $a_{n+1} \leq x \leq b_n$.

Indeed, for $a_{n+1} \leq x \leq b_n$ we have

$$\begin{aligned} g_n(x) &= \int_{-\infty}^{+\infty} G(y)\psi_{\epsilon_{n+1}}(y - x)dy \\ &= \frac{1}{\epsilon_{n+1}} \int_{-\infty}^{+\infty} G(y)\psi(\frac{y - x}{\epsilon_{n+1}})dy \\ &= \int_{-1}^{+1} G(x + t\epsilon_{n+1})\psi(t)dt, \end{aligned}$$

where we use a change of variable $t = \frac{y-x}{\epsilon_{n+1}}$.

If $a_n \leq x \leq b_n$, then $a_n - \epsilon_n < a_n - \epsilon_{n+1} \leq x + t\epsilon_{n+1} \leq b_n + \epsilon_{n+1} < b_n + \epsilon_n$ for all $-1 \leq t \leq 1$. Therefore,

$$g_n(x) = \int_{-1}^{+1} G(x + t\epsilon_{n+1})\psi(t)dt = -2n \int_{-1}^{+1} \psi(t)dt = -2n,$$

which proves (a). Similarly, if $a_{n+1} \leq x \leq b_{n+1}$, then $a_{n+1} - \epsilon_{n+1} \leq x + t\epsilon_{n+1} \leq b_{n+1} + \epsilon_{n+1}$ for every $-1 \leq t \leq 1$. Hence,

$$g_n(x) = \int_{-1}^{+1} G(x + t\epsilon_{n+1})\psi(t)dt = -2(n + 1) \int_{-1}^{+1} \psi(t)dt = -2(n + 1),$$

which finishes (b). Moreover, we have the following estimate

$$\begin{aligned}|g_n^{(k)}(x)| &= \frac{1}{\epsilon_{n+1}^{k+1}}\left|\int_{-\infty}^{+\infty} G(y)\psi^{(k)}\left(\frac{y-x}{\epsilon_{n+1}}\right)dy\right| \\ &= \frac{1}{\epsilon_{n+1}^{k}}\left|\int_{-1}^{+1} G(x+t\epsilon_{n+1})\psi^{(k)}(t)dt\right| \\ &\le \frac{1}{\epsilon_{n+1}^{k}}\int_{-1}^{+1} |G(x+t\epsilon_{n+1})||\psi^{(k)}(t)|dt \\ &\le \frac{2(n+1)}{\epsilon_{n+1}^{k}}\int_{-1}^{+1} |\psi^{(k)}(t)|dt \\ &= \frac{2(n+1)\|\psi^{(k)}\|_1}{\epsilon_{n+1}^{k}}\end{aligned}$$

for $a_{n+1} \le x \le b_n$, where we use again a change of variable $t = \frac{x-y}{\epsilon_{n+1}}$ and the last inequality in the previous equation follows from the fact that $|G(y)| \le 2(n+1)$ for all $a_{n+1} - \epsilon_{n+1} \le y \le b_n + \epsilon_n$. So, the assertion (c) is shown.

Now because of properties (a) and (b) the function

$$g(x) = \begin{cases} -8 & \text{if } x \ge \frac{9}{40} \\ g_n(x) & \text{if } a_{n+1} \le x \le b_n,\ n = 4, 5, \dots, \end{cases}$$

is well-defined. From the property (c), it is easy to show that $|g^{(k)}(x)| \lesssim \frac{1}{x^{3k+1}}$ for $k = 0, 1, \dots$ and for every $x \in (0, 1)$, where the constant depends only on k. Thus this proves (iii), and the assertions (i) and (ii) are obvious. Hence, the proof is complete. □

Lemma 3. *Let $h : (0, +\infty) \to \mathbb{R}$ be the piecewise linear function such that $h(a_n) = h(b_n) = 2^{2\cdot 4^{n-1}}$, $h(1/2) = \sqrt{2}$ and $h(t) = 0$ if $t \ge 1$, where $a_n = 1/2^{4^n}$, $a_0 = 1/2$, $b_n = (a_n + a_{n-1})/2$ for every $n \in \mathbb{N}^*$. Then the function $f : (0, 1) \to \mathbb{R}$ given by*

$$f(t) = -\int_t^1 h(\tau)d\tau$$

satisfies:

(i) $f'(a_n) = \frac{1}{\sqrt{a_n}}$ *for every* $n \in \mathbb{N}^*$;
(ii) $f'(b_n) \sim \frac{1}{4b_n^2}$ *as* $n \to \infty$;
(iii) $-\frac{1}{t} \lesssim f(t) \lesssim -\frac{1}{t^{1/16}}$, $\forall\ 0 < t < 1$.

Proof. We have $f'(a_n) = h(a_n) = 2^{2\cdot 4^{n-1}} = \frac{1}{\sqrt{a_n}}$, which proves (i). Since $b_n = (a_n + a_{n-1})/2 \sim a_{n-1}/2$ as $n \to \infty$, we have $f'(b_n) = h(b_n) = 2^{2\cdot 4^{n-1}} = \frac{1}{a_{n-1}^2} \sim \frac{1}{4b_n^2}$ as $n \to \infty$. So, the assertion (ii) follows. Now we shall show (iii).

For an arbitrary real number $t \in (0, 1/16)$, denote by N the positive integer such that

$$1/2^{4^{N+1}} \leq t < 1/2^{4^N}.$$

Then it is easy to show that

$$\begin{aligned} f(t) &\leq -\int_{a_N}^{b_N} h(\tau)d\tau = -\frac{1}{2}2^{2\cdot 4^{N-1}}(1/2^{4^{N-1}} - 1/2^{4^N}) \\ &\leq -\frac{1}{2}2^{4^{N-1}} + \frac{1}{8} \leq -\frac{1}{2}\frac{1}{t^{1/16}} + \frac{1}{8} \lesssim -\frac{1}{t^{1/16}}; \\ f(t) &\geq -2\int_{a_{N+1}}^{b_{N+1}} h(\tau)d\tau - \int_{a_N}^{1} h(\tau)d\tau \\ &\geq -2h(a_{N+1})(b_{N+1} - a_{N+1}) - h(a_N)(1 - a_N) \\ &\geq -2^{2\cdot 4^N}(1/2^{4^N} - 1/2^{4^{N+1}}) - 2^{2\cdot 4^{N-1}}(1 - 1/2^{4^N}) \\ &\gtrsim -\frac{1}{t} \end{aligned}$$

for any $0 < t < 1/16$. Thus (iii) is shown. □

Remark 4. i) We note that f is $\mathcal{C}^1$-smooth, increasing, and concave on the interval $(0, 1)$. By taking a suitable regularization of the function f as in the proof of Lemma 2, we may assume that it is $\mathcal{C}^\infty$-smooth and still satisfies the above properties (i), (ii), and (iii). In addition, for each $k \in \mathbb{N}$ there exist $C(k) > 0$ and $d(k) > 0$, depending only on k, such that $|f^{(k)}(t)| \leq \frac{C(k)}{t^{d(k)}}, \forall\, t \in (0, 1)$. Thus the function $R(z)$ defined by

$$R(z) := \begin{cases} \exp(f(|z|^2)) & \text{if } 0 < |z| < 1 \\ 0 & \text{if } z = 0 \end{cases}$$

is $\mathcal{C}^\infty$-smooth and vanishes to infinite order at the origin. Moreover, we have $\liminf_{z\to 0} |R'(z)/R(z)| < +\infty$ and $\limsup_{z\to 0} |R'(z)/R(z)| = +\infty$.

ii) Since the functions P, R are rotational, they do not satisfy the condition (I) (cf. Remark 2). On the other hand, the functions $\tilde{P}(z) := P(\text{Re}(z))$ and $\tilde{R}(z) := R(\text{Re}(z))$ satisfy the condition (I). Indeed, a simple calculation shows

$$\tilde{R}'(z) = \tilde{R}(z)f'(|\text{Re}(z)|^2)\text{Re}(z)$$

for any $z \in \mathbb{C}$ with $|\text{Re}(z)| < 1$. By the above property (ii), it follows that $\limsup_{z\to 0} |\tilde{R}'(z)|/\tilde{R}(z) = +\infty$. Moreover, for each $k \in \mathbb{N}^*$ and each $b \in \mathbb{C}^*$ if we choose a sequence $\{z_n\}$ with $z_n := \sqrt{b_n} + i(\sqrt{b_n})^\beta$, where $0 < \beta < \min\{1, 2/(k-1)\}$ if $k > 1$ and $\beta = 1/2$ if $k = 1$, then $z_n \to 0$ as $n \to \infty$ and

$$|\text{Re}\Big(bz_n^k \frac{\tilde{R}'(z_n)}{\tilde{R}(z_n)}\Big)| \gtrsim \frac{(\sqrt{b_n})^{(k-1)\beta+2}}{b_n^2} \to +\infty$$

as $n \to \infty$. Hence, $\tilde{R}$ satisfies the condition (I). Now it follows from the construction of the function g in the proof of Lemma 2 that $g'(\frac{1}{n}) \sim 3n^2$ as

$n \to \infty$. Therefore, using the same argument as above we conclude that $\tilde{P}$ also satisfies the condition (I).

It is not hard to show that the above functions such as $P, R, \tilde{P}, \tilde{R}$ are not subharmonic. To the author's knowledge, it is unknown that there exists a $\mathcal{C}^\infty$-smooth subharmonic function P defined on the unit disc such that $\nu_0(P) = +\infty$ and $\liminf_{z\to 0} |P'(z)/P(z)| < +\infty$.

3. Proof of Theorem 1

This section is entirely devoted to the proof of Theorem 1. Let $M = \{(z_1, z_2) \in \mathbb{C}^2 \colon \text{Re } z_1 + P(z_2) + (\text{Im } z_1)Q(z_2, \text{Im } z_1) = 0\}$ be the real hypersurface germ at 0 described in the hypothesis of Theorem 1. Our present goal is to show that there is no non-trivial holomorphic vector field vanishing at the origin and tangent to M.

For the sake of smooth exposition, we shall present the proof in two subsections. In Subsection 3.1, several technical lemmas are introduced. Then the proof of Theorem 1 is presented in Subsection 3.2. Throughout what follows, for $r > 0$ denote by $\tilde{\Delta}_r := \{z_2 \in \Delta_r \colon P(z_2) \neq 0\}$.

3.1. Technical lemmas

Since P satisfies the condition (I), it is not hard to show the following two lemmas.

Lemma 4. *Let P be a function defined on Δ_{ϵ_0} $(\epsilon_0 > 0)$ satisfying the condition* (I). *If a, b are complex numbers and if g_0, g_1, g_2 are $\mathcal{C}^\infty$-smooth functions defined on Δ_{ϵ_0} satisfying*:

(i) $g_0(z) = O(|z|)$, $g_1(z) = O(|z|^{\ell+1})$, $g_2(z) = o(|z|^m)$, *and*

(ii) $\text{Re}\Big[az^m + \frac{b}{P^n(z)}\Big(z^{\ell+1}\big(1 + g_0(z)\big)\frac{P'(z)}{P(z)} + g_1(z)\Big)\Big] = g_2(z)$

for every $z \in \tilde{\Delta}_{\epsilon_0}$ and for any non-negative integers ℓ, m, except the case that $m = 0$ and $\text{Re}(a) = 0$, then $a = b = 0$.

Proof. The proof follows easily from the condition (I.1). □

Lemma 5. *Let P be a function defined on Δ_{ϵ_0} $(\epsilon_0 > 0)$ satisfying the condition* (I). *Let $B \in \mathbb{C}^*$ and $m \in \mathbb{N}^*$. Then there exists $\alpha \in \mathbb{R}$ small enough such that*

$$\limsup_{\tilde{\Delta}_{\epsilon_0} \ni z \to 0} |\text{Re}\Big(B(i\alpha - 1)^m P'(z)/P(z)\Big)| = +\infty.$$

Proof. Since P satisfies the condition (I.2), there exists a sequence $\{z_k\} \subset \tilde{\Delta}_{\epsilon_0}$ converging to 0 such that $\lim_{k\to\infty} P'(z_k)/P(z_k) = \infty$. We can write

$$BP'(z_k)/P(z_k) = a_k + ib_k, \quad k = 1, 2, \ldots;$$
$$(i\alpha - 1)^m = a(\alpha) + ib(\alpha).$$

We note that $|a_k| + |b_k| \to +\infty$ as $k \to \infty$. Therefore, passing to a subsequence if necessary, we only consider two following cases.

Case 1. $\lim_{k\to\infty} a_k = \infty$ **and** $|\frac{b_k}{a_k}| \lesssim 1$. Since $a(\alpha)\to(-1)^m$ and $b(\alpha)\to 0$ as $\alpha\to 0$, if α is small enough, then

$$\begin{aligned}\mathrm{Re}\Big(B(i\alpha-1)^m P'(z_k)/P(z_k)\Big) &= a(\alpha)a_k - b(\alpha)b_k\\ &= a_k\Big(a(\alpha)-b(\alpha)\frac{b_k}{a_k}\Big)\to\infty\end{aligned}$$

as $k\to\infty$.

Case 2. $\lim_{k\to\infty} b_k = \infty$ **and** $\lim_{k\to\infty} |\frac{a_k}{b_k}| = 0$. Fix a real number α such that $b(\alpha)\neq 0$. Then we have

$$\begin{aligned}\mathrm{Re}\Big(B(i\alpha-1)^m P'(z_k)/P(z_k)\Big) &= a(\alpha)a_k - b(\alpha)b_k\\ &= b_k\Big(a(\alpha)\frac{a_k}{b_k}-b(\alpha)\Big)\to\infty\end{aligned}$$

as $k\to\infty$. Hence, the proof is complete. □

3.2. Proof of Theorem 1

The CR hypersurface germ $(M,0)$ at the origin in $\mathbb{C}^2$ under consideration is defined by the equation $\rho(z_1,z_2)=0$, where

$$\rho(z_1,z_2)=\mathrm{Re}\ z_1+P(z_2)+(\mathrm{Im}\ z_1)\ Q(z_2,\mathrm{Im}\ z_1)=0,$$

where P,Q are $\mathcal{C}^1$-smooth functions satisfying the three conditions specified in the hypothesis of Theorem 1, stated in Section 1. Recall that P is flat at $z_2=0$ in particular.

Then we consider a holomorphic vector field $H=h_1(z_1,z_2)\frac{\partial}{\partial z_1}+h_2(z_1,z_2)\frac{\partial}{\partial z_2}$ defined on a neighborhood of the origin. We only consider H that is tangent to M, which means that they satisfy the identity

$$(\mathrm{Re}\ H)\rho(z)=0,\ \forall z\in M. \tag{1}$$

The goal is to show that $H\equiv 0$. Indeed, striving for a contradiction, suppose that $H\not\equiv 0$. We notice that if $h_2\equiv 0$, then (1) shows that $h_1\equiv 0$. Thus, $h_2\not\equiv 0$.

Now we are going to prove that $h_1\equiv 0$. Indeed, suppose that $h_1\not\equiv 0$. Then we can expand h_1 and h_2 into the Taylor series at the origin so that

$$h_1(z_1,z_2)=\sum_{j,k=0}^{\infty} a_{jk}z_1^j z_2^k \text{ and } h_2(z_1,z_2)=\sum_{j,k=0}^{\infty} b_{jk}z_1^j z_2^k,$$

where $a_{jk},b_{jk}\in\mathbb{C}$. We note that $a_{00}=b_{00}=0$ since $h_1(0,0)=h_2(0,0)=0$.

By a simple computation, one has

$$\begin{aligned}\rho_{z_1}(z_1,z_2) &= \frac{1}{2}+\frac{Q(z_2,\mathrm{Im}\ z_1)}{2i}+(\mathrm{Im}\ z_1)Q_{z_1}(z_2,\mathrm{Im}\ z_1)\\ &= \frac{1}{2}+\frac{Q_0(z_2)}{2i}+\frac{2(\mathrm{Im}\ z_1)Q_1(z_2)}{2i}+\frac{3(\mathrm{Im}\ z_1)^2Q_2(z_2)}{2i}+\cdots;\\ \rho_{z_2}(z_1,z_2) &= P'(z_2)+(\mathrm{Im}\ z_1)Q_{z_2}(z_2,\mathrm{Im}\ z_1),\end{aligned}$$

and the equation (1) can thus be re-written as

$$\tag{2} \begin{aligned} \mathrm{Re}\Big[\Big(\frac{1}{2}+\frac{Q(z_2,\mathrm{Im}\ z_1)}{2i}+(\mathrm{Im}\ z_1)Q_{z_1}(z_2,\mathrm{Im}\ z_1)\Big)h_1(z_1,z_2)\\ +\Big(P'(z_2)+(\mathrm{Im}\ z_1)Q_{z_2}(z_2,\mathrm{Im}\ z_1)\Big)h_2(z_1,z_2)\Big]=0 \end{aligned}$$

for all $(z_1, z_2) \in M$.

Since $\Big(it - P(z_2) - tQ(z_2, t), z_2\Big) \in M$ for any $t \in \mathbb{R}$ with t small enough, the above equation again admits a new form

$$\tag{3} \begin{aligned} &\mathrm{Re}\Big[\Big(\frac{1}{2}+\frac{Q_0(z_2)}{2i}+\frac{2tQ_1(z_2)}{2i}+\frac{3t^2Q_2(z_2)}{2i}+\cdots\Big)\times\\ &\Big(\sum_{j,k=0}^{\infty}\big(it-P(z_2)-tQ_0(z_2)-t^2Q_1(z_2)-\cdots\big)^j a_{jk}z_2^k\Big)\\ &+\Big(P'(z_2)+tQ_{0_{z_2}}(z_2)+t^2Q_{1_{z_2}}(z_2)+\cdots\Big)\times\\ &\Big(\sum_{m,n=0}^{\infty}\big(it-P(z_2)-tQ_0(z_2)-t^2Q_1(z_2)-\cdots\big)^m b_{mn}z_2^n\Big)\Big]=0 \end{aligned}$$

for all $z_2 \in \mathbb{C}$ and for all $t \in \mathbb{R}$ with $|z_2| < \epsilon_0$ and $|t| < \delta_0$, where $\epsilon_0 > 0$ and $\delta_0 > 0$ are small enough.

Next, let us denote by j_0 the smallest integer such that $a_{j_0k} \neq 0$ for some integer k. Then let k_0 be the smallest integer such that $a_{j_0k_0} \neq 0$. Similarly, let m_0 be the smallest integer such that $b_{m_0n} \neq 0$ for some integer n. Then denote by n_0 the smallest integer such that $b_{m_0n_0} \neq 0$. One remarks that $j_0 \geq 1$ if $k_0 = 0$ and $m_0 \geq 1$ if $n_0 = 0$.

Notice that one may choose $t = \alpha P(z_2)$ in (3) (with α to be chosen later on), and since $P(z_2) = o(|z_2|^{n_0})$, one has

$$\tag{4} \begin{aligned} &\mathrm{Re}\Big[\frac{1}{2}a_{j_0k_0}(i\alpha-1)^{j_0}(P(z_2))^{j_0}z_2^{k_0}+b_{m_0n_0}(i\alpha-1)^{m_0}(z_2^{n_0}+o(|z_2|^{n_0})(P(z_2))^{m_0}\\ &\times\Big(P'(z_2)+\alpha P(z_2)Q_{z_2}(z_2,\alpha P(z_2))\Big)\Big]=o(P(z_2)^{j_0}|z_2|^{k_0}) \end{aligned}$$

for all $|z_2| < \epsilon_0$ and for any $\alpha \in \mathbb{R}$. We remark that in the case $k_0 = 0$ and $\mathrm{Re}(a_{j_00}) = 0$, α can be chosen in such a way that $\mathrm{Re}\big((i\alpha-1)^{j_0}a_{j_00}\big) \neq 0$. Then the above equation yields that $j_0 > m_0$.

We now divide the argument into two cases as follows.

Case 1. $\boldsymbol{n_0 \geq 1}$. In this case (4) contradicts Lemma 4.

Case 2. $\boldsymbol{n_0 = 0}$. Since P satisfies the condition (I) and $m_0 \geq 1$, by Lemma 5 we can choose a real number α such that

$$\limsup_{\tilde{\Delta}_{\epsilon_0}\ni z_2\to 0} |\mathrm{Re}\Big(b_{m0}(i\alpha-1)^m P'(z_2)/P(z_2)\Big)| = +\infty,$$

where $\epsilon_0 > 0$ is small enough. Therefore, (4) is a contradiction, and thus $h_1 \equiv 0$ on a neighborhood of $(0, 0)$ in $\mathbb{C}^2$.

Since $h_1 \equiv 0$, it follows from (3) with $t = 0$ that

$$\mathrm{Re}\Big[\sum_{m,n=0}^{\infty} b_{mn} z_2^n P'(z_2)\Big] = 0$$

for every z_2 satisfying $|z_2| < \epsilon_0$, for some $\epsilon_0 > 0$ sufficiently small. Since P satisfies the condition (I.1), we conclude that $b_{mn} = 0$ for every $m \geq 0, n \geq 1$. We now show that $b_{m0} = 0$ for every $m \in \mathbb{N}^*$. Indeed, suppose otherwise. Then let m_0 be the smallest positive integer such that $b_{m_0 0} \neq 0$. It follows from (3) with $t = \alpha P(z_2)$ that

$$\mathrm{Re}\Big(b_{m_0 0}(i\alpha - 1)^{m_0} P'(z_2)/P(z_2)\Big)$$

is bounded on $\tilde{\Delta}_{\epsilon_0}$ with $\epsilon_0 > 0$ small enough for any $\alpha \in \mathbb{R}$ small enough. By Lemma 5, this is again impossible.

Altogether, the proof of Theorem 1 is complete. □

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