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# THE PROBABILISTIC METHOD MEETS GO

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Abstract. Go is an ancient game of great complexity and has a huge following in East Asia. It is also very rich mathematically, and can be played on any graph, although it is usually played on a square lattice. As with any game, one of the most fundamental problems is to determine the number of legal positions, or the probability that a random position is legal. A random Go position is generated using a model previously studied by the author, with each vertex being independently Black, White or Uncoloured with probabilities $q, q, 1-2q$ respectively. In this paper we consider the probability of legality for two scenarios. Firstly, for an $N \times N$ square lattice graph, we show that, with $q = cN^{-\alpha}$ and $c$ and $\alpha$ constant, as $N \to \infty$ the limiting probability of legality is 0, $\exp(-2c^5)$, and 1 according as $\alpha < 2/5$, $\alpha = 2/5$ and $\alpha > 2/5$ respectively. On the way, we investigate the behaviour of the number of captured chains (or chromons). Secondly, for a random graph on $n$ vertices with edge probability $p$ generated according to the classical Gilbert-Erdős-Rényi model $\mathcal{G}(n;p)$, we classify the main situations according to their asymptotic almost sure legality or illegality. Our results draw on a variety of probabilistic and enumerative methods including linearity of expectation, second moment method, factorial moments, polyomino enumeration, giant components in random graphs, and typicality of random structures. We conclude with suggestions for further work.

## 1. Introduction

*Go* is among the oldest, deepest, and most elegant board games in the world. Here we use the name it goes by in the West, and among mathematicians and computer scientists, who have been much attracted to the game over the last century or so. It originated in China over 2,000 years ago [9] and is very popular in East Asia, where it is known as *Wéiqí* in China, *Go* or *Igo* in Japan, and *Baduk* in Korea. According to the International Go Federation, "the total

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number of players is well over 40 million" [14]. The game is the subject of extensive media coverage including a dedicated 24-hour TV channel [3].

Go is mathematically rich and deep. It is entirely graph-theoretic and contains one of the earliest formal uses of the notion of a connected component, long before the origin of graph theory itself. Around 1970, it inspired the development of surreal numbers and combinatorial game theory by Berlekamp, Conway, and Guy [6, Prologue], and remains an important topic in that field. Determining which player can win from a given position is PSPACE-hard [18]. It has a much larger search space than chess, and appears to be more complex. Computer programs have been written for it [5, 20], but only this year has a computer, Google DeepMind's alphaGo, reached the level of the best human players [12, 13], in contrast to the situation for chess (where the best computer players have defeated the best human players since 1997). The recent International Congress of Mathematicians (ICM) in Seoul, Korea, August 2014, included a special event on Go and its mathematics [16, pp. 8, 813], [21].

The game is played on a fixed graph, usually a $19 \times 19$ square lattice of $19^2 = 361$ vertices. Two players, Black and White, take turns. Each player, on their turn, colours a vertex with their colour (done physically by placing a black or white lens-shaped stone on it), with the aim of surrounding connected sets of uncoloured vertices and connected sets of stones of the opposite colour. We will not give the rules; for further information, see, e.g., [15]. One key notion we use, though, is capture. If a connected set $X$ of vertices all occupied by stones of one colour is entirely surrounded (in that no vertex in $X$ has a neighbouring uncoloured vertex) by stones of the opposite colour, then all stones on $X$ are said to be *captured* and are removed from the board. All the vertices of $X$ are now uncoloured. Since a captured set of stones cannot remain on the board, a position is only *legal* if none of its stones is captured.

One of the most fundamental questions that can be asked about any game is: how many legal positions are there? Another is its close relative: given a random assignment of game elements (in this case, black and white stones) to some locations on the board, what is the probability that the resulting assignment is a legal position in the game? In this paper, we address this question for Go.

In previous work, we introduced *Go polynomials* [10], which give the probability that a random Go position on a given graph is legal. The main contributions were recursive algorithms to compute Go polynomials, connections with the chromatic polynomial, and an application of transcendental number theory to show that the polynomials are #P-hard to compute. In a self-contained sequel, we used transfer matrix methods to obtain deterministic upper and lower bounds for numbers of legal Go positions on lattice graphs (square, triangular, hexagonal, square-octagonal) of up to certain sizes [11]. That work also considered asymptotic behaviour, as the board dimensions tend to infinity. We showed that some appropriate limits exist and gave bounds for them.

At around the same time and independently, Tromp and Farnebäck [23] developed related but stronger methods and applied them in a computational *tour de force*. They gave exact values of numbers of legal positions for specific square lattice graphs of size up to $17 \times 17$, and remarkably accurate estimates of the limiting constants for board size tending to infinity. Very recently, Tromp announced the computation of the exact number of legal positions for $18 \times 18$ and $19 \times 19$ square lattice graphs as well [22].

The present paper takes a new direction by investigating transitions between random Go positions being asymptotically almost surely legal or not. Topics drawn on include linearity of expectation, second moment method, the method of factorial moments, enumeration of polyominoes, giant components in random graphs, and typicality of random structures.

## 2. Definitions and overview

We now give formal definitions, in graph-theoretic language, for the Go concepts that we will use.

A *position* on $G$ is a function $f : V(G) \to \{\mathrm{B}, \mathrm{W}, \mathrm{U}\}$, where we use B, W, U as abbreviations for Black, White and Uncoloured, respectively. It partitions $V(G)$ into two *colour classes* $B := f^{-1}(\mathrm{B})$ and $W := f^{-1}(\mathrm{W})$ and the set $U := f^{-1}(\mathrm{U}) = V(G) \setminus (B \cup W)$ of uncoloured vertices. A *chromon* is a connected component of the subgraph of $G$ induced by one of the colour classes (using the terminology of [8]). This is often called a *monochromatic component* [1]. In Go, it is called a *chain*, or sometimes a *group* (as done in [10]) though the latter term has a broader meaning too. A chromon is *free* if at least one of its vertices is adjacent to an uncoloured vertex, and *captured* otherwise. A position is *legal* if every chromon is free. Equivalently, for each coloured vertex $v$ there is a path from $v$ to an uncoloured vertex $w$ such that all vertices on the path except $w$ have the same colour as $v$. A position is *superlegal* if every coloured vertex has an uncoloured neighbour (i.e., we can require that the aforementioned paths that link coloured vertices to uncoloured vertices all consist just of a single edge). Superlegality implies legality, and is of no particular relevance in playing Go, but it is useful to us as it is easier to analyse than ordinary legality.

To illustrate, see Figure 1, where all chromons are free except the lower right Black chromon, which is captured. Because of this captured chromon, the position is illegal as it stands. If this capture were carried out, so that the two stones of that chromon were removed (i.e., the vertices were uncoloured), then the position would become legal. It would still not be superlegal, but then the only obstacle to superlegality would be the middle vertex of the L-shaped black chromon in the upper right. A Go player would regard the upper right Black chromon as doomed, but it is not currently captured so is not illegal. A Go player would also regard the two Black chromons at lower left as combining to form a single “group” (using that term in its wider Go sense), but they are

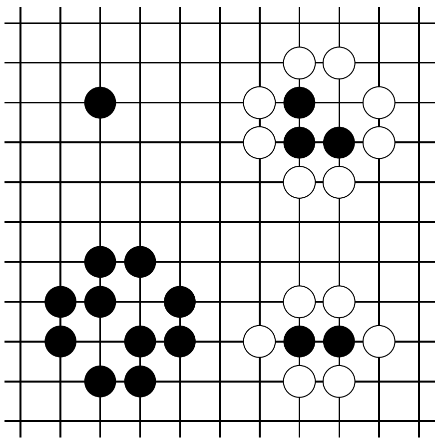

FIGURE 1. A Go position with eight White chromons and five Black chromons. The position is not legal since the lower right Black chromon is captured.

still separate chromons (or chains). A coloured vertex with no neighbours of the same colour is a *singleton chromon*; there are three singleton chromons in the figure.

Random positions on $G$ are generated as follows. Let $q \in [0, \frac{1}{2}]$ be a probability. Each vertex $v \in V(G)$ is independently coloured Black or White, or left uncoloured, with probabilities $q$, $q$ and $1-2q$ respectively. Under this model, $B$, $W$ and $U$ are set-valued random variables. Their sizes $|B|$, $|W|$, $|U|$ are binomially distributed with means $nq$, $nq$, $n(1-2q)$ respectively.

We write $\mathrm{Go}(G;q)$ for the probability that the random position so generated is legal. This function was introduced in [10], where it was shown to be a polynomial in $q$ (for fixed $G$). On the interval $[0, \frac{1}{2}]$ it is decreasing, with $\mathrm{Go}(G;0) = 1$ and $\mathrm{Go}(G;\frac{1}{2}) = 0$. Our focus on the probability of legality, $\mathrm{Go}(G;q)$, complements the work of [11,23] where the focus is on the number of legal positions (i.e., $\mathrm{Go}^{\#}(G;2)$, which equals $3^{|V(G)|} \cdot \mathrm{Go}(G;\frac{1}{3})$, in the notation of [10]).

We investigate the asymptotic behaviour of $\mathrm{Go}(G;q)$ when $G$ is (a) the $N \times N$ square lattice graph, as $N \to \infty$, and (b) a random graph generated according to the classical Gilbert-Erdős-Rényi model $\mathcal{G}(n;p)$ with $n$ vertices and edge probability $p$, as $n \to \infty$.

For the $N \times N$ square lattice, we first estimate the expected number of captured chromons, and determine its behaviour as $N \to \infty$. We then look at the probability of legality. We show that, with $q = cN^{-\alpha}$, there are three cases: if $\alpha < 2/5$, then $\lim_{N\to\infty} \mathrm{Go}(G;cN^{-\alpha}) = 0$, i.e., the position is asymptotically almost surely (a.a.s.) illegal, and in fact has a captured singleton chromon (i.e., a coloured vertex surrounded by vertices of the opposite colour); if $\alpha > 2/5$, then $\lim_{N\to\infty} \mathrm{Go}(G; cN^{-\alpha}) = 1$, i.e., the position is a.a.s. legal; if $\alpha = 2/5$, we have $\lim_{N\to\infty} \mathrm{Go}(G; cN^{-2/5}) = \exp(-2c^5)$. The first two of these cases are considered in §3, and the third in §4.

For random graphs, we consider in §5 several ranges of values for $p$ and $q$, and show that for some, a random position is a.a.s. superlegal, while for others, it is a.a.s. illegal due to the asymptotically almost certain existence of a captured singleton chromon. The case when $p = c(\log n)n^{-1}$, $c > (1-q)^{-1}$, and $q$ is a constant, $0 \le q < \frac{1}{2}$, is more subtle, and we show that a.a.s. the position is legal; it becomes a.a.s. superlegal if $c > (1-2q)^{-1}$.

We conclude in §6 with some suggestions for future work.

Throughout, $G$ is a graph and $n = |V(G)|$. If $U \subseteq V(G)$, then the subgraph of $G$ induced by $U$ is denoted by $\langle U\rangle$. If $X, Y \subseteq V(G)$, then $E(X, Y)$ is the set of edges with one endpoint in $u$ and the other in $v$.

We write $f(n) \sim g(n)$ when $\lim_{n\to\infty} f(n)/g(n) = 1$.

## 3. Large square lattice graphs

We now look at the asymptotic behaviour of $\mathrm{Go}(G; q)$ when $G$ is the $N \times N$ square lattice graph, $q = cN^{-\alpha}$ for fixed $c, \alpha$, and $N \to \infty$.

We begin with some notation, terminology and basic facts for finite and infinite square lattice graphs and their connected induced subgraphs.

The infinite square lattice graph $\mathrm{Sq}_{\infty,\infty}$ has vertex set $\mathbb{Z}^2$, with vertices $(x_1, y_1)$ and $(x_2, y_2)$ being adjacent if and only if they are identical in one coordinate and differ by $\pm 1$ in the other. We consider this graph to be embedded in the plane in the natural way. We treat the $N \times N$ square lattice graph $\mathrm{Sq}_{N,N}$ as the subgraph of $\mathrm{Sq}_{\infty,\infty}$ induced by $\{-\lfloor (N-1)/2\rfloor, -\lfloor (N-1)/2\rfloor + 1, \ldots, \lfloor N/2\rfloor - 1, \lfloor N/2\rfloor\}^2$, so it has $n = N^2$ vertices. Clearly $M \le N$ implies $\mathrm{Sq}_{M,M} \le \mathrm{Sq}_{N,N}$. The *border* of $\mathrm{Sq}_{N,N}$ is the set of vertices $(x, y)$ with $x \in \{-\lfloor (N-1)/2\rfloor, \lfloor N/2\rfloor\}$ or $y \in \{-\lfloor (N-1)/2\rfloor, \lfloor N/2\rfloor\}$. These are precisely the vertices of degree $< 4$ in $\mathrm{Sq}_{N,N}$, and are $4N - 4$ in number if $N \ge 2$, with just one if $N = 1$. If $N \ge 2$, the *corner* vertices of $\mathrm{Sq}_{N,N}$ are the four vertices with $x, y \in \{-\lfloor (N-1)/2\rfloor, \lfloor N/2\rfloor\}$ (which are those of degree 2), and the *side* vertices are the border vertices that are not corner vertices (i.e., those of degree 3, which are $4N - 8$ in number).

Two sets $V_1, V_2 \subseteq \mathbb{Z}^2$ of vertices of $\mathrm{Sq}_{\infty,\infty}$ are *translation-equivalent* if there is a translation of the plane which, when restricted to $V_1$, gives a bijection from $V_1$ to $V_2$. This bijection is then an isomorphism from $\langle V_1\rangle$ to $\langle V_2\rangle$.

A *pattern* is an equivalence class, under translation-equivalence, of vertex sets of connected induced subgraphs of $\mathrm{Sq}_{\infty,\infty}$. We usually refer to a pattern by specifying a single member of its equivalence class. Each member of a pattern *represents* the pattern, and is an *instance* of it. We use bold type for patterns, e.g., $\mathbf{A}$, and italic for its instances. So $A \in \mathbf{A}$ indicates that the vertex set $A$ is an instance of the pattern $\mathbf{A}$.

The *restriction* of a pattern to $\mathrm{Sq}_{N,N}$ is the set of all instances of the pattern that are subsets of $V(\mathrm{Sq}_{N,N})$. Note that the restriction may be empty if $N$ is too small. We write $\mathcal{P}_N$ for the set of all non-empty restrictions of patterns to $\mathrm{Sq}_{N,N}$. We extend pattern notation and terminology ($\mathbf{A}$, instance, ...) to restrictions of patterns in the natural way. Observe that patterns are infinite sets while restrictions of patterns are finite sets.

If $U \subseteq \mathbb{Z}^2$, we speak of the set of vertices in $U$ with a given $x$-co-ordinate (respectively, $y$-co-ordinate) as a *column* (resp., *row*) of $U$.

If $U$ is an instance of a pattern, then the $x$-co-ordinates of its vertices form an integer interval (i.e., a sequence of consecutive integers), and likewise for the $y$-co-ordinates. We write $x_U$ and $y_U$ for the numbers of different $x$-co-ordinates and $y$-co-ordinates, respectively, in $U$. These give the horizontal and vertical extents of $U$, and the numbers of different columns and rows. It is easy to see that

$$x_U + y_U - 1 \le |U| \le x_U y_U. \tag{1}$$

This notation allows us to describe the number of instances of a restriction $\mathbf{A} \in \mathcal{P}_N$ to $\mathrm{Sq}_{N,N}$ of a pattern with instance $U$. For sufficiently large $N$,

$$\begin{aligned}
|\mathbf{A}| &= (N - x_U + 1)(N - y_U + 1) && (2)\\
&= N^2 - (x_U + y_U - 2)N + (x_U y_U - x_U - y_U + 1)\\
&\ge N^2 - (|U| - 1)N + 2|U| \text{ (using (1))}\\
&> N^2 - (|U| - 1)N. && (3)
\end{aligned}$$

A pattern has a unique instance $U$, called its *canonical instance*, such that

$$U \subseteq \{-\lfloor (x_U - 1)/2 \rfloor, \ldots, \lfloor x_U/2 \rfloor\} \times \{-\lfloor (y_U - 1)/2 \rfloor, \ldots, \lfloor y_U/2 \rfloor\}.$$

This can be thought of as a most centrally located instance of the pattern. The canonical instance of the pattern $\mathbf{A}$ is denoted by $[\mathbf{A}]$. If $\mathbf{A}$ is a restriction of a pattern, then $|\mathbf{A}|$, its number of instances, is not to be confused with $|[\mathbf{A}]|$, the number of vertices in its canonical instance. If a restriction of a pattern is nonempty, then it must contain the canonical instance of the pattern.

The number of different patterns in $\mathrm{Sq}_{\infty,\infty}$ whose canonical instance has size $i$ is just the number of $i$-cell fixed polyominoes, which we denote by $a_i$:

$$a_i = |\{\mathbf{A} : |[\mathbf{A}]| = i\}|\,. \tag{4}$$

It is known that $a := \lim_{i\to\infty} a_i^{1/i}$ exists: this is Klarner's constant. In fact $a_i \le a^i$ for all $i$, and $a < 4.65$ [17].

If $U \subseteq \mathbb{Z}^2$, then the *boundary* of $U$, denoted by $\partial U$, is the set of all vertices in $\mathrm{Sq}_{\infty,\infty}$ that are not in $U$ but are adjacent to a vertex in $U$. Note that, in our usage, the boundary of $U$ is always disjoint from it. See Figure 2, where the vertices of the boundary $\partial U$ of the set $U$ of black vertices are indicated by **X**. If $v$ is a vertex, we use $\partial v$ as a shorthand for $\partial\{v\}$; this is just the neighbourhood of $v$. If $U \subseteq \mathrm{Sq}_{N,N}$, then $\partial_N U := \partial U \cap V(\mathrm{Sq}_{N,N})$ is the boundary of $U$ in

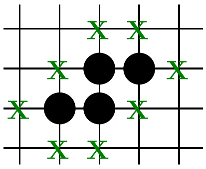

FIGURE 2. A black chromon $U$ and its boundary $\partial U$.

$\mathrm{Sq}_{N,N}$. If $U$ is the canonical instance of a pattern, then $\partial_N U = \partial U$ if and only if $N \geq \max\{x_U, y_U\} + 2$.

We will need some bounds on $|\partial U|$, in the case where $\langle U \rangle$ is connected. Our interest in this is because, in that case, $\langle U \rangle$ is a captured chromon if and only if the vertices in $U$ all get one colour and no vertex of $\partial U$ gets the same colour or is uncoloured.

We first establish an upper bound on $|\partial U|$. Since the maximum degree of $\langle U \rangle$ is $\leq 4$, it is clear that $|\partial U| \leq 4|U|$. Since $4|U|$ counts each edge of $\langle U \rangle$ twice, we have $|\partial U| \leq 4|U| - 2|E(\langle U \rangle)|$. If $U$ is an instance of a pattern, then by connectivity, $|E(\langle U \rangle)| \geq |U| - 1$. So, in fact, $|\partial U| \leq 4|U| - 2(|U| - 1) = 2|U| + 2$.

We now establish a lower bound. For every row of $U$, there is at least one leftmost boundary vertex, and at least one rightmost boundary vertex, and these are distinct. Furthermore, there is a boundary row just above the highest row of $U$, and another just below the lowest row of $U$. Therefore $|\partial U| \geq 2y_U + 2$. Similarly, by considering columns, $|\partial U| \geq 2x_U + 2$. So $|\partial U| \geq 2\max\{x_U, y_U\} + 2$. Now, by the right side of (1), $\max\{x_U, y_U\} \geq \lceil \sqrt{|U|} \rceil$. Hence $|\partial U| \geq 2\lceil \sqrt{|U|} \rceil + 2$.

Summarising,

$$2\lceil \sqrt{|U|} \rceil + 2 \leq |\partial U| \leq 2|U| + 2. \tag{5}$$

We now analyse the expected number of captured chromons.

**Theorem 1.** *The expected number of captured chromons* $\sim 2N^2q^5$, *provided* $q \to 0$ *and* $Nq \to \infty$ *as* $N \to \infty$. *Its limit as* $N \to \infty$ *is*

$$\begin{array}{ll} \infty, & \text{if } q = cN^{-\alpha} \text{ where } \alpha < 2/5; \\ 2c^5, & \text{if } q = cN^{-2/5}; \\ 0, & \text{if } q = cN^{-\alpha} \text{ where } \alpha > 2/5, \end{array} \tag{6}$$

*where* $c$ *is any positive real constant.*

The first two cases of (6) satisfy $Nq \to \infty$ as $N \to \infty$, but the third does not if $\alpha \geq 1$. Nonetheless, in that case ($\alpha > 2/5$), the limit of the expectation is still 0 even if $Nq \not\to \infty$.

*Proof.* For every instance $A$, in $\mathrm{Sq}_{N,N}$, of any pattern, define the indicator random variable

$$X_A = \begin{cases} 1, & \text{if } A \text{ is a captured chromon;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X_A) = \Pr(X_A = 1) = 2q^{|A|+|\partial A|}$. The number of captured chromons is given by the random variable

$$Z = \sum_A X_A,$$

where the sum is over all pattern instances in $\mathrm{Sq}_{N,N}$. We have

$$\begin{aligned} E(Z) &= \sum_A E(X_A) \\ &= 2 \sum_A q^{|A|+|\partial_N A|} \\ &= 2 \sum_{\mathbf{A}\in\mathcal{P}_N} \sum_{A\in\mathbf{A}} q^{|A|+|\partial_N A|} \\ &= 2 \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} \sum_{A\in\mathbf{A}} q^{i+|\partial_N A|}. \end{aligned} \tag{7}$$

We now derive upper and lower bounds on $E(Z)$.

For the lower bound, observe first that $|\partial_N A| \le |\partial[\mathbf{A}]|$ for any instance $A$ of $\mathbf{A}$ (with equality unless $A$ includes vertices on the border of the board), so $q^{|\partial_N A|} \ge q^{|\partial[\mathbf{A}]|}$. Therefore,

$$\begin{aligned} E(Z) &\ge 2 \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} \sum_{A\in\mathbf{A}} q^{i+|\partial[\mathbf{A}]|} \\ &= 2 \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} |\mathbf{A}|\, q^{i+|\partial[\mathbf{A}]|} \\ &\ge 2 \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} (N^2-(i-1)N)\, q^{i+|\partial[\mathbf{A}]|} \text{ (by (3))} \\ &\ge 2 \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} (N^2-(i-1)N)\, q^{3i+2} \text{ (by (5), right side)} \\ &= 2N^2 \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} q^{3i+2} \;-\; 2N \sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N: \\ |[\mathbf{A}]|=i}} (i-1)\, q^{3i+2} \end{aligned}$$

$$\geq 2N^2q^5 \;-\; 2N \sum_{i=1}^{\infty} |\{\mathbf{A} \in \mathcal{P}_N : |[\mathbf{A}]| = i\}| \;\; (i-1)\, q^{3i+2}$$

(dropping all terms except the first from the first sum)

$$\geq 2N^2q^5 \;-\; 2N \sum_{i=1}^{\infty} a_i \;(i-1)\, q^{3i+2} \text{ (using (4))}$$

$$\geq 2N^2q^5 \;-\; 2Nq^2 \sum_{i=1}^{\infty} (i-1)\,(aq^3)^i \text{ (using } a_i \leq a^i)$$

$$= 2N^2q^5 \;-\; 2N \frac{a^2q^8}{(1-aq^3)^2} \text{ (since } a < 4.65 \text{ and } q < \tfrac{1}{2}\text{, so that } aq^3 < 1)$$

$$= 2N^2q^5(1 - O(q^3N^{-1})) \text{ (since } aq^3 < 1-\varepsilon \text{ for some constant } \varepsilon > 0).$$

Now we turn to an upper bound on $E(Z)$.

Of all the instances $A \in \mathbf{A}$ of a pattern restriction $\mathbf{A} \in \mathcal{P}_N$ (for sufficiently large $N$): $\leq (N-2)^2$ of them do not include a border vertex of $\mathrm{Sq}_{N,N}$ (so they also belong to the further restriction of $\mathbf{A}$ to $\mathrm{Sq}_{N-2,N-2}$); $\leq 2(N-2)$ include a top or bottom border vertex only; $\leq 2(N-2)$ include a left or right border vertex only; and $\leq 4$ include both a top/bottom border vertex and a left/right border vertex (i.e., they are pushed as far into a corner as they can go). These cases determine $|\partial_N A|$, for a given $A$.

Observe that, if $A$ is a singleton on a side of the board, then $|\partial_N A| = 3$, while if it is in a corner, $|\partial_N A| = 2$. If $|A| = 2$, then $|\partial_N A| \geq 4$ if $A$ is on a side and $|\partial_N A| \geq 3$ if it is in a corner. If $|A| \geq 3$, then $|\partial_N A| \geq 5$ in the side case and $|\partial_N A| \geq 3$ in the corner case.

From the observations of the previous two paragraphs and (7), we have

$$\begin{aligned}
E(Z) \leq\; & 2\Bigg(\sum_{i=1}^{\infty} \sum_{\substack{\mathbf{A}\in\mathcal{P}_N:\\ |[\mathbf{A}]|=i}} N^2\, q^{i+|\partial[\mathbf{A}]|}\Bigg) \\
& + 2\Bigg(4(N-2)q^4 + 4(N-2)a_2q^6 + 4(N-2)\sum_{i=3}^{\infty} a_i q^{i+5}\Bigg) \\
& + 2\Bigg(4q^3 + 4a_2q^5 + 4\sum_{i=3}^{\infty} a_i q^{i+3}\Bigg)
\end{aligned}$$

(treating middle, side, and corner instances in turn)

$$\begin{aligned}
&\leq 2N^2q^5 + 2N^2 \sum_{i=2}^{\infty} a_i\, q^{i+2\lceil\sqrt{i}\rceil+2} + 8(N-2)q^4 + 8q^3 + 16(N-2)q^6 \\
&\quad + 16q^5 + 8(N-2)q^5 \sum_{i=3}^{\infty} a_i q^i + 8q^3 \sum_{i=3}^{\infty} a_i q^i
\end{aligned}$$

(by (5), left side, applied to the double summation)

$$
\begin{aligned}
&\le 2N^2q^5 + 2N^2\sum_{i=2}^{\infty} a_i\, q^{i+2\lceil\sqrt{2}\rceil+2} \\
&\quad + 2N^2q^5(4(Nq)^{-1} + 4(Nq)^{-2} + 8N^{-1}q + 8N^{-2}) \\
&\quad + 8Nq^5\sum_{i=3}^{\infty}(aq)^i + 8q^3\sum_{i=3}^{\infty}(aq)^i \\
&\quad \text{(using } i \ge 2 \text{ (second summand) and } a_i \le a^i \text{ (last two summands))} \\
&\le 2N^2q^5 + 2N^2q^6\sum_{i=2}^{\infty}(aq)^i \\
&\quad + 2N^2q^5(4(Nq)^{-1} + 4(Nq)^{-2} + 8N^{-1}q + 8N^{-2}) \\
&\quad + 8Nq^5\sum_{i=3}^{\infty}(aq)^i + 8q^3\sum_{i=3}^{\infty}(aq)^i \\
&\quad \text{(using } a_i \le a^i \text{ again)} \\
&= 2N^2q^5 + 2N^2q^6\frac{(aq)^2}{1-aq} \\
&\quad + 2N^2q^5(4(Nq)^{-1} + 4(Nq)^{-2} + 8N^{-1}q + 8N^{-2}) \\
&\quad + 8Nq^5\frac{(aq)^3}{1-aq} + 8q^3\frac{(aq)^3}{1-aq} \\
&\quad \text{(if } aq < 1-\varepsilon \text{ for some constant } \varepsilon > 0) \\
&\le 2N^2q^5 + 2a^2\varepsilon^{-1}N^2q^8 \\
&\quad + 2N^2q^5(4(Nq)^{-1} + 4(Nq)^{-2} + 8N^{-1}q + 8N^{-2}) \\
&\quad + 8a^3\varepsilon^{-1}Nq^8 + 8a^3\varepsilon^{-1}q^6 \\
&= 2N^2q^5(1 + a^2\varepsilon^{-1}q^3 + 4(Nq)^{-1} + 4(Nq)^{-2} + 8N^{-1}q + 8N^{-2} \\
&\quad + 4a^3\varepsilon^{-1}N^{-1}q^3 + 4a^3\varepsilon^{-1}N^{-2}q) \\
&= 2N^2q^5(1 + O(q^3) + O((Nq)^{-1}) + O((Nq)^{-2}) + O(N^{-1}q)).
\end{aligned}
$$

We conclude that $E(Z) \sim 2N^2q^5$, if $q \to 0$ and $Nq \to \infty$ as $N \to \infty$.
In any case, we have

$$
\begin{aligned}
E(Z) &\le 2N^2q^5 + 2a^2\varepsilon^{-1}N^2q^8 + 8Nq^4 + 8q^3 + 16Nq^6 + 16q^5 \\
&\quad + 8a^3\varepsilon^{-1}Nq^8 + 8a^3\varepsilon^{-1}q^6 \\
&\to 0, \text{ if } q = cN^{-\alpha} \text{ and } \alpha > 2/5.
\end{aligned}
$$

The result follows. □

We are now ready to state and prove the main result of this section. In parts (a) and (b), we will see that there is a sharp threshold between extreme legality and extreme illegality. Part (c) will be established as an incidental outcome of the methods used for (a) and (b). A more precise result for that case, and with no special restriction on $c$, will be established in the next section.

**Theorem 2.** (a) *If* $\alpha > \frac{2}{5}$*, then* $\lim_{N\to\infty} \mathrm{Go}(\mathrm{Sq}_{N,N}; cN^{-\alpha}) = 1$*, and in fact the position is a.a.s. superlegal.*

(b) *If* $\alpha < \frac{2}{5}$*, then* $\lim_{N\to\infty} \mathrm{Go}(\mathrm{Sq}_{N,N}; cN^{-\alpha}) = 0$*, and in fact, a.a.s., there is a captured singleton chromon.*

(c) *If* $\alpha = \frac{2}{5}$ *and* $c > 2^{-1/5}$*, then* $\lim_{N\to\infty} \mathrm{Go}(\mathrm{Sq}_{N,N}; cN^{-\alpha})$ *is bounded above by a constant* $< 1$.

*Proof.* We again write $Z$ for the number of captured chromons.

(a) From the third case of (6) in Theorem 1 (which only needs the upper bound on $E(Z)$), it follows that $\Pr(Z \geq 1) \to 0$ as $\infty$, so that, asymptotically almost surely, there are no captured chromons, so that the position is legal. But to prove asymptotic almost sure *super*legality, we use a different, and simpler, approach.

For each $v \in V(G)$, define

$$X_v = \begin{cases} 1, & \text{if } v \text{ is coloured and has no uncoloured neighbour;} \\ 0, & \text{otherwise.} \end{cases}$$

Then $E(X_v) = \Pr(X_v = 1) = (2q)^{\deg v + 1}$. Define

$$Z = \sum_v X_v.$$

Then the position is superlegal if and only if $Z = 0$. So we analyse $E(Z)$. If $N \geq 2$,

$$\begin{aligned} E(Z) &= \sum_v E(X_v) \\ &= (N-2)^2 2^5 q^5 + (4N-8)2^4 q^4 + 4 \cdot 2^3 q^3 \\ &\quad \text{(considering interior, side, and corner vertices in turn)} \\ &= (1-2N^{-1})^2 2^5 c^5 N^{2-5\alpha} + (1-2N^{-1}) 4 \cdot 2^4 c^4 N^{1-4\alpha} + 4 \cdot 2^3 c^3 N^{-3\alpha} \\ &\to 0 \end{aligned}$$

as $N \to \infty$ if $\alpha > 2/5$. Therefore $\Pr(Z = 0) \to 1$ as $N \to \infty$. So, a.a.s., the position is superlegal.

(b), (c): It remains to apply the Second Moment Method to deal with the first two cases of (6) and so prove cases (b) and (c) of the theorem.

It is convenient now to make some observations about captured chromons of size at least two, which we will need later. An analysis very similar to that given in the proof of Theorem 1 (but taking into account the two different orientations that a chromon of size 2 can have) shows that the number $Z_{\geq 2}$ of

(a) (b)

FIGURE 3. A captured singleton chromon in (a) a square lattice graph, and (b) some other graph.

such captured chromons satisfies

$$\lim_{n\to\infty} E(Z_{\geq 2}) = \begin{cases} \infty, & \text{if } q = cN^{-\alpha} \text{ where } \alpha < 1/4; \\ 4c^8, & \text{if } q = cN^{-1/4}; \\ 0, & \text{if } q = cN^{-\alpha} \text{ where } \alpha > 1/4. \end{cases}$$

It follows that, if $\alpha > 1/4$ (which includes $\alpha = 2/5$), then, a.a.s., there are no captured chromons of size $\geq 2$ although, as we shall see shortly, if $\alpha < 2/5$, then a.a.s. there is a captured singleton chromon (see, e.g., Figure 3(a)) so the position is illegal.

Let $Z_1$ be the number of captured singleton chromons in $\mathrm{Sq}_{N,N}$, and for all $v \in V(\mathrm{Sq}_{N,N})$, let $X_v$ be the indicator random variable for the event that $v$ is a captured singleton chromon. Then $E(X_v) = \Pr(X_v = 1) = 2q^{\deg v+1}$ and

$$E(Z_1) = (N-2)^2 \cdot 2q^5 + 4(N-2) \cdot 2q^4 + 4 \cdot 2q^3 \sim 2N^2q^5.$$

Chebyshev's Inequality gives

$$\Pr(Z_1 = 0) \leq \frac{E(Z_1^2)}{E(Z_1)^2} - 1.$$

So we examine $E(Z_1^2)$. We have

$$Z_1^2 = \sum_{u,v} X_u X_v = \sum_u X_u + \sum_{u,v:u\neq v} X_u X_v\,,$$

so

$$E(Z_1^2) = E(Z_1) + \sum_{u,v:u\neq v} \Pr(X_u X_v = 1).$$

For some vertex pairs, there is overlap — the boundary of one vertex includes the other vertex or part of its boundary — but more often they are disjoint.

$$\sum_{u,v:u\neq v} \Pr(X_uX_v = 1)$$

$$= \sum_{\substack{u,v:u\neq v,\\ (\{u\}\cup\partial u)\cap(\{v\}\cup\partial v)\neq\emptyset}} \Pr(X_uX_v = 1) \;+\; \sum_{\substack{u,v:\\ (\{u\}\cup\partial u)\cap(\{v\}\cup\partial v)=\emptyset}} \Pr(X_uX_v = 1). \tag{8}$$

We consider these two sums in turn.

Firstly, consider the overlapping case.

For any vertex $u$, the number of $v$ such that $(\{u\} \cup \partial u) \cap (\{v\} \cup \partial v) \neq \emptyset$ is $\leq 12$ (with equality except near the border).

Up to symmetry, there are just three ways of overlapping, and in each case,

$$|\{u\} \cup \partial u \cup \{v\} \cup \partial v| \in \{8, 9\},$$

provided $u$ and $v$ are not close to the border.

We now bound the first sum in (8).

$$\begin{aligned} \sum_{\substack{u,v:u\neq v,\\ (\{u\}\cup\partial u)\cap(\{v\}\cup\partial v)\neq\emptyset}} \Pr(X_u X_v = 1) &= 2 \sum_{\substack{u,v:u\neq v,\\ (\{u\}\cup\partial u)\cap(\{v\}\cup\partial v)\neq\emptyset}} q^{|\{u\}\cup\partial u\cup\{v\}\cup\partial v|} \\ &\leq 24N^2q^8 + h_0Nq^6 + h_1q^5, \end{aligned}$$

where the second and third terms take care of cases near the border, and $h_0$ and $h_1$ are constants.

Secondly, consider the non-overlapping case.

If neither $u$ nor $v$ is a boundary vertex of $\mathrm{Sq}_{N,N}$ and $(\{u\}\cup\partial u)\cap(\{v\}\cup\partial v) = \emptyset$, then $|\{u\} \cup \partial u \cup \{v\} \cup \partial v| = 10$.

We now bound the second sum in (8).

$$\begin{aligned} &\sum_{\substack{u,v:\\ (\{u\}\cup\partial u)\cap(\{v\}\cup\partial v)=\emptyset}} \Pr(X_u X_v = 1) \\ &= 4 \sum_{\substack{u,v:\\ (\{u\}\cup\partial u)\cap(\{v\}\cup\partial v)=\emptyset}} q^{|\{u\}\cup\partial u\cup\{v\}\cup\partial v|} \\ &\leq 4(N^4q^{10} + k_1N^3q^9 + k_2N^2q^8 + k_3Nq^7 + 12q^6) \\ &\quad \text{(for suitable constants } k_1, k_2, k_3\text{, by grouping the pairs } u, v \\ &\quad \text{according to their degrees in } \mathrm{Sq}_{N,N}) \\ &= E(Z_1)^2 + 4(k_1N^3q^9 + k_2N^2q^8 + k_3Nq^7 + 12q^6). \end{aligned}$$

Combining (8) with our bounds on those two sums, and putting it all back into Chebyshev's Inequality, we have

$$\begin{aligned} &\Pr(Z_1 = 0) \\ &\leq \frac{E(Z_1) + \sum_{u,v:u\neq v} \Pr(X_u X_v = 1)}{E(Z_1)^2} - 1 \\ &\leq \frac{1}{E(Z_1)} + \frac{24N^2q^8+h_0Nq^6+h_1q^5+E(Z_1)^2+4(k_1N^3q^9+k_2N^2q^8+k_3Nq^7+12q^6)}{E(Z_1)^2} - 1 \\ &\leq \frac{1}{E(Z_1)} + \frac{4(k_1N^3q^9 + k_2'N^2q^8 + k_3Nq^7 + (h_0'N+12)q^6 + h_1'q^5)}{(2N^2q^5)^2} \\ &\quad \text{(where } k_2' = k_2 + 6,\ h_0' = h_0/4, \text{ and } h_1' = h_1/4) \\ &= \frac{1}{E(Z_1)} + \frac{k_1N^3q^9 + k_2'N^2q^8 + k_3Nq^7 + (h_0'N+12)q^6 + h_1'q^5}{N^4q^{10}}. \end{aligned}$$

The second summand $\to 0$ as $N \to \infty$ if $q \geq cN^{-\alpha}$ and $\alpha < \frac{3}{4}$.

If $q = cN^{-\alpha}$ and $\alpha < 2/5$, then $E(Z_1) \to \infty$ as $N \to \infty$, so in this case $\Pr(Z_1 = 0) \to 0$ as $N \to \infty$. So, a.a.s., $Z_1 > 0$, i.e., there is a captured singleton chromon. This proves (b).

In the borderline case $q = cN^{-2/5}$, we have $E(Z_1) = 2c^5$ so, a.a.s.,

$$\Pr(Z_1 = 0) \leq \frac{1}{2c^5}.$$

Since (a.a.s.) there are no captured chromons of size $\geq 2$ for such $q$, we see that the probability that the position is legal also has this same upper bound. This proves (c). □

## 4. The borderline case

We now continue the work of the previous section by considering the case $q = cN^{-2/5}$ in more detail. We know from Theorem 2(c) that

$$\lim_{N\to\infty} \mathrm{Go}(\mathrm{Sq}_{N,N}; cN^{-2/5})$$

is bounded above by a constant $< 1$. In this section we determine this limit.

**Theorem 3.**

$$\lim_{N\to\infty} \mathrm{Go}(\mathrm{Sq}_{N,N}; cN^{-2/5}) = e^{-2c^5}.$$

*Proof.* When $q = cN^{-2/5}$, a.a.s. there is no captured chromon of size $\geq 2$, as we saw in the proof of Theorem 2. So the position is a.a.s. legal if and only if a.a.s. it has no captured singletons. So we use again the random variable $Z_1$ which equals the number of captured singleton chromons.

Consider $(Z_1)_k := Z_1(Z_1 - 1)\cdots(Z_1 - (k-1))$. This is the number of sequences of $k$ distinct captured singleton chromons. We will use the following concept.

Let $\tau$ be any sequence of distinct vertices of $\mathrm{Sq}_{N,N}$. Let $\mathrm{Sq}^{(2)}_{N,N}$ be the graph obtained from $\mathrm{Sq}_{N,N}$ by adding an edge between every pair of vertices that are at distance 2 in $\mathrm{Sq}_{N,N}$. Consider the subgraph $\langle\tau\rangle_2$ of $\mathrm{Sq}^{(2)}_{N,N}$ induced by the set of vertices in $\tau$. The vertex sets of components of $\langle\tau\rangle_2$ will be called the *2-components* of $\tau$. Their importance to us is that, if all vertices in a 2-component are to be captured singleton chromons, and some vertex $v$ in the 2-component is given a colour, then the colour of every other vertex in the 2-component is determined by the parity of its distance in $\mathrm{Sq}_{N,N}$ from $v$.

For our purposes, 2-components are of three types: *normal* 2-components, which include a vertex of degree 4 in $\mathrm{Sq}_{N,N}$ (which therefore cannot be a side or corner vertex, although its 2-component may have other vertices on a side or corder of $\mathrm{Sq}_{N,N}$); *side* 2-components, which consist only of vertices on one side; and *corner* 2-components, which either (i) include a corner vertex or (ii) have no corner vertex but include vertices on two sides that are adjacent to a corner (and, in (ii), must include two side vertices adjacent to a corner vertex).

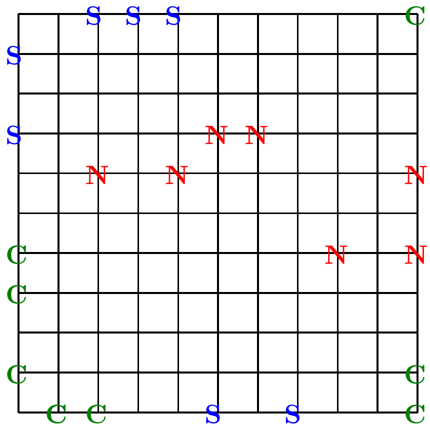


FIGURE 4. $\mathrm{Sq}_{11,11}$ with two normal 2-components (red Ns), three side 2-components (blue Ss) and three corner 2-components (green Cs).

Side 2-components have no vertices of degree 4 and no corner vertices, while corner 2-components may have side vertices from up to two sides but have no vertices of degree 4. The three types of 2-component are illustrated in Figure 4, with vertices of the components marked by letters N, S, or X according to 2-component type. These three types may not exhaust all 2-components for a given $\tau$ and $\mathrm{Sq}_{N,N}$; the other possibility is 2-components that include two or more corner vertices or vertices from three or more sides. But, in the cases that interest us, the length of the sequence $\tau$ is fixed and $N$ can be made sufficiently large, and in such a scenario we can ignore this fourth type of 2-component.

It is easy to show that, if $A$ is the vertex set of a side 2-component, then $|A \cup \partial A| \geq 2|A| + 2$, and if $A$ is the vertex set of a corner 2-component, then $|A \cup \partial A| \geq 2|A|$. The probability that every vertex in a side or corner 2-component $A$ is a captured singleton chromon is therefore $\leq 2q^{2|A|+2}$ for the side case and $\leq 2q^{2|A|}$ for the corner case.

Each 2-component has a *principal* vertex, defined as follows. For a normal 2-component, the principal vertex is its earliest (in $\tau$) degree-4 vertex. For a side 2-component, its principal vertex is its earliest vertex (in $\tau$). For a corner 2-component, its principal vertex is its corner vertex or, if it has no corner vertex, then the earliest (in $\tau$) of the two vertices that (in this case) must be adjacent to the corner

The number of ways of choosing a sequence of $k_1$ distinct vertices that has $i_1$ normal 2-components, and no other 2-components, is the product of the following:

- the number of ways of choosing the positions in the sequence for the $i_1$ principal vertices, which is $\binom{k_1}{i_1}$;
- the number of ways of choosing the $i_1$ principal vertices in order, from among the non-border vertices of $\mathrm{Sq}_{N,N}$, which is $\le ((N-2)^2)^{i_1}$;
- for each such choice, the number of ways of choosing the remaining $k_1 - i_1$ vertices so that each belongs to one of the $i_1$ 2-components associated with the principal vertices, which is $\le (i_1 \cdot (8(k_1-i_1)^2 + 4(k_1-i_1)))^{k_1-i_1}$ (since each remaining vertex must be within distance $2(k_1-i_1)$ of one of the $i_1$ principal vertices, with the factor of 2 here allowing for the possibility of paths that alternate between sequence vertices and their boundary vertices, and the number of vertices within distance $d$ of a specific principal vertex is $2d^2+2d$).

Once these choices are made, the probability that each of these vertices is a captured singleton chromon is $(2q^5)^{i_1}$, if $k_1 = i_1$ (when each normal 2-component is a singleton not on the border), or $\le (2q^5)^{i_1}q^3$, if $k_1 > i_1$, since each of these 2-components includes a vertex of degree 4, and some non-principal vertex of some 2-component must contribute at least a further three vertices to that 2-component. This latter is a very loose upper bound in general, and more precise counting is needed for side and corner 2-components.

The number of ways of choosing a sequence of $k_2$ distinct vertices that has $i_2$ side 2-components, and no other 2-components, is the product of the following:

- the number of ways of choosing the positions in the sequence for the $i_2$ principal vertices, which is $\binom{k_2}{i_2}$;
- the number of ways of choosing the $i_2$ principal vertices, from among the non-border vertices of $\mathrm{Sq}_{N,N}$, which is $\le (4N-8)^{i_2}$;
- for each such choice, the number of ways of choosing the remaining $k_2 - i_2$ vertices so that each belongs to one of the $i_2$ 2-components associated with the principal vertices, which is $\le (i_2 \cdot 4(k_2-i_2))^{k_2-i_2}$ (since each remaining vertex must be within distance $2(k_2-i_2)$, along a side, of one of the $i_2$ principal vertices).

Once these choices are made, the probability that each of these vertices is a captured singleton chromon is $\le 2^{i_2}q^{2k_2+2i_2}$.

The number of ways of choosing a sequence of $k_3$ distinct vertices that has $i_3 \le 4$ corner 2-components, and no other 2-components, is the product of the following:

- the number of ways of choosing the positions in the sequence for the $i_3$ principal vertices, which is $\binom{k_3}{i_3}$;
- the number of ways of choosing the $i_3$ principal vertices in order, which is $\le 12^{i_3}$ (since there are four corner vertices with eight neighbours in total, making 12 options altogether for each principal vertex);
- for each such choice, the number of ways of choosing the remaining $k_3 - i_3$ vertices so that each belongs to one of the $i_3$ corner 2-components,

which is $\leq (i_3 \cdot 4(k_3 - i_3))^{k_3 - i_3}$ (since each remaining vertex must be within distance $2(k_3 - i_3)$ of one of the $i_3$ principal vertices).

The probability that each of these vertices is a captured singleton chromon is $\leq 2^{i_3} q^{2k_3}$.

The number $(Z_1)_k$ of sequences of $k$ distinct captured singleton chromons may be written

$$(Z_1)_k = S_k + T_k, \tag{9}$$

where $S_k$ is the number of such sequences that contain no border vertices and such that every two vertices in the sequence are distance $\geq 3$ apart, and $T_k$ is the number of all other such sequences, i.e., where some singleton chromon lies on the boundary of the board, or some pair of singleton chromons are distance 1 or 2 apart (so that the boundary of one meets the other singleton chromon or its boundary).

We consider $S_k$ first. The number of border vertices is $4N - 4$. The number $s_k$ of sequences of $k$ principal vertices satisfies

$$s_k = (N^2 - 4N + 4)(N^2 - 4N + 4 - \Theta(1)) \cdots (N^2 - 4N + 4 - (k-1)\Theta(1)),$$

with the quantities $\Theta(1)$ each having lower and upper bounds that are constant with respect to both $N$ and $k$. (In fact, for a typical non-border vertex, there are 13 vertices which are distance $\leq 2$ from it, although this quantity is 12 or 11 for vertices at distance 1 from the edge of the board.) Therefore

$$\begin{aligned} E(S_k) &= \left( \prod_{j=0}^{k-1} (N^2 - 4N + 4 - j\Theta(1)) \right) \cdot (2q^5)^k \\ &= \left( \prod_{j=0}^{k-1} (N^2 - 4N + 4 - j\Theta(1)) \right) \cdot (2c^5)^k N^{-2k} \text{ (substituting } q = cN^{-2/5}) \\ &= \left( \prod_{j=0}^{k-1} (1 - 4N^{-1} + (4 - j\Theta(1))N^{-2}) \right) \cdot (2c^5)^k \\ &\to (2c^5)^k \text{ as } N \to \infty. \qquad (10) \end{aligned}$$

Now consider $T_k$. We consider these sequences according to their numbers $i_1, i_2, i_3$ of normal, side, and corner 2-components respectively, with the constraints that $i_1 \neq k$ (else the sequence was considered in $S_k$) and $i_3 \leq 4$. The number of ways of apportioning the $k$ sequence positions among the three types of 2-components — normal, side, corner — so that $k_1$ are in normal 2-components, $k_2$ are in side 2-components and $k_3$ are in corner 2-components, is the trinomial coefficient $\binom{k}{k_1, k_2, k_3}$. For each $j \in \{1, 2, 3\}$, the number of ways of choosing $i_j$ principal vertices — one for each of $i_j$ components of the $j$-th type — is $\binom{k_j}{i_j}$. For each of the three types, with their positions in the sequence now determined, the number of possible subsequences with all vertices at those positions belonging to 2-components of that type has been bounded from above in previous paragraphs.

We now put all this together, with the aid of some notation.

Write $\sum_{k_1,k_2,k_3:...}$ for the sum over all $k_1, k_2, k_3$ such that $k_1 + k_2 + k_3 = k$, $k_1 \geq i_1$, $k_2 \geq i_2$, and $k_3 \geq i_3$. Define

$$\begin{aligned} & h(k, k_1, k_2, k_3, i_1, i_2, i_3) \\ = & \binom{k}{k_1, k_2, k_3}\binom{k_1}{i_1}\binom{k_2}{i_2}\binom{k_3}{i_3}(i_1 \cdot (8(k_1 - i_1)^2 + 4(k_1 - i_1)))^{k_1 - i_1} \\ & \times 2^{i_1} \cdot (i_2 \cdot 4(k_2 - i_2))^{k_2 - i_2} 8^{i_2} \cdot 12^{i_3}(i_3 \cdot 4(k_3 - i_3))^{k_3 - i_3} 2^{i_3}. \end{aligned}$$

The expression $[k_1 > i_1]$ is 1 if $k_1 > i_1$ and 0 otherwise (using the Iverson bracket). We have

$$\begin{aligned} E(T_k) \leq & \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{\min\{4,k-i_1-i_2\}}\sum_{k_1,k_2,k_3:...}\binom{k}{k_1, k_2, k_3}\binom{k_1}{i_1}\binom{k_2}{i_2}\binom{k_3}{i_3} \\ & ((N-2)^2)^{i_1}(i_1 \cdot (8(k_1 - i_1)^2 + 4(k_1 - i_1)))^{k_1 - i_1}(2q^5)^{i_1} q^{3[k_1 > i_1]} \cdot \\ & (4N-8)^{i_2}(i_2 \cdot 4(k_2 - i_2))^{k_2 - i_2} 2^{i_2} q^{2k_2 + 2i_2} \cdot \\ & 12^{i_3}(i_3 \cdot 4(k_3 - i_3))^{k_3 - i_3} 2^{i_3} q^{2k_3} \\ & \text{(with the last three lines here pertaining to the three types of} \\ & \text{2-component; see previous paragraphs)} \\ = & \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{\min\{4,k-i_1-i_2\}}\sum_{k_1,k_2,k_3:...} h(k, k_1, k_2, k_3, i_1, i_2, i_3) \cdot \\ & ((N-2)^2)^{i_1}(N-2)^{i_2} \cdot (q^5)^{i_1} q^{3[k_1 > i_1]} q^{2k_2 + 2i_2} q^{2k_3} \\ \leq & \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{\min\{4,k-i_1-i_2\}}\sum_{k_1,k_2,k_3:...} h(k, k_1, k_2, k_3, i_1, i_2, i_3) \cdot \\ & N^{2i_1 + i_2} \cdot q^{5i_1 + 3[k_1 > i_1] + 2k_2 + 2i_2 + 2k_3} \\ = & \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{\min\{4,k-i_1-i_2\}}\sum_{k_1,k_2,k_3:...} h(k, k_1, k_2, k_3, i_1, i_2, i_3) \cdot \\ & c' \cdot N^{2i_1 + i_2 - 2i_1 - (6/5)[k_1 > i_1] - 4k_2/5 - 4i_2/5 - 4k_3/5} \\ & \text{(substituting } q = cN^{-2/5} \text{ and setting } c' := c^{5i_1 + 3[k_1 > i_1] + 2k_2 + 2i_2 + 2k_3}) \\ = & \sum_{i_1=0}^{k-1}\sum_{i_2=0}^{k-i_1}\sum_{i_3=0}^{\min\{4,k-i_1-i_2\}}\sum_{k_1,k_2,k_3:...} h(k, k_1, k_2, k_3, i_1, i_2, i_3) \cdot \\ & c' \cdot N^{-(6/5)[k_1 > i_1] + i_2/5 - 4k_2/5 - 4k_3/5} \end{aligned}$$

$$\le \sum_{i_1=0}^{k-1} \sum_{i_2=0}^{k-i_1} \sum_{i_3=0}^{\min\{4,k-i_1-i_2\}} \sum_{k_1,k_2,k_3:\ldots} h(k,k_1,k_2,k_3,i_1,i_2,i_3)\cdot$$
$$c' \cdot N^{-(6/5)[k_1>i_1]-3k_2/5-4k_3/5}$$
(using $i_2 \le k_2$).

Observe that the total number of summands in this expression is independent of $N$, being bounded above by a function only of $k$. Furthermore, $h$ and $c'$ are independent of $N$. Each summand has a factor $N^{-(6/5)[k_1>i_1]-3k_2/5-4k_3/5}$ which is the only part of it that depends on $N$. Now consider the exponent of $N$ here. If either $k_2 > 0$ or $k_3 > 0$, then the exponent of $N$ is clearly negative. On the other hand, if $k_2 = k_3 = 0$, then $k = k_1$, since $k_1 + k_2 + k_3 = k$. But $i_1 < k_1$, since $i_1 \le k-1$ else we are in the $S_k$ case, as remarked earlier. So $[k_1 > i_1] = 1$ and the exponent of $N$ is again negative. So, in every summand, the exponent of $N$ is negative. This, with the number of summands being fixed (for fixed $k$), implies that the whole expression $\to 0$ as $N \to \infty$. Since the expression is an upper bound for the nonnegative $E(T_k)$, we have

$$\lim_{N\to\infty} E(T_k) = 0.$$

Combining this with (9) and (10), we conclude that

$$\lim_{N\to\infty} E((Z_1)_k) = (2c^5)^k.$$

Since this holds for all $k$, we find that $Z_1$ approaches a Poisson distribution with mean $2c^5$ as $N \to \infty$ (see, e.g., [2, p. 100]). In particular,

$$\lim_{N\to\infty} \Pr(Z_1 = 0) = e^{-2c^5}.$$

□

## 5. Random graphs

We now look at random Go positions on the random graph $\mathcal{G}(n;p)$. We use the same random colouring model as before, where vertices are independently Black, White, Uncoloured, with probabilities $q$, $q$, $1-2q$, respectively.

We prove three theorems that cover the main asymptotic situations giving almost sure legality or illegality. We do not attempt to cover every combination of possibilities for $p$ and $q$, as functions of $n$.

**Theorem 4.** *If any of the following holds, then the position is a.a.s. superlegal*:

(a) $q = o(n^{-1})$, *or*
(b) $p = \omega(1)n^{-1}$ *and* $q = cn^{-1}$, *or*
(c) $p \ge (1+\varepsilon)(\log f(n))n^{-1}$ *and* $q = f(n)n^{-1}$ *for some unbounded function* $f(n) = o(n)$ *and some* $\varepsilon > 0$, *or*
(d) $p \ge c(\log n)n^{-1}$ *and* $q \le \frac{1}{2} - \varepsilon$ *for some* $\varepsilon > 0$, *where* $c > (1-2q)^{-1}$.

*Proof.* Put

$$X_v := \begin{cases} 1, & \text{if } v \text{ is coloured and has no uncoloured neighbour,} \\ 0, & \text{otherwise,} \end{cases}$$

so that

$$E(X_v) = \Pr(X_v = 1) = 2q(1 - p(1-2q))^{n-1},$$

and

$$Z \;:=\; \#\text{ coloured vertices with no uncoloured neighbour} \;=\; \sum_v X_v\,.$$

Then

$$E(Z) \;=\; \sum_v E(X_v) = \sum_v 2q(1-p(1-2q))^{n-1} = \; 2nq(1-p(1-2q))^{n-1}.$$

In each of the four cases in the hypothesis of the theorem, $E(Z) \to 0$ as $n \to \infty$. For example, considering case (d) (since it abuts another situation we look closely at later), we have

$$E(Z) \le 2nq\left(1 - \frac{c(1-2q)\log n}{n}\right)^{n-1} \sim 2nq\, n^{-c(1-2q)} \;=\; 2q\, n^{1-c(1-2q)} \;\to\; 0.$$

In each case, $\Pr(Z = 0) \to 1$ as $n \to \infty$. So, a.a.s., the position is superlegal. □

Note that case (d) of the theorem includes the possibility that $p$ and $q$ are both constant with $p > 0$ and $q < \frac{1}{2}$.

In our second theorem of this section, we find again that illegality mostly arises from captured singleton chromons (See Figure 3(b)).

**Theorem 5.** *If $p \le c(\log n)n^{-1}$ where $c < (1-q)^{-1}$ and $q$ is constant with $0 < q < \frac{1}{2}$, then a.a.s. there is a captured singleton chromon, so the position is illegal.*

*Proof.* We have

$$X_v := \begin{cases} 1, & \text{if } v \text{ is a captured singleton chromon,} \\ 0, & \text{otherwise,} \end{cases}$$

$$E(X_v) = \Pr(X_v = 1) \;=\; 2q(q + (1-q)(1-p))^{n-1} \;=\; 2q(1-p(1-q))^{n-1},$$

$$Z := \#\text{ captured singleton chromons} \;=\; \sum_v X_v\,,$$

$$E(Z) = \sum_v 2q(1-p(1-q))^{n-1} \;=\; 2nq(1-p(1-q))^{n-1}.$$

Suppose $p, q$ satisfy the hypotheses of the theorem. Then

$$E(Z) \ge 2nq\left(1 - \frac{c(1-q)\log n}{n}\right)^{n-1} \;\sim\; 2q\, n^{1-c(1-q)} \;\to\; \infty \tag{11}$$

as $n \to \infty$. From Chebyshev's Inequality,

$$\Pr(Z=0) \le \frac{1}{E(Z)} + \frac{\sum_{v \ne w} \Pr(v, w \text{ are captured singletons})}{\left(\sum_v \Pr(v \text{ is a captured singleton})\right)^2} - 1. \tag{12}$$

Now,

$$\begin{aligned}
&\sum_{v \ne w} \Pr(v, w \text{ are captured singletons}) \\
&= n(n-1)\ \big(\ 2q^2(1-p)(q+(1-q)(1-p)^2)^{n-2} \\
&\quad + 2q^2(2q(1-p)+(1-2q)(1-p)^2)^{n-2}\ \big) \\
&= 2q^2 n(n-1)\ \big(\ (1-p)(1-p(1-q)(2-p))^{n-2} \\
&\quad + (1-2p(1-q)+p^2(1-2q))^{n-2}\ \big) \\
&= 2q^2 n(n-1)\ \big(\ (1-p)(1-\tfrac{c\log n}{n}(1-q)(2-p))^{n-2} \\
&\quad + (1-2\tfrac{c\log n}{n}(1-q)+p^2(1-2q))^{n-2}\ \big) \\
&\sim 2q^2 n(n-1)\ \big(\ e^{-2c(1-q)\log n} + e^{-2c(1-q)\log n}\ \big) \\
&= 4q^2 n(n-1) e^{-2c(1-q)\log n}.
\end{aligned}$$

Also,

$$\begin{aligned}
\left(\sum_v \Pr(v \text{ is a captured singleton})\right)^2 &= (2nq(1-p(1-q))^{n-1})^2 \\
&= 4q^2n^2(1-p(1-q))^{2n-2} \\
&= 4q^2n^2(1-\tfrac{c\log n}{n}(1-q))^{2n-2} \\
&\sim 4q^2n^2e^{-2c(1-q)\log n}.
\end{aligned}$$

So, in (12), the main fraction $\to 1$ as $n \to \infty$. This, together with $E(Z) \to \infty$ (see (11)), establishes that $\Pr(Z=0) \to 0$ as $n \to \infty$. So, a.a.s, $Z \ge 1$, i.e., there is a captured singleton chromon, so the position is illegal. □

We now consider the gap between Theorems 4 and 5. This is when $p = c(\log n)n^{-1}$ and $(1-q)^{-1} \le c \le (1-2q)^{-1}$.

**Theorem 6.** *If $p = c(\log n)n^{-1}$ where $c > (1-q)^{-1}$ and $q$ is constant with $0 < q < \frac{1}{2}$, then a.a.s. the position is legal.*

*Proof.* Write $F$ for a random position generated according to our model, and $f$ for any specific position.

Put $\varepsilon_n = n^{-\beta}$ with $\beta$ constant and $0 < \beta < \frac{1}{2}$. We say the position is *typical* if

$$(||B| - qn| \le \varepsilon_n qn) \wedge (||W| - qn| \le \varepsilon_n qn) \wedge (||U| - (1-2q)n| \le \varepsilon_n (1-2q)n).$$

We will need the limiting probability of typicality.

$\Pr(F \text{ is typical})$

$$
\begin{aligned}
&= \Pr((||B| - qn| \le \varepsilon_n qn) \wedge (||W| - qn| \le \varepsilon_n qn) \\
&\quad \wedge (||U| - (1-2q)n| \le \varepsilon_n (1-2q)n)) \\
&= 1 - \Pr((||B| - qn| > \varepsilon_n qn) \vee (||W| - qn| > \varepsilon_n qn) \\
&\quad \vee (||U| - (1-2q)n| > \varepsilon_n (1-2q)n)) \\
&\ge 1 - (\Pr(||B| - qn| > \varepsilon_n qn) + \Pr(||W| - qn| > \varepsilon_n qn) \\
&\quad + \Pr(||U| - (1-2q)n| > \varepsilon_n (1-2q)n)) \\
&> 1 - (2e^{-2nq^2\varepsilon_n^2} + 2e^{-2nq^2\varepsilon_n^2} + 2e^{-2n(1-2q)^2\varepsilon_n^2}) \\
&\quad \text{(by large deviation bound for binomial distribution,} \\
&\quad \text{e.g., [2, Theorem A.4])} \\
&\to 1 \text{ as } n \to \infty, \text{ using the definition of } \varepsilon_n.
\end{aligned}
$$

Now,

$$
\begin{aligned}
&\Pr(F \text{ is legal}) \\
&\ge \Pr((F \text{ is legal}) \cap (F \text{ is typical})) \\
&= \sum_{f \text{ is typical}} \Pr((F \text{ is legal}) \cap (F = f)) \\
&= \sum_{f \text{ is typical}} \Pr((f \text{ is legal}) \cap (F = f)) \\
(13)\quad &= \sum_{f \text{ is typical}} \Pr(f \text{ is legal}) \cdot \Pr(F = f) \\
&= \sum_{f \text{ is typical}} \begin{array}{l} \Pr((\text{every component of } \langle B \rangle \text{ has an uncoloured neighbour}) \\ \cap(\text{every component of } \langle W \rangle \text{ has an uncoloured neighbour})) \\ \cdot \Pr(F = f) \end{array} \\
(14)\quad &= \sum_{f \text{ is typical}} \begin{array}{l} \Pr(\text{every component of } \langle B \rangle \text{ has an uncoloured neighbour}) \\ \cdot \Pr(\text{every component of } \langle W \rangle \text{ has an uncoloured neighbour}) \\ \cdot \Pr(F = f). \end{array}
\end{aligned}
$$

Here, (13) follows from the observation that, for a given position $f$, the event that $f$ is legal depends only on the random choices of edges for $G$ under the model $\mathcal{G}(n,p)$, while the event that $F = f$ depends only on the random choices of state, from {B,W,U}, made for each vertex. So these two events are independent.

Consider

$$\Pr(\text{every component of } \langle B \rangle \text{ has an uncoloured neighbour})$$

when $f$ is a fixed typical position, so that $|B|$ lies within the required bounds. The randomness is now entirely in the choice of edge set $E(G)$ for our set of $n$ vertices.

For $X \in \{B, W\}$, denote the vertex set of the largest component of $\langle X \rangle$ by $C_X$.

The edge probability $p$ may be expressed as a function of $|B|$:

$$p = \frac{c \log n}{n} = \frac{c \log(|B|n/|B|)}{|B|n/|B|} = \frac{c|B|}{n} \cdot \frac{\log|B| - \log(|B|/n)}{|B|}.$$

Since $f$ is typical, $(1-\varepsilon_n)qn \le |B| \le (1+\varepsilon_n)qn$. It follows that

$$c(1-\varepsilon_n)q \cdot \frac{\log|B| - \log((1+\varepsilon_n)q)}{|B|} \le p \le c(1+\varepsilon_n)q \cdot \frac{\log|B| - \log((1-\varepsilon_n)q)}{|B|},$$

so

$$\begin{aligned} &c(1-\varepsilon_n)q \cdot \frac{\log|B|}{|B|} - \frac{c(1-\varepsilon_n)q\log((1+\varepsilon_n)q)}{|B|} \\ &\le p \\ &\le c(1+\varepsilon_n)q \cdot \frac{\log|B|}{|B|} - \frac{c(1+\varepsilon_n)q\log((1-\varepsilon_n)q)}{|B|}, \end{aligned}$$

so

$$p = \frac{cq\log|B| - \Theta(1)}{|B|} = \frac{c'}{|B|},$$

where $c' = cq\log|B| - \Theta(1)$.

Since $p|B|$ is unbounded (since $c'$ is), the subgraph $\langle B \rangle$ almost certainly has a giant component (see, e.g., [2, §10.5]). Write $y$ for the limiting proportion (as $|B| \to \infty$) of vertices of $\langle B \rangle$ that are not in the giant component, so that the giant component's size approaches $(1-y)|B|$ vertices in the limit. Standard analysis of the size of the giant component [2, pp. 152, 155] gives, a.a.s.,

$$|B|(1-|B|^{-cq}) < |C_B| < |B|(1-|B|^{-cq+o(1)}), \tag{15}$$

$$|B|^{1-cq} < |B \setminus C_B| < |B|^{1-cq+o(1)}, \tag{16}$$

$$|B|^{-cq} < y < |B|^{-cq+o(1)}. \tag{17}$$

We will shortly need some of these bounds in terms of $n$. The left-hand side of (15) gives

$$|C_B| > (1-\varepsilon_n)qn(1-((1-\varepsilon_n)qn)^{-cq}) \ge (1-\varepsilon_n)qn(1-y_1) \tag{18}$$

for any $y_1 \ge ((1-\varepsilon_n)qn)^{-cq}$. We could take $y_1 = \frac{1}{2}$, for example. The right-hand side of (16) gives

$$|B \setminus C_B| < ((1+\varepsilon_n)qn)^{1-cq+o(1)} \le dn^{1-cq+o(1)} \tag{19}$$

for suitable constant $d \ge ((1+\varepsilon_n)q)^{1-cq+o(1)}$.

Given $B, W, U$ (which are determined by our fixed $f$) and the choice of the edge set $E(\langle B \rangle)$ (which, along with $B$, determines $C_B$), the choice of $E(C_B, U)$

is independent of the choice of $E(B\setminus C_B, U)$. So the event that the largest component $\langle C_B\rangle$ of $\langle B\rangle$ has an uncoloured neighbour is independent of any event relating to whether or not the vertices of $B \setminus C_B$ have uncoloured neighbours. Therefore,

$$
\begin{aligned}
&\Pr(\text{every component in } \langle B\rangle \text{ has an uncoloured neighbour})\\
\geq\ &\Pr((\langle C_B\rangle \text{ has an uncoloured neighbour})\\
&\cap (\text{every vertex in } B \setminus C_B \text{ has an uncoloured neighbour}))\\
=\ &\Pr(\langle C_B\rangle \text{ has an uncoloured neighbour})\\
&\cdot \Pr(\text{every vertex in } B \setminus C_B \text{ has an uncoloured neighbour})\\
=\ &(1-(1-p)^{|U|\cdot|C_B|})\cdot(1-(1-p)^{|U|})^{|B\setminus C_B|}\\
&(\text{where } |B|, |W|, |U| \text{ must satisfy the requirements of typicality})\\
\geq\ &(1-(1-p)^{(1-\varepsilon_n)(1-2q)n\cdot|C_B|})\cdot(1-(1-p)^{(1-\varepsilon_n)(1-2q)n})^{|B\setminus C_B|}\\
&(\text{using typicality})\\
=\ &\left(1-\left(1-\frac{c\log n}{n}\right)^{(1-\varepsilon_n)(1-2q)n\cdot|C_B|}\right)\\
&\cdot\left(1-\left(1-\frac{c\log n}{n}\right)^{(1-\varepsilon_n)(1-2q)n}\right)^{|B\setminus C_B|}\\
\geq\ &\left(1-e^{-c(1-\varepsilon_n)(1-2q)\log n\cdot|C_B|}\right)\cdot\left(1-e^{-c(1-\varepsilon_n)(1-2q)\log n}\right)^{|B\setminus C_B|}\\
=\ &\left(1-\frac{1}{n^{c(1-\varepsilon_n)(1-2q)\cdot|C_B|}}\right)\cdot\left(1-\frac{1}{n^{c(1-\varepsilon_n)(1-2q)}}\right)^{|B\setminus C_B|}\\
=\ &\left(1-\frac{1}{n^{c(1-\varepsilon_n)^2(1-2q)q(1-y_1)n}}\right)\cdot\left(1-\frac{1}{n^{c(1-\varepsilon_n)(1-2q)}}\right)^{dn^{1-cq+o(1)}}\\
&(\text{using (18) and (19)})\\
=\ &\left(1-\frac{1}{n^{c(1-\varepsilon_n)^2(1-2q)q(1-y_1)n}}\right)\\
&\cdot\left(1-\frac{1}{n^{c(1-\varepsilon_n)(1-2q)}}\right)^{n^{c(1-\varepsilon_n)(1-2q)}dn^{-c(1-\varepsilon_n)(1-2q)+1-cq+o(1)}}\\
\geq\ &\left(1-\frac{1}{n^{c(1-\varepsilon_n)^2(1-2q)q(1-y_1)n}}\right)\cdot e^{-dn^{1-c(1-q)+o(1)}}\\
&(\text{using } c\varepsilon_n(1-2q)=o(1) \text{ in the last exponent}).
\end{aligned}
$$

The first factor here goes to 1 as $n\to\infty$. The last factor will do so too if $1-c(1-q)<0$, i.e., if $c>1/(1-q)$. So, as $n\to\infty$,

$$\Pr(\text{every component in } \langle B\rangle \text{ has an uncoloured neighbour}) \to 1,$$

and similarly,

$$\Pr(\text{every component in } \langle W \rangle \text{ has an uncoloured neighbour}) \to 1.$$

Substituting into (14), we find that, as $n \to \infty$,

$$\begin{aligned}\Pr((F \text{ is legal}) \cap (F \text{ is typical})) &\sim \sum_{f \text{ is typical}} \Pr(F = f) \\ &= \Pr(F \text{ is typical}) \\ &\to 1,\end{aligned}$$

as we saw early in the proof.

Hence

$$\Pr(F \text{ is legal}) \to 1$$

as $n \to \infty$, i.e., the position is asymptotically almost surely legal. □

## 6. Future work

There are many possible directions for further research.

**1. Other lattice graphs**

It is a routine exercise to modify the results and proofs of §§3–4 to deal with other lattice graphs such as regular or semi-regular tessellations of the plane.

**2. Random planar graphs**

We looked at Go positions on classical random graphs in §5. Since Go is traditionally played on planar graphs, it might be interesting to look at $\mathrm{Go}(G;q)$ when $G$ is a random planar graph [7, 19].

**3. Random regular graphs**

Since the usual Go board is "nearly" 4-regular, it would be natural to investigate $\mathrm{Go}(G;q)$ when $G$ is a random regular graph [24].

**4. Complexity of approximating $\mathrm{Go}(G;q)$.**

We know that $\mathrm{Go}(G;q)$ is #P-hard to compute exactly [10]. But what about efficient approximations? Is there a Fully Polynomial Randomised Approximation Scheme (FPRAS) for it? Is there a rapidly-mixing Markov chain among legal Go positions?

**5. Random generation of legal positions**

Related to the previous item is the question of how easy it is to generate legal positions uniformly at random. For a large board, generating arbitrary random positions until a legal one is found may take too many iterations to be useful. Another method might be to generate an arbitrary random position and then make it legal, using some process of removing captured stones (although this may be tricky since, in such situations, stones could be mutually capturing).

**6. Different probabilities for Black and White.**

In our scenario, each colour Black/White has probability $q$ of being assigned to a given vertex. Suppose instead that we use two different probabilities, $q_{\text{Black}}$ and $q_{\text{White}}$ for a vertex being Black or White (where $q_{\text{Black}} + q_{\text{White}} \leq 1$). It would be routine to extend our results to this situation. Of greater interest might be the following question. If a random legal position is used to start play, the player with the higher probability can expect to have an advantage. How great is this advantage, in terms of $q_{\mathrm{Black}}$, $q_{\mathrm{White}}$? This is really an empirical question, and would require appropriate measures of advantage.

## 7. Random play

We have focused on random *positions*. But it is natural to ask about random *play*. Suppose each player, in turn, chooses uniformly at random from among all vertices on the board on which it is legal for them to place a stone at that time. Alternatively, the player could choose from all unoccupied vertices, but an illegal move is treated as a Pass, with the usual Go rule that two consecutive passes — one by each player — define the end of the game. What can we say about the length of the game, the time to the first capture, the number of stones captured, the final scores (whose expectations will be very nearly equal), and so on?

## 8. The Go constant

Define

$$\begin{aligned} g(N,N) &:= \text{number of legal Go positions on the } N \times N \text{ square lattice} \\ &= \mathrm{Go}(\mathrm{Sq}(N,N); \tfrac{1}{3}) \cdot 3^{N^2} \\ g &:= \lim_{N\to\infty} g(N,N)^{1/N^2}. \end{aligned}$$

It is not difficult to show that the limit $g$ exists, using log-superadditivity and Fekete's Lemma [11, 23].

The following problems are discussed in [11, 23].

(a) Determine $g_{19,19}$, the number of legal positions on the standard Go board.
(b) Determine the limiting constant $g$.

Random sampling of positions on the $19 \times 19$ board and testing for legality indicates that about 1.2% of partial colour assignments are legal positions, so $g_{19,19} \approx 0.012 \times 3^{19^2} \approx 2.1 \times 10^{170}$. This estimate is attributed in [23] to A. Flammenkamp.

Transfer matrix methods were used in [11] to establish the following (deterministic) bounds:

$$1.743\ldots \times 10^{169} \leq g_{19,19} \leq 6.432\ldots \times 10^{170}.$$

We did exact computations for $k \times n$ strips, $k \leq 6$ and obtain the bounds by partitioning the $19 \times 19$ board into strips.

Independently, Tromp and Farnebäck [23] have pushed the implementation of the transfer matrix method further, with impressive and intricate computations. That paper reports their exact determination of $g_{17,17}$ and gives the estimate

$$g_{19,19} \approx 2.08168199382 \times 10^{170}.$$

At that time, the estimate relied on a conjecture of theirs on a rate of convergence. They suggested that exact calculation of $g_{19,19}$ would be possible using their methods "within a decade", and so it has turned out to be. Tromp completed the exact computation of $g_{19,19}$ on 20 January 2016 [22], obtaining the 171-digit value

20816 81993 81979 98469 94786 33344 86277 02865 22453 88453
05484 25639 45682 09274 19612 73801 53785 25648 45169 85196
43907 25991 60156 28128 54608 98883 14427 12971 53193 17557
73662 03972 47064 84093 5

which accords with their previous estimate.

For the limiting constant $g$, our bounds, from [11], are:

$$2.958\ldots \leq g \leq 2.983\ldots.$$

Tromp and Farnebäck [23] give the estimate

$$g \approx 2.9757341920433572493.$$

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